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HARMONIC ANALYSIS ASSOCIATED WITH THE MODIFIED CHEREDNIK TYPE OPERATOR ON THE REAL LINE AND PALEY–WIENER THEOREMS FOR ITS HARTLEY TRANSFORM

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Abstract. We consider a new differential-difference operator Λ on the real line. We study the harmonic analysis associated with this operator. Next, we establish the Paley–Wiener theorems for its Hartley transform on $\mathbb{R}$.

Dedicated to Professor Emeritus Khalifa Trimèche.

1. Introduction

The Hartley transform is an integral transform, attributed to Ralph Vinton Lyon Hartley (cf. [3, 11]). This transform permits a function to be decomposed into two independent sets of sinusoidal components; these sets are represented in terms of positive and negative frequency components, respectively. This is in contrast to the complex exponential, $\exp(i\lambda x)$, used in classical Fourier analysis. The Hartley transform has advantages over the Fourier transform: it transforms real functions into real functions (as opposed to requiring complex numbers); also this transform has complementary symmetry properties with respect to its real and imaginary axis. Moreover, it is well known that the Hartley transform is connected with various applications in mathematical physics.

The familiar reciprocal pair of the Hartley transforms for f in a suitable function class is given by

$$\begin{aligned} \mathcal{H}_c(f)(\lambda) &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \operatorname{cas}(\lambda x)\, dx \\ f(x) &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \mathcal{H}_c(f)(\lambda) \operatorname{cas}(\lambda x)\, d\lambda, \end{aligned} \tag{1}$$

where

$$\operatorname{cas}(y) := \cos y + \sin y. \tag{2}$$

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The cas function (2) can be considered as a generalization of the exponential function exp. A simple computation shows that the cas function is the unique C^∞ solution of the differential-reflection problem

$$Y(f)(x) = \lambda f(x), \quad f(0) = 1,$$

where Y is the differential-reflection operator defined by

$$Y(f)(x) = \left(\frac{d}{dx} \circ s\right) f(x) := f'(-x).$$

Here s is the reflection, acting on the function f as

$$s(f)(x) := f(-x).$$

Further, the function $\operatorname{cas}(x)$ satisfies the product formula

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}\left((1-s)\operatorname{cas}\right)(x+y) + \frac{1}{2}\left((1+s)\operatorname{cas}\right)(x-y).$$

This allows us to define the generalized translation operator related to the differential-reflection operator Y by

$$\sigma_x f(y) = \frac{1}{2}\left((1-s)f\right)(x+y) + \frac{1}{2}\left((1+s)f\right)(x-y),$$

and the convolution product by

$$f*g(x) = \int_{\mathbb{R}} \sigma_x f(y) g(y)\, dy. \tag{3}$$

The Hartley transform has the following properties:

$$\mathcal{H}_c(Yf)(\lambda) = -\lambda \mathcal{H}_c(f)(\lambda),$$
$$\mathcal{H}_c(\sigma_x f)(\lambda) = \operatorname{cas}(\lambda x)\mathcal{H}_c(f)(\lambda),$$
$$\mathcal{H}_c(f*g)(\lambda) = \mathcal{H}_c(f)(\lambda)\mathcal{H}_c(g)(\lambda).$$

In this paper, we consider the first-order singular differential-difference operator on $\mathbb{R}$

$$\Lambda f(x) = Yf(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \rho f(-x),$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

i) For all $x \geq 0, A(x)$ is increasing and $\lim_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\frac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty)$, $x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\delta x} D(x), & \text{if } \varrho > 0 \\ \frac{2\alpha+1}{x} + e^{-\delta x} D(x), & \text{if } \varrho = 0, \end{cases}$$

where D is a C^∞ function, bounded together with its derivatives.

The purpose of the present paper is twofold. On one hand, we want to provide a new harmonic analysis on the real line corresponding to the differential-difference operator Λ. More precisely, we study the Hartley transform associated with the operator Λ, we prove an inversion formula and a Plancherel theorem. On the other hand, we want to highlight the close connection between Roe's theorem and real Paley–Wiener theorems.

The Paley–Wiener theorems associated with partial differential operators are one of the most useful subjects in harmonic analysis. In this paper we present only three themes for this subject.

The first theme is the real Paley–Wiener theorems. The fundamental theorem given by Bang (cf. [2]) can be stated as follows. Let f be a C^∞ function on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, the function $\frac{d^n}{dx^n} f$ belongs to the Lebesgue space $L^p(\mathbb{R})$, then the limit $R_f := \lim_{n\to\infty} \|\frac{d^n}{dx^n} f\|_p^{1/n}$ exists and we have

$$R_f = \sup \{|\lambda| : \lambda \in \operatorname{supp} \mathcal{F}(f)\},$$

where $\mathcal{F}(f)$ is the classical Fourier transform of f. Next the analogue of this theorem was established for many other integral transforms; see for example [7, 13, 21].

The second theme is the study of tempered distributions with spectral gaps. More precisely, a tempered distribution on $\mathbb{R}$ whose Fourier transform is supported in an interval $[-M, M]$, where $M > 0$, can be characterized by the behaviour of its successive derivatives. On the other hand, a tempered distribution on $\mathbb{R}$ whose Fourier transform vanishes in an interval $(-M, M)$, where $M > 0$, can be characterized by the behaviour of a particular sequence of successive antiderivatives.

The third theme is Roe's theorem. The fundamental theorem given by Roe (cf. [15]) can be stated as follows. If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π.

The remaining part of the paper is organized as follows. In Section 2, we study the harmonic analysis associated with the modified Cherednik type operator on the real line. More precisely, we study the eigenfunctions of the operator Λ. In particular we prove the Laplace integral representation for these eigenfunctions. Next, we prove the inversion theorems and Plancherel's formula for the Hartley transform associated with the operator Λ, noted $\mathcal{H}_\Lambda$. Section 3 is devoted to characterizing the functions in the generalized Schwartz spaces such that their Hartley transform $\mathcal{H}_\Lambda$ vanishes outside a polynomial domain. In Section 4 we prove new versions of real Paley–Wiener theorems for the Hartley transform $\mathcal{H}_\Lambda$. In Section 5 we prove Roe's theorem for the modified Cherednik type operator on $\mathbb{R}$. Finally, in the last section we study the tempered distributions with spectral gaps.

2. Harmonic analysis associated with the modified Cherednik type operator on the real line

In this section we study the harmonic analysis related to the operator Λ.

Notations. We denote by

- $\mathbb{R}_\varrho := \{x \in \mathbb{R} : |x| \geq \varrho\}$.
- $\mathcal{E}(\mathbb{R})$, the space of C^∞ functions on $\mathbb{R}$.
- $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $D_e(\mathbb{R})$, the space of even C^∞ functions on $\mathbb{R}$ which are of compact support.
- $D(\mathbb{R})$, the space of C^∞ functions on $\mathbb{R}$ which are of compact support.
- $D'(\mathbb{R})$, the space of distributions on $\mathbb{R}$.

2.1. **The eigenfunctions of the operator Λ.** To study the eigenfunctions of Λ, we consider first those of the second-order singular differential operator on $\mathbb{R}$ defined by

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

Our basic reference about L will be the papers [19, 20] from which we recall the following result.

Lemma 2.1. *(i) For each $\lambda \in \mathbb{C}$ the differential equation*

$$Lu = -(\lambda^2 + \varrho^2)u, \quad u(0) = 1, \quad u'(0) = 0, \tag{4}$$

admits a unique even C^∞ solution on $\mathbb{R}$, denoted φ_λ.

(ii) For every $x \in \mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic.

(iii) For every $x \in \mathbb{R}$,

$$e^{-\varrho|x|} \leq \varphi_0(x) \leq 1.$$

(iv) For all $x > 0$ and $\lambda \in \mathbb{C}$, the function φ_λ possesses the Laplace type integral representation

$$\varphi_{\sqrt{\lambda^2-\varrho^2}}(x) = \int_0^x \mathcal{K}(x,y)\cos(\lambda y)\,dy, \tag{5}$$

where $\mathcal{K}(x,.)$ is a positive continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$.

(v) There is a positive constant C such that for all $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| \leq C|x|^n e^{(|\operatorname{Im}\lambda|-\varrho)\,|x|}.$$

Remark 2.2. If $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, then the differential operator L reduces to the so-called Jacobi operator. The eigenfunction φ_λ is the Jacobi function of index (α,β) given by

$$\varphi_\lambda(x) = F\left(\frac{1}{2}(\rho+\lambda), \frac{1}{2}(\rho-\lambda);\ \alpha+1;\ -(\sinh(x))^2\right), \tag{6}$$

where F is Gauss's hypergeometric function ${}_2F_1$.

Proposition 2.3. *For each $\lambda \in \mathbb{C}$ the differential-difference equation*

$$\Lambda u = \lambda u, \quad u(0) = 1, \tag{7}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ and given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) + \frac{1}{\lambda-\varrho} Y\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x), & \text{if } \lambda \neq \varrho \\ 1 + \operatorname{sgn}(x)\frac{2\varrho}{A(x)} \int_0^{|x|} A(t)\,dt, & \text{if } \lambda = \varrho. \end{cases} \tag{8}$$

Proof. Write $u = u_e + u_o$ with

$$u_e(x) = \frac{u(x)+u(-x)}{2} \quad \text{and} \quad u_o(x) = \frac{u(x)-u(-x)}{2}.$$

Then (7) is equivalent to the system

$$\begin{cases} Yu_o + \dfrac{A'(x)}{A(x)} u_o = (\lambda+\rho)u_e \\ Yu_e = (\lambda-\rho)u_o \\ u_e(0) = 1. \end{cases} \tag{9}$$

If $\lambda \neq \varrho$, the identity (8) is now immediate from Lemma 2.1 and the relation (9). On the other hand, we have

$$Y\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) = \operatorname{sgn}(x)\frac{(\lambda^2-\varrho^2)}{A(x)} \int_0^{|x|} \varphi_{\sqrt{\lambda^2-2\varrho^2}}(t)A(t)\,dt.$$

Then, for $\lambda \neq \varrho$

$$\frac{1}{\lambda-\varrho} Y\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) = \operatorname{sgn}(x)\frac{(\lambda+\varrho)}{A(x)} \int_0^{|x|} \varphi_{\sqrt{\lambda^2-2\varrho^2}}(t)A(t)\,dt.$$

Thus if $\lambda = \varrho$, we have

$$\begin{aligned} u(x) &= u_e(x) + u_o(x) \\ &= \lim_{\lambda\to\varrho} \left(\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) + \frac{1}{\lambda-\varrho} Y\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) \right) \\ &= \varphi_{i\varrho}(x) + \operatorname{sgn}(x)\frac{2\varrho}{A(x)} \int_0^{|x|} \varphi_{i\varrho}(t)A(t)\,dt \\ &= 1 + \operatorname{sgn}(x)\frac{2\varrho}{A(x)} \int_0^{|x|} A(t)\,dt. \end{aligned}$$

□

Remark 2.4. For all $x \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}$, we can write

$$\Phi_\lambda(x) = \varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) + \operatorname{sgn}(x)\,\frac{\lambda+\rho}{A(x)} \int_0^{|x|} \varphi_{\sqrt{\lambda^2-2\varrho^2}}(t)A(t)\,dt. \tag{10}$$

The eigenfunctions Φ_λ possess the following properties:

i) There exists a positive constant C such that for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, with $|\lambda| \geq \sqrt{2}\varrho$ and $n \in \mathbb{N}_0$, we have

$$\left| \frac{d^n}{d\lambda^n} \Phi_\lambda(x) \right| \leq C(1+|\lambda|+\rho)(1+|x|)^n e^{-\varrho\,|x|}. \tag{11}$$

ii) There exists a positive constant C such that for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, with $|\lambda| \leq \sqrt{2}\varrho$, we have

$$\left|\frac{d^n}{d\lambda^n}\Phi_\lambda(x)\right| \leq C(1+|\lambda|+\rho)(1+|x|)^n e^{(\sqrt{2\varrho^2-\lambda^2}-\rho)\,|x|}.$$

The following lemma shall be useful.

Lemma 2.5. *(i) For all $x > 0$ and $\lambda \in \mathbb{C}$, we have*

$$\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x) = \int_0^x \mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy, \tag{12}$$

where $\mathcal{K}(x,.)$ is the positive continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$, given in (5).

(ii) For all $x \in \mathbb{R}\backslash\{0\}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\frac{\lambda+\varrho}{A(x)}&\int_0^{|x|}\varphi_{\sqrt{\lambda^2-2\varrho^2}}(t)A(t)\,dt\\ &= \frac{1}{2A(x)}\int_{-|x|}^{|x|}\left(\varrho\mathcal{G}_\mathcal{K}(x,y)-\frac{\partial}{\partial y}\mathcal{G}_\mathcal{K}(x,y)\right)\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}\,y)\,dy,\end{aligned} \tag{13}$$

where

$$\mathcal{G}_\mathcal{K}(x,y) = \begin{cases}\displaystyle\int_{|y|}^{|x|}\mathcal{K}(t,y)A(t)\,dt & \textit{if } |y|<|x|,\\ 0 & \textit{if } |y|\geq |x|.\end{cases}$$

Proof. The first assertion (i) is immediate from (5).

Now we want to prove (ii). From the relation (12) and Fubini's theorem, we obtain

$$\begin{aligned}\frac{\lambda+\varrho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-\varrho^2}}(t)A(t)\,dt &= \frac{\lambda+\varrho}{A(x)}\int_0^{|x|}\left(\int_0^t\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy\,\mathcal{K}(t,y)\right)A(t)\,dt\\ &= \frac{\lambda+\varrho}{A(x)}\int_0^{|x|}\left(\int_{|y|}^{|x|}\mathcal{K}(t,y)A(t)\,dt\right)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy\\ &= \frac{\lambda+\varrho}{A(x)}\int_0^{|x|}\mathcal{G}_\mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy.\end{aligned} \tag{14}$$

Moreover, by integration by parts, we obtain

$$\frac{\lambda}{A(x)}\int_0^{|x|}\mathcal{G}_\mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy = \frac{-1}{2A(x)}\int_{-|x|}^{|x|}\frac{\partial}{\partial y}\mathcal{G}_\mathcal{K}(x,y)\sin(\sqrt{\lambda^2-\varrho^2}\,y)\,dy.$$

On the other hand,

$$\frac{\varrho}{A(x)}\int_0^{|x|}\mathcal{G}_\mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy = \frac{\varrho}{2A(x)}\int_{-|x|}^{|x|}\mathcal{G}_\mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy.$$

Thus

$$\frac{\lambda+\varrho}{A(x)}\int_0^{|x|}\mathcal{G}_{\mathcal{K}}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy$$
$$=\frac{1}{2A(x)}\int_{-|x|}^{|x|}\left(\varrho\mathcal{G}_{\mathcal{K}}(x,y)-\frac{\partial}{\partial y}\mathcal{G}_{\mathcal{K}}(x,y)\right)\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}\,y)\,dy.$$

This finishes the proof. □

Theorem 2.6. *The eigenfunction $\Phi_\lambda(x)$ has the following Laplace integral representation:*

$$\forall\, x\in\mathbb{R}\backslash\{0\},\ \forall\,\lambda\in\mathbb{C},\quad \Phi_\lambda(x)=\int_{-|x|}^{|x|}K(x,y)\,\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}\,y)\,dy,$$

where $K(x,y)$ is a continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$, given by

$$K(x,y)=\frac{1}{2}\mathcal{K}(x,y)+\varrho\frac{\mathrm{sgn}(x)}{2A(x)}\mathcal{G}_{\mathcal{K}}(x,y)-\frac{\mathrm{sgn}(x)}{2A(x)}\frac{\partial}{\partial y}\mathcal{G}_{\mathcal{K}}(x,y). \tag{15}$$

Proof. For all $x\in\mathbb{R}\backslash\{0\}$ and $\lambda\in\mathbb{C}$, the relations (10), (12) and (13) give that

$$\begin{aligned}\Phi_\lambda(x)&=\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x)+\mathrm{sgn}(x)\,\frac{\lambda+\rho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-2\varrho^2}}(y)A(y)\,dy\\&=\int_0^{|x|}\mathcal{K}(x,y)\cos(\sqrt{\lambda^2-\varrho^2}\,y)\,dy+\mathrm{sgn}(x)\,\frac{\lambda+\rho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-2\varrho^2}}(y)A(y)\,dy\\&=\int_{-|x|}^{|x|}\left(\frac{1}{2}\mathcal{K}(x,y)+\frac{\varrho\,\mathrm{sgn}(x)}{2A(x)}\mathcal{G}_{\mathcal{K}}(x,y)-\frac{\mathrm{sgn}(x)}{2A(x)}\frac{\partial}{\partial y}\mathcal{G}_{\mathcal{K}}(x,y)\right)\\&\qquad\cdot\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}\,y)A(y)\,dy.\end{aligned}$$

Thus

$$\Phi_\lambda(x)=\int_{-|x|}^{|x|}K(x,y)\,\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}\,y)\,dy.$$ □

Using the similar method presented in [20], we prove that for all $x\in\mathbb{R}\backslash\{0\}$ and $y\in\mathbb{R}$ the kernel $K(x,y)$ is positive.

We consider the generalized intertwining operator V, defined on $\mathcal{E}(\mathbb{R})$ by

$$V\,f(x)=\begin{cases}\displaystyle\int_{-|x|}^{|x|}K(x,y)f(y)\,dy & \text{if } x\in\mathbb{R}\setminus\{0\},\\ f(0) & \text{if } x=0,\end{cases} \tag{16}$$

where $K(x,y)$ is the continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$, defined by relation (15).

We have

$$\forall\lambda\in\mathbb{C},\ \forall x\in\mathbb{R},\quad \Phi_\lambda(x)=V\,(\mathrm{cas}(\sqrt{\lambda^2-\varrho^2}))(x). \tag{17}$$

Using the similar method presented in [19, 20], we prove the following results.

The operator V is a topological automorphism of $\mathcal{E}(\mathbb{R})$ satisfying

$$\forall f \in \mathcal{E}(\mathbb{R}), \quad \Lambda(Vf) = V(Yf).$$

The operator ${}^tV$ is defined on $D(\mathbb{R})$ by

$$\forall y \in \mathbb{R}, \quad {}^tV(f)(y) = \int_{|x|\geq|y|} K(x,y) f(x) A(x)\,dx.$$

The operator ${}^tV$ is a topological automorphism of $D(\mathbb{R})$. The operators V and ${}^tV$ possess the following properties:

(i) For all $f \in D(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$,

$$\int_{\mathbb{R}} {}^tV(f)(y) g(y)\,dy = \int_{\mathbb{R}} f(x) Vg(x) A(x)\,dx. \tag{18}$$

(ii) The inverse operator ${}^tV^{-1}$ is given by

$${}^tV^{-1} f = {}^t\chi^{-1}(f_e) + (\varrho I - Y)\,{}^t\chi^{-1}(Jf_o),$$

where ${}^t\chi^{-1}$ is the inverse of the transmutation operator associated with the operator L and J is the integral operator defined by

$$Jf(x) = \int_{-\infty}^{x} f(t)\,dt, \quad x \in \mathbb{R}. \tag{19}$$

(iii) The generalized intertwining operator V and its dual ${}^tV$ are positive.

Corollary 2.7. *For all $\lambda, x \in \mathbb{R}$, $|\lambda| \geq \varrho$, we have*

$$|\Phi_\lambda(x)| \leq 2. \tag{20}$$

Proof. The relations (16), (17) and the positivity of the kernel $K(x,y)$ give the result. □

2.2. **Hartley transform associated with the modified Cherednik type operator.**

Notations. We denote by

- $L^p_A(\mathbb{R})$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{L^p_A(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p A(x)\,dx \right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty$$
$$\|f\|_{L^\infty_A(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty.$$

- $\mathcal{S}^2(\mathbb{R})$, the space of C^∞ functions on $\mathbb{R}$ such that for all $m, n \in \mathbb{N}$

$$q_{n,m}(f) := \sup_{x\in\mathbb{R}} (\cosh x)^\varrho (1+x^2)^m \left| \frac{d^n}{dx^n} f(x) \right| < \infty.$$

  The topology of $\mathcal{S}^2(\mathbb{R})$ is defined by the semi-norms $q_{n,m}$, $m, n \in \mathbb{N}$.
- $\mathcal{S}^2_e(\mathbb{R})$ (resp. $\mathcal{S}^2_o(\mathbb{R})$), the subspace of $\mathcal{S}^2(\mathbb{R})$ consisting of even (resp. odd) functions.
- $\mathcal{S}'(\mathbb{R})$, the space of temperate distributions on $\mathbb{R}$; it is the dual space of $\mathcal{S}(\mathbb{R})$.

- ${\mathcal{S}'}^2(\mathbb{R})$, the space of generalized temperate distributions on $\mathbb{R}$; it is the dual space of $\mathcal{S}^2(\mathbb{R})$.

Remark 2.8. When $\varrho = 0$, $\mathcal{S}^2(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ and ${\mathcal{S}'}^2(\mathbb{R}) = \mathcal{S}'(\mathbb{R})$.

Definition 2.9. The Hartley transform, associated with Λ, of a function $f \in D(\mathbb{R})$ is defined by

$$\mathcal{H}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\lambda(x)A(x)\,dx, \quad \text{for all } \lambda \in \mathbb{R}. \tag{21}$$

Proposition 2.10. *For all $f \in D(\mathbb{R})$ we have*

$$\mathcal{H}_\Lambda(f) = \mathcal{H}_m \circ {}^tV\,(f), \tag{22}$$

where $\mathcal{H}_m$ is the modified Hartley transform defined on $D(\mathbb{R})$ by

$$\forall\,\lambda \in \mathbb{C}, \quad \mathcal{H}_m(f)(\lambda) = \int_{\mathbb{R}} f(x)\operatorname{cas}(\sqrt{\lambda^2-\varrho^2}x)\,dx = \mathcal{H}_c(f)(\sqrt{\lambda^2-\varrho^2}).$$

Proof. The result is immediate from the relations (21), (17) and (18). □

Proposition 2.11. *For all $f \in D(\mathbb{R})$, we have the decomposition*

$$\mathcal{H}_\Lambda f(\lambda) = 2\,\mathcal{F}_L(f_e)(\sqrt{\lambda^2-2\varrho^2}) - 2(\lambda+\varrho)\,\mathcal{F}_L J(f_o)(\sqrt{\lambda^2-2\varrho^2}), \tag{23}$$

where J is the integral operator defined by (19) and $\mathcal{F}_L$ stands for the Fourier transform related to the differential operator L, defined on $\mathcal{S}^2_e(\mathbb{R})$ by

$$\mathcal{F}_L(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)A(x)\,dx, \quad \lambda \in \mathbb{R},$$

φ_λ being the eigenfunction of L as defined by (4).

Proof. If $f \in D_e(\mathbb{R})$, identity (23) is obvious. Assume $f \in D_o(\mathbb{R})$ (the space of odd C^∞ functions on $\mathbb{R}$ which are of compact support). By using (21), (8), (4), and by integrating by parts we obtain

$$\begin{aligned}
\mathcal{H}_\Lambda f(\lambda) &= \frac{1}{\lambda-\varrho}\int_{\mathbb{R}} f(x)Y\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x)A(x)\,dx\\
&= \frac{1}{\lambda-\varrho}\int_{\mathbb{R}} \frac{d}{dx}\left(A(x)\frac{d}{dx}\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x)\right)J(f)(x)\,dx\\
&= \frac{1}{\lambda-\varrho}\int_{\mathbb{R}} L\varphi_{\sqrt{\lambda^2-2\varrho^2}}(x)J(f)(x)A(x)\,dx\\
&= -(\lambda+\varrho)\int_{\mathbb{R}} \varphi_{\sqrt{\lambda^2-2\varrho^2}}(x)J(f)(x)A(x)\,dx\\
&= -2(\lambda+\varrho)\mathcal{F}_L(Jf)(\sqrt{\lambda^2-2\varrho^2}),
\end{aligned}$$

which completes the proof. □

We shall need the following properties.

Proposition 2.12 (Transmutation formula)**.** *(i) Let $f \in \mathcal{S}^2(\mathbb{R})$ and g a nice function. Then*

$$\int_{\mathbb{R}} \Lambda f(x)g(x)A(x)\,dx = \int_{\mathbb{R}} f(x)\Lambda g(x)A(x)\,dx. \tag{24}$$

(ii) For $f \in \mathcal{S}^2(\mathbb{R})$,

$$\mathcal{H}_\Lambda\left(\Lambda f\right)(\xi) = \xi \mathcal{H}_\Lambda f(\xi), \quad \xi \in \mathbb{R}_\varrho.$$

(iii) For $f \in \mathcal{S}^2(\mathbb{R})$,

$$\mathcal{H}_\Lambda(\triangle_A f)(y) = y^2\mathcal{H}_\Lambda(f)(y), \quad \textit{for all } y \in \mathbb{R}_\varrho,$$

where $\triangle_A$ is the generalized Laplace operator on $\mathbb{R}$ given by

$$\forall\, x \in \mathbb{R}, \quad \triangle_A f(x) := \Lambda^2 f(x).$$

Proof. Let $f \in \mathcal{S}^2(\mathbb{R})$ and g a nice function, and consider the bracket

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)A(x)\,dx.$$

First, we have

$$\begin{aligned}
\langle Yf, g\rangle &= \int_{\mathbb{R}} f'(-x)g(x)A(x)\,dx = -\int_{\mathbb{R}} f(x)\frac{d}{dx}[g(-x)A(x)]\,dx \\
&= -\int_{\mathbb{R}} f(x)g(-x)A'(x)\,dx + \int_{\mathbb{R}} f(x)g'(-x)A(x)\,dx \\
&= \langle f, Yg\rangle - \langle f, \check{g}\frac{A'}{A}\rangle,
\end{aligned}$$

where $\check{h}(x) = h(-x)$.

Second, we have

$$\begin{aligned}
\langle \frac{1}{2}\frac{A'}{A}(f - \check{f}), g\rangle &= \int_{\mathbb{R}} \frac{A'(x)}{A(x)}\frac{f(x) - f(-x)}{2} g(x)A(x)\,dx \\
&= \int_{\mathbb{R}} (\frac{f(x) - f(-x)}{2}) g(x)A'(x)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left(A'(x)f(x)g(x) - A'(x)f(-x)g(x)\right)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left(A'(x)f(x)g(x) - A'(-x)f(x)g(-x)\right)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \left(A'(x)f(x)g(x) + A'(x)f(x)g(-x)\right)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} A'(x)f(x)\left(g(-x) + g(x)\right)\,dx \\
&= \int_{\mathbb{R}} \frac{A'(x)}{A(x)} f(x)(\frac{g(x) + g(-x)}{2})A(x)\,dx \\
&= \langle f, \frac{1}{2}\frac{A'}{A}(g + \check{g})\rangle.
\end{aligned}$$

Finally,

$$\begin{aligned}\langle(-\varrho\check{f}),g\rangle &= -\varrho\int_{\mathbb{R}} f(-x)g(x)A(x)\,dx\\ &= -\varrho\int_{\mathbb{R}} f(x)g(-x)A(x)\,dx\\ &= \langle f,(-\varrho\check{g})\rangle.\end{aligned}$$

All together, this gives

$$\begin{aligned}\langle\Lambda f,g\rangle &= \langle Yf+\frac{1}{2}\frac{A'}{A}(f-\check{f})-\varrho\check{f},g\rangle\\ &= \langle f,Yg-\check{g}\frac{A'}{A}+\frac{1}{2}\frac{A'}{A}(g+\check{g})-\varrho\check{g}\rangle\\ &= \langle f,Yg+\frac{1}{2}\frac{A'}{A}(g-\check{g})-\varrho\check{g}\rangle\\ &= \langle f,\Lambda g\rangle.\end{aligned}$$

Assertion (ii) follows by substituting in (24) g by Φ_λ. Assertion (iii) is immediate from (ii). □

2.3. Inversion formulas and Plancherel's theorem for $\mathcal{H}_\Lambda$.

Definition 2.13. The generalized translation operators τ_x, $x\in\mathbb{R}$, are defined on $D(\mathbb{R})$ by

$$\forall\,\lambda\in\mathbb{R},\quad \mathcal{H}_\Lambda(\tau_x f)(\lambda)=\Phi_\lambda(x)\mathcal{H}_\Lambda(f)(\lambda). \tag{25}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

Definition 2.14. For $f,g\in D(\mathbb{R})$, the generalized convolution product $f*_\Lambda g$ is defined by

$$f*_\Lambda g(x)=\int_{\mathbb{R}}\tau_x f(y)g(y)A(y)\,dy,\quad \text{for all } x\in\mathbb{R}.$$

Proposition 2.15. *(i) Let f and g be in $D(\mathbb{R})$. Then the generalized convolution product $f*_\Lambda g$ belongs to $D(\mathbb{R})$ and we have*

$$\mathcal{H}_\Lambda(f*_\Lambda g)(\xi)=\mathcal{H}_\Lambda(f)(\xi)\mathcal{H}_\Lambda(g)(\xi),\quad \textit{for all } \xi\in\mathbb{R}.$$

(ii) We have the relation

$$\forall\,f,g\in D(\mathbb{R}),\quad {}^tV(f*_\Lambda g)={}^tV(f)*{}^tV(g),$$

where $*$ is the convolution product on $\mathbb{R}$ given by (3)*.*

Proof. We obtain the results by using ideas similar to those in the context of Chébli–Trimèche hypergroups (cf. [19]). □

Definition 2.16. Let $f\in L^\infty_A(\mathbb{R})$. f is called $\mathcal{H}_\Lambda$-function of positive type if for all $\varphi\in D(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x)\varphi*_\Lambda\varphi(x)A(x)\,dx\geq 0.$$

Lemma 2.17. *Let u in $D'(\mathbb{R})$. The following assertions are equivalent:*

(i) For all $\varphi \in D(\mathbb{R})$, we have $\langle u, \varphi^2 \rangle \geq 0$.
(ii) u is a positive distribution.
(iii) u is a positive measure.

Proof. By using the proof of [17, Theorem XVIII, p. 276–277], we obtain the result. □

Theorem 2.18 (Bochner's theorem)**.** *Let $f \in L^\infty_A(\mathbb{R})$, if f is $\mathcal{H}_\Lambda$-function of positive type, there exists a positive bounded measure μ on $\mathbb{R}$, such that $f(x) = \mathcal{H}_\Lambda^{-1}(\mu)(x)$, for all $x \in \mathbb{R}$.*

Proof. Let $f \in L^\infty_A(\mathbb{R})$ be $\mathcal{H}_\Lambda$-function of positive type and put $u = \mathcal{H}_\Lambda(f)$. For all $\varphi \in D(\mathbb{R})$, we have

$$\begin{aligned}\langle u, \varphi^2 \rangle &= \langle \mathcal{H}_\Lambda(f), \varphi^2 \rangle \\ &= \langle f, \mathcal{H}_\Lambda^{-1}(\varphi^2) \rangle \\ &= \langle f, {\mathcal{H}_\Lambda}^{-1}(\varphi) *_\Lambda {\mathcal{H}_\Lambda}^{-1}(\varphi) \rangle \geq 0.\end{aligned}$$

Hence u is a positive distribution. Again, by using Lemma 2.17, it is a measure of positive type. As $u \in \mathcal{H}_\Lambda(L^\infty_A(\mathbb{R}))$, by a standard analysis, we prove that this measure is bounded and we deduce the result. □

Definition 2.19. Let $u \in D'(\mathbb{R})$. u is called $\mathcal{H}_\Lambda$-distribution of positive type if for all $\varphi \in D(\mathbb{R})$ we have

$$\langle u, \varphi *_\Lambda \varphi \rangle \geq 0.$$

Theorem 2.20 (Bochner–Schwartz's theorem)**.** *Let $u \in D'(\mathbb{R})$. The following assertions are equivalent:*

(i) u is of positive type.
(ii) u is a tempered distribution, and there exists a tempered positive measure μ on $\mathbb{R}$ such that $u = \mathcal{H}_\Lambda^{-1}(\mu)$.

Proof. We want to prove that (i) $\Rightarrow$ (ii).

It is easy to see that for all $\varphi \in D(\mathbb{R})$, the function $\varphi \mapsto u *_\Lambda \varphi *_\Lambda \varphi$ is of positive type. Then by Theorem 2.18, there exists a bounded positive measure μ_φ such that $\mu_\varphi = \mathcal{H}_\Lambda(u *_\Lambda \varphi *_\Lambda \varphi)$. Let $\chi \in D(\mathbb{R})$ such that $\mathcal{H}_\Lambda(\chi)(\lambda) \neq 0$, for all $\lambda \in \mathbb{R}$. We consider the measure μ defined by $\mu = \frac{\mu_\chi}{|\mathcal{H}_\Lambda(\chi)(\lambda)|^2}$. It is clear that μ is a positive measure; we can write:

$$\mathcal{H}_\Lambda(u *_\Lambda \varphi *_\Lambda \varphi *_\Lambda \chi *_\Lambda \chi) = \big(\mathcal{H}_\Lambda(\chi)\big)^2 \mu_\varphi = \big(\mathcal{H}_\Lambda(\varphi)\big)^2 \mu_\chi.$$

Thus

$$\mu_\varphi = \big(\mathcal{H}_\Lambda(\varphi)\big)^2 \mu, \quad \text{for all } \varphi \in D(\mathbb{R}).$$

Then, we have

$$\begin{aligned}\langle u, \varphi *_\Lambda \varphi\rangle &= \langle u *_\Lambda \varphi *_\Lambda \varphi, \delta\rangle \\ &= \mathcal{H}_\Lambda^{-1}(\mu_\varphi)(0) \\ &= \int_{\mathbb{R}} \left(\mathcal{H}_\Lambda(\varphi)\right)^2(\lambda)\, d\mu(\lambda).\end{aligned}$$

Thus, we deduce that

$$\langle u, \psi\rangle = \int_{\mathbb{R}} \mathcal{H}_\Lambda(\psi)(\lambda)\, d\mu(\lambda) = \langle \mathcal{H}_\Lambda^{-1}(\mu), \psi\rangle.$$

So the result follows.

Now we want to prove that (ii) $\Rightarrow$ (i).

If $\mu = \mathcal{H}_\Lambda(u)$, we have for all $\varphi \in D(\mathbb{R})$,

$$\begin{aligned}\langle \mathcal{H}_\Lambda^{-1}(\mu), \varphi *_\Lambda \varphi\rangle &= \langle \mu, \mathcal{H}_\Lambda(\varphi *_\Lambda \varphi)\rangle \\ &= \langle \mu, \left(\mathcal{H}_\Lambda(\varphi)\right)^2\rangle \geq 0.\end{aligned}$$

The theorem is then proved. □

Remark 2.21. As above, we prove the Bochner and Bochner–Schwartz's theorems for the classical Hartley transform $\mathcal{H}_c$ defined by (1).

Notation. We denote by $L^p_c(\mathbb{R}_+)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\|f\|_{L^p_c(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(x)|^p \frac{dx}{|c(x)|^2}\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_c(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty,$$

where $c(s)$ is a continuous function on $(0,\infty)$ such that

$$c^{-1}(s) \sim k_1\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \to \infty,$$
$$c^{-1}(s) \sim \begin{cases} k_2\, s, & \text{as } s \to 0, \text{ if } \varrho > 0 \\ k_3\, s^{\alpha+\frac{1}{2}}, & \text{as } s \to 0, \text{ if } \varrho = 0\end{cases}$$

for some $k_1\, k_2, k_3 \in \mathbb{C}$. The measure $\frac{dx}{|c(x)|^2}$ is the Plancherel measure for the Fourier transform $\mathcal{F}_L$.

$L^p_\nu(\mathbb{R})$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{L^p_\nu(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p d\nu(x)\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_\nu(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty,$$

where $d\nu$ is the measure given by

$$d\nu(\lambda) = \frac{|\lambda|}{4\sqrt{\lambda^2 - 2\varrho^2}\,|c(\sqrt{\lambda^2 - 2\varrho^2})|^2} 1_{\mathbb{R}\setminus(-\sqrt{2}\varrho, \sqrt{2}\varrho)}\, d\lambda, \tag{26}$$

with 1_A the characteristic function of the set A.

Theorem 2.22. *For all $f \in D(\mathbb{R})$, we have*

$$f(x) = \int_{\mathbb{R}} \mathcal{H}_\Lambda(f)(\lambda)\Phi_\lambda(x)\, d\nu(\lambda),$$

where $d\nu$ is given by (26).

Proof. By using Proposition 2.15 we see that the functional

$$\langle S, f\rangle = {}^tV^{-1}(f)(0), \quad f \in D(\mathbb{R})$$

is $\mathcal{H}_c$-positive type distribution in $D'(\mathbb{R})$. From Remark 2.21, we have the Bochner–Schwartz theorem for $\mathcal{H}_c$. Thus, we deduce that there exists a positive measure ν, such that for all f in $D(\mathbb{R})$, we have

$$f(0) = \int_{\mathbb{R}} \mathcal{H}_m({}^t\, V(f))(\lambda)\, d\nu(\lambda).$$

Using the relation (22) we obtain

$$f(0) = \int_{\mathbb{R}} \mathcal{H}_\Lambda(f)(\lambda)\, d\nu(\lambda).$$

By substituting in this relation f by $\tau_x f$, $x \in \mathbb{R}$, and by using (25) we obtain

$$f(x) = \int_{\mathbb{R}} \mathcal{H}_\Lambda(\tau_x f)(\lambda) d\nu(\lambda) = \int_{\mathbb{R}} \mathcal{H}_\Lambda(f)(\lambda)\Phi_\lambda(x) d\nu(\lambda). \tag{27}$$

If we suppose that f is even, then by the inversion formula for the transform $\mathcal{F}_L$ (see [19]), we have

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_L(f)(\mu)\varphi_\mu(x)\frac{d\mu}{2|c(\mu)|^2}. \tag{28}$$

But the parametrization $\mu^2 = \lambda^2 - 2\varrho^2$ shows that $\mu \in [0, \infty)$ if and only if $\lambda \in \mathbb{R}\backslash(-\sqrt{2}\varrho, \sqrt{2}\varrho)$, thus from (23) we deduce that (28) can also be written in the form

$$f(x) = \int_{\mathbb{R}\backslash(-\sqrt{2}\varrho, \sqrt{2}\varrho)} \mathcal{H}_\Lambda(f)(\lambda)\Phi_\lambda(x)\frac{|\lambda| d\lambda}{4\sqrt{\lambda^2 - 2\varrho^2}|c(\sqrt{\lambda^2 - 2\varrho^2})|^2}.$$

Using the relation (27) we deduce that the measure $d\nu$ is supported by $\mathbb{R}\backslash(-\sqrt{2}\varrho, \sqrt{2}\varrho)$ and given by (26). $\square$

Theorem 2.23. *Let $f \in L^1_A(\mathbb{R})$ such that $\mathcal{H}_\Lambda(f)$ belongs to $L^1_\nu(\mathbb{R})$. Then we have the inversion formula*

$$f(x) = \int_{\mathbb{R}} \mathcal{H}_\Lambda(f)(\lambda)\Phi_\lambda(x)\, d\nu(\lambda), \quad a.e.\ x \in \mathbb{R}.$$

Proof. It is easy to see that if $\mathcal{H}_\Lambda(f)$ belongs to $L^1_\nu(\mathbb{R})$ then $\mathcal{F}_L(f_e)$ belongs to $L^1_c(\mathbb{R}_+)$, where f_e is the even part of f. For $|\lambda| \geq \sqrt{2}\varrho$ the function $\lambda \mapsto \frac{1}{\lambda}[\mathcal{H}_\Lambda(f)(\lambda) - \mathcal{H}_\Lambda(f)(-\lambda)]$ belongs to $L^1_\nu(\mathbb{R})$, then $\mathcal{F}_L(Jf_o) \in L^1_c(\mathbb{R}_+)$, where f_o is the odd part of f and J the transform given by (19). On the other hand, using (20) we obtain for $\mu^2 = \lambda^2 - 2\varrho^2$

$$|Y\varphi_\mu(x)| \leq |(\lambda - \varrho)\frac{\Phi_\lambda(x) - \Phi_\lambda(-x)}{2}| \leq 2(|\lambda| + \rho).$$

Furthermore, the function $\mu \mapsto \sqrt{\mu^2 + 2\varrho^2}\mathcal{F}_L(Jf_o)(\mu)$ is in $L^1_c(\mathbb{R}_+)$, so by the dominated convergence theorem we obtain

$$\int_{\mathbb{R}_+} \mathcal{F}_L(Jf_o)(\lambda) Y\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2} = Y\left(\int_{\mathbb{R}_+} \mathcal{F}_L(Jf_o)(\lambda)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}\right).$$

We get the result by using the relation

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}_\Lambda(f)(\lambda)\Phi_\lambda(x)\, d\nu(\lambda) = &\int_{\mathbb{R}_+} \mathcal{F}_L(f_e)(\mu)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2} \\ &- Y\left(\int_{\mathbb{R}_+} \mathcal{F}_L(Jf_o)(\mu)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}\right)\end{aligned}$$

and the analogous theorem corresponding to $\mathcal{F}_L$. ☐

Theorem 2.24 (Plancherel formula)**.** *For all $f \in \mathcal{S}^2(\mathbb{R})$, we have*

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{H}_\Lambda(f)(\lambda)|^2 d\nu(\lambda) = \int_{\mathbb{R}} |f(x)|^2 A(x)\, dx &+ 2\varrho^2 \int_{\mathbb{R}} |Jf_o(x)|^2 A(x)\, dx \\ &- 2\varrho \mathrm{Re}\left(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)\, dx\right),\end{aligned}$$

where $d\nu$ is the measure given by (26)*.*

Proof. By (23),

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{H}_\Lambda(f)(\lambda)|^2 d\nu(\lambda) &= 4\int_{\mathbb{R}} |\mathcal{F}_L(f_e)(\sqrt{\lambda^2 - 2\varrho^2})|^2 d\nu(\lambda) \\ &\quad + 4\int_{\mathbb{R}} (\lambda + \varrho)^2 |\mathcal{F}_L(Jf_o)(\sqrt{\lambda^2 - 2\varrho^2})|^2 d\nu(\lambda) \\ &\quad - 8\varrho \mathrm{Re}\left(\int_{\mathbb{R}} \mathcal{F}_L(f_e)(\sqrt{\lambda^2 - 2\varrho^2})\overline{\mathcal{F}_L(Jf_o)(\sqrt{\lambda^2 - 2\varrho^2})} d\nu(\lambda)\right) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

By a Plancherel formula for the Fourier transform $\mathcal{F}_L$,

$$I_1 = \int_{\mathbb{R}} |f_e(x)|^2 A(x)\, dx \quad \text{and} \quad I_3 = -2\varrho \mathrm{Re}\left(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)\, dx\right).$$

Moreover,

$$\begin{aligned}
I_2 &= 4\int_{\mathbb{R}} (\lambda+\varrho)^2 |\mathcal{F}_L(Jf_o)(\sqrt{\lambda^2-2\varrho^2})|^2 d\nu(\lambda)\\
&= 4\int_{\mathbb{R}} (\lambda^2+\varrho^2) |\mathcal{F}_L(Jf_o)(\sqrt{\lambda^2-2\varrho^2})|^2 d\nu(\lambda)\\
&= \int_{\mathbb{R}} (\mu^2+\varrho^2)|\mathcal{F}_L(Jf_o)(\mu)|^2 \frac{d\mu}{|c(\mu)|^2} + 2\varrho^2\int_{\mathbb{R}} |\mathcal{F}_L(Jf_o)(\mu)|^2\frac{d\mu}{|c(\mu)|^2}.
\end{aligned}$$

Using now the identity

$$\mathcal{F}_L(Lh)(\mu) = -(\mu^2+\varrho^2)\mathcal{F}_L(h)(\mu), \quad h\in\mathcal{S}^2_e(\mathbb{R}),$$

and integration by parts, we deduce that

$$\begin{aligned}
I_2 &= -\int_{\mathbb{R}} \mathcal{F}_L(LJf_o)(\mu)\overline{\mathcal{F}_L(Jf_o)(\mu)}\frac{d\mu}{|c(\mu)|^2} + 2\varrho^2\int_{\mathbb{R}} |\mathcal{F}_L(Jf_o)(\mu)|^2\frac{d\mu}{|c(\mu)|^2}\\
&= -\int_{\mathbb{R}} LJf_o(x)\overline{Jf_o(x)}A(x)\,dx + 2\varrho^2\int_{\mathbb{R}} |Jf_o(x)|^2 A(x)\,dx\\
&= -\int_{\mathbb{R}} \frac{d}{dx}\left(A(x)\frac{d}{dx}Jf_o(x)\right)\overline{Jf_o(x)}dx + 2\varrho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)\,dx\\
&= \int_{\mathbb{R}} |f_o(x))|^2A(x)\,dx + 2\varrho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
I_1+I_2+I_3 &= \int_{\mathbb{R}} |f_e(x)|^2A(x)\,dx + \int_{\mathbb{R}} |f_o(x))|^2A(x)\,dx\\
&\quad + 2\varrho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)\,dx - 2\varrho\mathrm{Re}\left(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)\,dx\right)\\
&= \int_{\mathbb{R}} |f(x)|^2A(x)\,dx + 2\varrho^2\int_{\mathbb{R}} |Jf_o(x)|^2\\
&\quad - 2\varrho\mathrm{Re}\left(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)\,dx\right).
\end{aligned}$$

This completes the proof. □

In the rest of this paper we assume that $\varrho = 0$.

Using Bloom-Xu's approach [6] we prove the next result.

Theorem 2.25. *The transform $\mathcal{H}_\Lambda$ is a bijection from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.*

Definition 2.26. The transform $\mathcal{H}_\Lambda$ of a distribution τ in $\mathcal{S}'(\mathbb{R})$ is defined by

$$\langle\mathcal{H}_\Lambda(\tau),\phi\rangle = \langle\tau,\mathcal{H}_\Lambda^{-1}(\phi)\rangle, \quad \text{for all } \phi\in\mathcal{S}(\mathbb{R}).$$

From the above, it is easy to obtain the following corollary.

Corollary 2.27. *The transform $\mathcal{H}_\Lambda$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R})$ onto itself. Moreover, for all $\tau\in\mathcal{S}'(\mathbb{R})$, we have*

$$\mathcal{H}_\Lambda(\Lambda\,\tau) = y\mathcal{H}_\Lambda(\tau)$$

and

$$\mathcal{H}_\Lambda(\triangle_A \tau) = y^2 \mathcal{H}_\Lambda(\tau). \tag{29}$$

3. Real Paley–Wiener theorems for the transform $\mathcal{H}_\Lambda$

Notation. We denote by $\mathbb{C}[x]$ the set of polynomials on $\mathbb{R}$ with complex coefficients.

Definition 3.1. Let u be a distribution on $\mathbb{R}$ and P a polynomial. Then we let

$$R(P,u) = \sup\{|P(y)| : y \in \operatorname{supp} u\} \in [0,\infty],$$

where by convention $R(P,u) = 0$ if $u = 0$.

We will now study the real L^2-Paley–Wiener theorem for the transform $\mathcal{H}_\Lambda$, for which we need the following key propositions.

Proposition 3.2. *Let P be a polynomial and $f \in \mathcal{S}(\mathbb{R})$. Then in the extended positive real numbers*

$$\limsup_{n\to\infty} \|P^n(\Lambda) f\|_{L^2_A(\mathbb{R})}^{\frac{1}{n}} \le R(P, \mathcal{H}_\Lambda(f)). \tag{30}$$

Proof. Suppose firstly that $R(P,\mathcal{H}_\Lambda(f)) = 0$. Then $\mathcal{H}_\Lambda(f) = 0$, and hence from Theorem 2.25 $f = 0$. Thus (30) is immediate.

Moreover, the inequality (30) is clear when $R(P,\mathcal{H}_\Lambda(f)) = \infty$. So we can assume that

$$0 < R(P,\mathcal{H}_\Lambda(f)) < \infty.$$

Hölder's inequality gives

$$\|f\|^2_{L^2_A(\mathbb{R})} \le C \sup_{x\in\mathbb{R}} (1+x^2)^{2m} |f(x)|^2,$$

for $m \ge 1$. Thus

$$\|f\|_{L^2_A(\mathbb{R})} \le C \sup_{x\in\mathbb{R}} (1+x^2)^{m} |f(x)|.$$

Consequently, we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|P^n(\Lambda) f\|_{L^2_A(\mathbb{R})} &\le C \sup_{x\in\mathbb{R}} (1+x^2)^m |P^n(\Lambda) f(x)| \\ &\le C \sup_{x\in\mathbb{R}} (1+x^2)^m \left|\left[\mathcal{H}_\Lambda^{-1}\left(P^n(\xi)\mathcal{H}_\Lambda(f)(x)\right)\right]\right|.\end{aligned}$$

Using the continuity of $\mathcal{H}_\Lambda^{-1}$ we can show that

$$\|P^n(\Lambda) f\|_{L^2_A(\mathbb{R})} \le C \sup_{\xi\in\mathbb{R}} \sum_{0\le l,j\le M} (1+\xi^2)^j \left| \frac{d^l}{d\xi^l}\left[P^n(\xi)\mathcal{H}_\Lambda(f)(\xi)\right]\right|,$$

with positive constants C and integer M, independent of n. Using Leibniz's rule we deduce that

$$\|P^n(\Lambda) f\|_{L^2_A(\mathbb{R})} \le C n^M \sup_{y\in \operatorname{supp} \mathcal{H}_\Lambda(f)} |P(y)|^{n-M},$$

with C a constant independent of n. Hence, from the previous inequalities we obtain

$$\limsup_{n\to\infty} \|P^n(\Lambda)f\|^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \le \sup_{y\in \operatorname{supp} \mathcal{H}_\Lambda(f)} |P(y)| = R(P, \mathcal{H}_\Lambda(f)). \qquad \square$$

Proposition 3.3. *Let P be a polynomial. Suppose that $P^n(\Lambda)f \in L^2_A(\mathbb{R})$ for all $n \in \mathbb{N}_0$. Then in the extended positive real numbers*

$$\liminf_{n\to\infty} \|P^n(\Lambda)f\|^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \ge R(P, \mathcal{H}_\Lambda(f)).$$

Proof. Fix $\xi_0 \in \operatorname{supp} \mathcal{H}_\Lambda(f)$. We can assume that $|P(\xi_0)| \neq 0$. We will show that

$$\liminf_{n\to\infty} \|P^n(\Lambda)f\|^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \ge |P(\xi_0)| - \varepsilon,$$

for any fixed $\varepsilon > 0$ such that $0 < 2\varepsilon < |P(\xi_0)|$. To this end, choose and fix $\chi \in D(\mathbb{R})$ such that $\langle \mathcal{H}_\Lambda(f), \chi\rangle \neq 0$, and

$$\operatorname{supp} \chi \subset \{\xi \in \mathbb{R} : |P(\xi_0)| - \varepsilon < |P(\xi)| < |P(\xi_0)| + \varepsilon\}.$$

For $n \in \mathbb{N}$, let $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$. Now we want to estimate $\|\mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^2_A(\mathbb{R})}$. Indeed as above we have

$$\begin{aligned}\|\mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^2_A(\mathbb{R})} &\le C \sup_{x\in\mathbb{R}} (1+x^2)^m |\mathcal{H}_\Lambda^{-1}(\chi_n)(x)| \\ &\le C \sup_{x\in\mathbb{R}} (1+x^2)^m \left|\left[\mathcal{H}_\Lambda^{-1}\left(P^{-n}(\xi)\chi\right)(x)\right]\right|,\end{aligned}$$

with $m \ge 1$. Using the continuity of $\mathcal{H}_\Lambda^{-1}$ we can show that

$$\|\mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^2_A(\mathbb{R})} \le C \sup_{\xi\in\mathbb{R}} \sum_{0\le l,j\le M} (1+\xi^2)^j \left|\frac{d^l}{d\xi^l}\left[P^{-n}(\xi)\chi(\xi)\right]\right|,$$

with positive constants C and integer M, independent of n. Using Leibniz's rule we deduce that

$$\|\mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^2_A(\mathbb{R})} \le C n^M (|P(\xi_0)| - \varepsilon)^{-n}.$$

Then,

$$\begin{aligned}\langle \mathcal{H}_\Lambda(f), \chi\rangle &= \langle \mathcal{H}_\Lambda(f), P^n(\xi)\chi_n\rangle = \langle P^n(\xi)\mathcal{H}_\Lambda(f), \chi_n\rangle \\ &= \langle \mathcal{H}_\Lambda(P^n(\Lambda)f), \chi_n\rangle = \langle (P^n(\Lambda)f), \mathcal{H}_\Lambda^{-1}(\chi_n)\rangle.\end{aligned}$$

Hence, from Hölder's inequality we obtain

$$\begin{aligned}|\langle \mathcal{H}_\Lambda(f), \chi\rangle| &\le C\|P^n(\Lambda)f\|_{L^2_A(\mathbb{R})} \|\mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^2_A(\mathbb{R})} \\ &\le C n^M (|P(\xi_0)| - \varepsilon)^{-n} \|P^n(\Lambda)f\|_{L^2_A(\mathbb{R})}.\end{aligned}$$

Since $|\langle \mathcal{H}_\Lambda(f), \chi\rangle| > 0$, we deduce that

$$\liminf_{n\to\infty} \|P^n(\Lambda)f\|^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \ge |P(\xi_0)| - \varepsilon.$$

Thus

$$\liminf_{n\to\infty} \|P^n(\Lambda)f\|^{\frac{1}{n}}_{L^2_A(\mathbb{R})} \ge \sup_{y\in \operatorname{supp} \mathcal{H}_\Lambda(f)} |P(y)| = R(P, \mathcal{H}_\Lambda(f)). \qquad \square$$

Combining Proposition 3.2 and Proposition 3.3, we get

Theorem 3.4. *Let P be a non-constant polynomial. For any function $f \in \mathcal{S}(\mathbb{R})$ the following relation holds:*

$$\lim_{n\to\infty} \|P^n(\Lambda)f\|_{L^2_A(\mathbb{R})}^{\frac{1}{n}} = R(P, \mathcal{H}_\Lambda(f)).$$

Definition 3.5. Let P be a non-constant polynomial. We define the polynomial domain U_P by

$$U_P := \{\xi \in \mathbb{R} : |P(\xi)| \leq 1\}.$$

We have the following result.

Corollary 3.6. *Let $f \in \mathcal{S}(\mathbb{R})$. The transform $\mathcal{H}_\Lambda(f)$ vanishes outside a domain U_P if and only if*

$$\limsup_{n\to\infty} \|P^n(\Lambda)f\|_{L^2_A(\mathbb{R})}^{\frac{1}{n}} \leq 1.$$

Remark 3.7. If we take $P(y) = y$, then $P(\Lambda) = \triangle_A$, and Theorem 3.4 and Corollary 3.6 characterize functions f such that the support of their transform $\mathcal{H}_\Lambda(f)$ is $[-1, 1]$.

4. Characterization of the functions whose transform $\mathcal{H}_\Lambda$ has support in antipodal points

Notation. We denote by $\mathcal{E}'(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support.

Definition 4.1. The inverse of the transform $\mathcal{H}_\Lambda$ of a distribution τ in $\mathcal{E}'(\mathbb{R})$ is defined by

$$\forall\, x \in \mathbb{R}, \quad \mathcal{H}_\Lambda^{-1}(\tau)(x) = \langle \tau_\lambda, \Phi_\lambda(x)\rangle.$$

Theorem 4.2. *Let $u \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$. Then the support of $\mathcal{H}_\Lambda(u)$ is contained in the compact $V_r := \{\xi \in \mathbb{R} : |P(\xi)| \leq r\}$ for a polynomial P and a constant $r \geq 0$ if and only if for each $R > r$ there exist $N_R \in \mathbb{N}_0$ and a positive constant $C(R)$ such that*

$$|P^n(\Lambda)(u)(x)| \leq C(R) R^n (1 + |x|)^{N_R}, \tag{31}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Proof. Assume that the support of $\mathcal{H}_\Lambda(u)$ is contained in the compact V_r. Let $R > r$ and let $\varepsilon \in (0, R - r)$. We choose $\chi \in D(\mathbb{R})$ such that $\chi \equiv 1$ on an open neighborhood of the support of $\mathcal{H}_\Lambda(u)$, and $\chi \equiv 0$ outside $V_{R-\frac{\varepsilon}{3}}$. As $\mathcal{H}_\Lambda(u)$ is of order N, there exists a positive constant C such that for all $x \in \mathbb{R}$

$$\begin{aligned} |P^n(\Lambda)(u)(x)| &= \left|\mathcal{H}_\Lambda^{-1}\left(P^n(\xi)\mathcal{H}_\Lambda(u)\right)(x)\right| \\ &= \left|\mathcal{H}_\Lambda^{-1}\left(\chi(\xi)P^n(\xi)\mathcal{H}_\Lambda(u)\right)(x)\right| \\ &= \left|\langle \chi(\xi)P^n(\xi)\mathcal{H}_\Lambda(u)(\xi), \Phi_\xi(x)\rangle\right| \\ &\leq C \sup_{\xi\in\mathbb{R}} \sum_{0\leq j\leq N} \left|D^j\left(\chi(\xi)P^n(\xi)\Phi_\xi(x)\right)\right|. \end{aligned}$$

Thus from the Leibniz formula and (11) we obtain that

$$\forall\, n \in \mathbb{N}_0, \quad |P^n(\Lambda)(u)(x)| \leq C_1(R) n^N (R - \frac{\varepsilon}{3})^n (1+|x|)^{N+2} \leq C_2(R) R^n (1+|x|)^{N+2}.$$

Conversely, assume that we have (31). Suppose that $\xi_0 \in \mathbb{R}$ is fixed and such that $|P(\xi_0)| \geq R + \varepsilon$, for some $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R})$ such that $\operatorname{supp} \chi \subset \left\{\xi \in \mathbb{R} : |P(\xi)| \geq R + \frac{\varepsilon}{3}\right\}$. For $n \in \mathbb{N}$, we introduce the function χ_n defined by $\chi_n(\xi) = P^{-n}(\xi)\chi(\xi)$. We have

$$\begin{aligned} \langle \mathcal{H}_\Lambda(u), \chi \rangle &= \langle \mathcal{H}_\Lambda(u), P^n(\xi)\chi_n \rangle = \langle P^n(\xi)\mathcal{H}_\Lambda(u), \chi_n \rangle \\ &= \langle \mathcal{H}_\Lambda(P^n(\Lambda)u), \chi_n \rangle \\ &= \langle ((1+|x|)^{-N} P^n(\Lambda)u), (1+|x|)^N \mathcal{H}_\Lambda^{-1}(\chi_n) \rangle. \end{aligned}$$

Hence, from the Hölder inequality we obtain

$$|\langle \mathcal{H}_\Lambda(u), \chi \rangle| \leq \|(1+|x|)^{-N} P^n(\Lambda)u\|_{L^\infty_A(\mathbb{R})} \|(1+|x|)^N \mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^1_A(\mathbb{R})}.$$

Proceeding as in Proposition 3.3, we prove that

$$\|(1+|x|)^N \mathcal{H}_\Lambda^{-1}(\chi_n)\|_{L^1_A(\mathbb{R})} \leq C n^M (R + \frac{\varepsilon}{3})^{-n}.$$

Thus

$$|\langle \mathcal{H}_\Lambda(u), \chi \rangle| \leq C(R) n^M \left(\frac{R}{R + \frac{\varepsilon}{3}} \right)^n.$$

Hence we deduce that $\langle \mathcal{H}_\Lambda(u), \chi \rangle = 0$, which implies that $\xi_0 \notin \operatorname{supp} \mathcal{H}_\Lambda(u)$. Thus the support of $\mathcal{H}_\Lambda(u)$ is contained in the compact V_r. □

Notations. Let $r > 0$, we denote by

$$B_r := \{\xi \in \mathbb{R} : |P(\xi)| < r\}, \quad S_r := \{\xi \in \mathbb{R} : |P(\xi)| = r\}.$$

A reasoning analogous to the one developed in Theorem 4.2 allows us to establish the following theorem.

Theorem 4.3. *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, and consider the infinite sequence $\{u_{-n}\}_{n \in \mathbb{N}}$ of tempered distributions defined as $u_{-n+1} = P(\Lambda)u_n$, for a polynomial P and for all $n \in \mathbb{N}$. Let $r > 0$. Assume that, for all $R \in (0, r)$, there exist constants $N_R \in \mathbb{N}_0$ and $C(R) > 0$ such that*

$$\forall\, x \in \mathbb{R}, \quad |u_{-n}(x)| \leq C(R) R^{-n} (1+|x|)^{N_R}, \tag{32}$$

for all $n \in \mathbb{N}$. Then $\operatorname{supp} \mathcal{H}_\Lambda(u) \cap B_r = \emptyset$. On the other hand, if $\operatorname{supp} \mathcal{H}_\Lambda(u) \cap B_r = \emptyset$ and $\operatorname{supp} \mathcal{H}_\Lambda(u)$ is compact, then (32) holds, for all $R \in (0, r)$.

Combining Theorem 4.2 and Theorem 4.3 together, we get

Corollary 4.4. *Let $u = u_0 \in \mathcal{E}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, and consider the infinite sequence $\{u_n\}_{n \in \mathbb{Z}}$ of tempered distributions defined as $u_{n+1} = P(\Lambda)u_n$, for a polynomial P and for all $n \in \mathbb{Z}$. Let $R > 0$. Then $\operatorname{supp} \mathcal{H}_\Lambda(u)$ is contained in S_R if and only if for all $\varepsilon > 0$ there exist constants $N_\varepsilon \in \mathbb{N}_0$ and $C_\varepsilon > 0$ such that*

$$\forall\, x \in \mathbb{R}, \quad |u_n(x)| \leq C_\varepsilon R^n (1+\varepsilon)^{|n|} (1+|x|)^{N_\varepsilon}$$

for all $n \in \mathbb{Z}$.

Remark 4.5. Theorem 4.3 and Corollary 4.4 are the analogue of the new real Paley–Wiener theorems for the Fourier transform, proved by Andersen (see [1]).

5. Roe's theorem associated with the modified Cherednik type operator

In [15] Roe proved that if a doubly-infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies $\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \leq M$ for all $j = 0, \pm 1, \pm 2, \dots$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x+b)$ where a and b are real constants.

The purpose of this section is to generalize this theorem for the modified Cherednik type operator Λ.

Theorem 5.1. *Suppose $P(\xi) = \sum_n a_n \xi^n$ is real-valued and let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(\Lambda) f_j.$$

Let $a \geq 0$, $R > 0$, and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)| \leq M_j R^j (1+|x|)^a, \tag{33}$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the sublinear growth condition

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{34}$$

Then $f_0 = f_+ + f_-$, where $P(\Lambda)f_+ = Rf_+$ and $P(\Lambda)f_- = -Rf_-$. If R (resp. $-R$) is not in the range of P then $f_+ = 0$ (resp. $f_- = 0$).

In order to prove Theorem 5.1 we need the following two lemmas.

Lemma 5.2. *Let $(f_j)_{j\in\mathbb{Z}}$ be a sequence of functions on $\mathbb{R}$ satisfying*

$$f_{j+1} = P(\Lambda) f_j, \tag{35}$$

$$|f_j(x)| \leq M_j R^j (1+|x|)^a, \tag{36}$$

and

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^{|j|}} = 0,$$

for all $\varepsilon > 0$. Then

$$\operatorname{supp}(\mathcal{H}_\Lambda(f_0)) \subset S_R := \{\xi : |P(\xi)| = R\}.$$

Proof. First we show that $\mathcal{H}_\Lambda(f_0)$ is supported in $\{\xi : |P(\xi)| \leq R\}$. To do this we need to show that $\langle \mathcal{H}_\Lambda(f_0), \phi\rangle = 0$ if $\phi \in D(\mathbb{R})$ and

$$\operatorname{supp}(\phi) \cap \{\xi : |P(\xi)| \leq R\} = \emptyset.$$

Since $\operatorname{supp}(\phi)$ is compact, there is some $r < \frac{1}{R}$ so that $\frac{1}{|P(\xi)|} \leq r$, for all $\xi \in \operatorname{supp}(\phi)$. Then

$$\begin{aligned}\langle \mathcal{H}_\Lambda(f_0), \phi\rangle &= \langle P^j \mathcal{H}_\Lambda(f_0), \frac{\phi}{P^j}\rangle \\ &= \langle \mathcal{H}_\Lambda\left(P^j(\Lambda) f_0\right), \frac{\phi}{P^j}\rangle \\ &= \langle P^j(\Lambda) f_0, \mathcal{H}_\Lambda^{-1}(\frac{\phi}{P^j})\rangle.\end{aligned}$$

Choose an integer m with $2m \geq a+1$. A calculation, using the hypothesis of the lemma and Cauchy–Schwarz's inequality, implies

$$\begin{aligned}|\langle \mathcal{H}_\Lambda(f_0), \phi\rangle| &\leq \int_{\mathbb{R}} |P^j(\Lambda) f_0(x)| \left|\mathcal{H}_\Lambda^{-1}(\frac{\phi}{P^j})(x)\right| A(x)\, dx \\ &\leq C M_j R^j \sup_{x\in\mathbb{R}} \left|(1+x^2)^m \mathcal{H}_\Lambda^{-1}(\frac{\phi}{P^j})(x)\right|.\end{aligned}$$

Using the continuity of $\mathcal{H}_\Lambda^{-1}$ and the fact that ϕ is supported in

$$\{\xi : |P(\xi)| \geq R+\varepsilon\}$$

for some fixed $\varepsilon > 0$, it is not hard to prove that the right-hand side of the above inequality goes to zero as $j \to \infty$ and so $\langle \mathcal{H}_\Lambda(f_0), \phi\rangle = 0$. To complete the proof we need to show that $\mathcal{H}_\Lambda(f_0)$ is also supported in $\{\xi : |P(\xi)| \geq R\}$, which means $\langle \mathcal{H}_\Lambda(f_0), \phi\rangle = 0$ if ϕ is supported in $\{\xi : |P(\xi)| \leq R\}$. Here we use (35) to obtain

$$\langle \mathcal{H}_\Lambda(f_0), \phi\rangle = \langle \mathcal{H}_\Lambda(f_{-j}), P^j \phi\rangle$$

and the argument proceeds as before. □

Lemma 5.3. *If $-R \notin P(\mathbb{R})$, then $P(\Lambda) f_0 = R f_0$.*

Proof. Using Lemma 5.2 and proceeding as in [12], we prove that there exists an integer N such that

$$(P(\xi) - R)^{N+1} \mathcal{H}_\Lambda(f_0) = 0. \tag{37}$$

Inverting the generalized Hartley transform in (37) yields

$$(P(\Lambda) - R)^{N+1} f_0 = 0. \tag{38}$$

This equation implies

$$\begin{aligned}\operatorname{span}\{f_0, f_1, f_2, \dots\} &= \operatorname{span}\{f_0, P(\Lambda) f_0, P(\Lambda) f_0, \dots\} \\ &= \operatorname{span}\left\{f_0, P(\Lambda) f_0, \dots, P^N(\Lambda) f_0\right\}.\end{aligned}$$

We shall now show that we can take $N = 0$ in (38). If not, then $(P(\Lambda) - R) f_0 \neq 0$. Let p be the largest positive integer so that $(P(\Lambda) - R)^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := (P(\Lambda) - R)^{p-1} f_0 \in \operatorname{span}\{f_0, f_1, \dots, f_N\}$$

will satisfy

$$(P(\Lambda) - R) f = 0 \quad \text{and} \quad (P(\Lambda) - R) f \neq 0. \tag{39}$$

Write

$$f = a_0 f_0 + \cdots + a_N f_N,$$

for constants $a_0, \ldots, a_N$. Then

$$P^j(\Lambda) f = a_0 f_j + \cdots + a_N f_{N+j}.$$

If

$$C_j = |a_0| R^0 M_j + \cdots + |a_N| R^N M_{j+N},$$

then by (33) we have

$$|P^j(\Lambda) f(x)| \leq C_j R^j (1 + |x|)^a. \tag{40}$$

By (34) the sequence $(C_j)_j$ satisfies the sublinear growth condition

$$\lim_{j \to \infty} \frac{C_j}{j} = 0. \tag{41}$$

An induction using (39) implies for $j \geq 2$ that

$$P^j(\Lambda) f = R^{j-1} j P(\Lambda) f - R^j (j-1) f = R^{j-1} j (P(\Lambda) - R) f + R^j f.$$

Thus

$$|(P(\Lambda) - R) f(x)| \leq \frac{1}{j R^{j-1}} |P^j(\Lambda) f(x)| + \frac{R|f(x)|}{j} \leq \frac{C_j R}{j} (1 + |x|)^a + \frac{R|f(x)|}{j}.$$

Letting $j \to \infty$ and using (41) implies $(P(\Lambda) - R) f = 0$. But this contradicts (39). Consequently, $N = 0$ in (38). This completes the proof. $\square$

Proof of Theorem 5.1. We distinguish three cases for R.

First case: $-R \notin P(\mathbb{R})$. In this case, we have proved in Lemma 5.3 that $P(\Lambda) f_0 = R f_0$.

Second case: $R \notin P(\mathbb{R})$. In this case, we apply the same argument to $-P(\Lambda)$ to conclude $P(\Lambda) f_0 = -R f_0$.

Third case: $R \in P(\mathbb{R})$. The arguments used above are not valid for the sequence $(f_p)_{p \in \mathbb{Z}}$. For this, let $\mathfrak{L} = P(\Lambda)$. We have

$$f_{2p+2} = P(\Lambda) f_{2p+1} = P(\Lambda) f_{2p} = \mathfrak{L} f_{2p} \quad \text{and} \quad \mathcal{H}_\Lambda(\mathfrak{L} f)(\xi) = P(\xi) \mathcal{H}_\Lambda(f)(\xi).$$

As $-R \notin P(\mathbb{R})$, if we apply the same arguments used in Lemma 5.3 for the sequence $(f_{2p})_{p \in \mathbb{Z}}$ we see that

$$\mathfrak{L} f_0 = P(\Lambda) f_0 = R f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + \frac{1}{R} P(\Lambda) f_0)$ and $f_- = \frac{1}{2}(f_0 - \frac{1}{R} P(\Lambda) f_0)$.

It is easy to see that $f_0 = f_+ + f_-$, and that $P(\Lambda) f_+ = R f_+$ and $P(\Lambda) f_- = -R f_-$. This completes the proof. $\square$

Theorem 5.4. *If we replace* (34) *with*

$$\lim_{j \to \infty} \frac{M_{|j|}}{(1 + \varepsilon)^{|j|}} = 0, \tag{42}$$

for all $j > 0$, then the span of $(f_j)_j$ is finite dimensional. Moreover, $f_0 = f_+ + f_-$, where, for some integer N, $(P(\Lambda) - R)^N f_+ = 0$ and $(P(\Lambda) + R)^N f_- = 0$. Thus f_+ (resp. f_-) is a generalized eigenfunction of $P(\Lambda)$ with eigenvalue R (resp. $-R$).

The proof will be based on the following result from linear algebra.

Lemma 5.5 ([5]). *Let X be a finite dimensional complex vector space, and let $T : X \to X$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_p$. Then $X = X_1 \oplus \cdots \oplus X_p$, where $X_j = \ker((T - \lambda_j)^N)$ and $\dim X = N$.*

Proof of Theorem 5.4. We first prove Theorem 5.4 under the assumption that $P(\xi) \neq -R$. Using the growth condition (42) and Lemma 5.5, we may still conclude that

$$\mathrm{supp}(\mathcal{H}_\Lambda(f_0)) \subset S_R := \{\xi : P(\xi) = R\}.$$

But then, as before, we can conclude that (38) holds. This is enough to complete the proof in this case. A similar argument shows that if $P(\xi) \neq R$, then $(P(\Lambda) + R)^N f_0 = 0$.

In the general case we again let $\mathfrak{L} = P(\Lambda)$ and $P_0 = P^2$. Then $P_0(\xi) \neq -R$ and the span of $(f_{2j})_j$ is finite dimensional. The map $P(\Lambda)$ takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is finite dimensional. Any $f \in X$ will have $\mathrm{supp}(f)$ inside the set defined by $P(\xi) = \pm R$. From this it is not hard to show that the only possible eigenvalues of $P(\Lambda)$ restricted to X are R and $-R$. The result now follows from the last lemma. □

Remark 5.6. If we take $P(y) = y^2$, then $P(\Lambda) = \triangle_A$ and Theorem 5.1 give $\triangle_A f_0 = R f_0$. This characterizes the eigenfunctions f of the generalized Laplace operator $\triangle_A$ with polynomial growth in terms of the size of the powers $\triangle_A^j f$, $-\infty < j < \infty$.

Theorem 5.7. *Suppose $P(\xi) = \sum_n a_n \xi^n$ is a non-constant polynomial with complex coefficients. Let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j \in \mathbb{Z}, \quad f_{j+1} = P(\Lambda) f_j.$$

Let $a \geq 0$ and let $R > 0$. Assume that for all $\varepsilon > 0$ there exist constants $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\forall\, x \in \mathbb{R}, \quad |f_n(x)| \leq C R^n (1+\varepsilon)^{|n|} (1+|x|)^N \tag{43}$$

is satisfied for all $n \in \mathbb{Z}$. Then

$$f_0 = \sum_{\lambda \in S_R} \sum_{j=0}^{N} c(\lambda, j) \frac{d^j}{d\xi^j}_{|\xi=\lambda} \Phi_\xi, \tag{44}$$

for constants $c(\lambda, j) \in \mathbb{C}$ and $N \in \mathbb{N}$.

Proof. Assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies (43). Then Corollary 4.4 implies that the support of $\mathcal{H}_\Lambda(f_0)$ is contained in the finite set S_R. A standard result in distribution theory (see e.g. [16, Theorem 6.25]) implies that

$$\mathcal{H}_\Lambda(f_0) = \sum_{\lambda\in S_R}\sum_{0\le j\le N} c(\lambda,j)\delta_\lambda^{(j)}$$

for constants $c(\lambda,j)\in\mathbb{C}$, and some integer N. Here δ_ξ^j denotes the jth distributional derivative of the delta function δ_ξ at ξ. The result follows with $f_0 = \mathcal{H}_\Lambda^{-1}\left(\sum_{\lambda\in S_R}\sum_{0\le j\le N} c(\lambda,j)\delta_\lambda^{(j)}\right)$. □

By proceeding in a similar way as in Theorem 5.1 and using Theorem 5.7, we can infer the following theorem.

Theorem 5.8. *Suppose $P(\xi)=\sum_n a_n\xi^n$ is a non-constant polynomial with complex coefficients. Let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of complex-valued functions on $\mathbb{R}$ so that*

$$\forall\, j\in\mathbb{Z},\quad f_{j+1}=P(\Lambda)f_j.$$

Let $a\ge 0$ and let $R>0$ and assume that $\{f_j\}_{-\infty}^{\infty}$ satisfies

$$|f_j(x)|\le M_jR^j(1+|x|)^a,$$

where $(M_j)_{j\in\mathbb{Z}}$ satisfies the subpotential growth condition

$$\lim_{j\to\infty}\frac{M_{|j|}}{j^m}=0, \tag{45}$$

for some $m\ge 0$. We have:

(i) If $P'(\lambda_p)\neq 0$ for all $\lambda_p\in S_R$, then $N<m$ in (44). In particular, if $m=1$, then

$$f_0=\sum_{\lambda_p\in S_R} f_{\lambda_p},\quad \textit{where } f_{\lambda_p}=c(\lambda_p)\Phi_{\lambda_p}.$$

(ii) If S_R consists of one point λ_0 and $m=1$ in (45), then $P(\Lambda)f_0=P(\lambda_0)f_0$.

Remark 5.9. The previous theorem is the analogue of [1, Theorem 6].

6. Real Paley–Wiener theorems for the transform $\mathcal{H}_\Lambda$ on $\mathcal{S}'(\mathbb{R})$

Let $u\in\mathcal{S}'(\mathbb{R})$. We put $\Gamma_u := \inf\{r\in(0,\infty] : \operatorname{supp}(\mathcal{H}_\Lambda(u))\subset[-r,r]\}$.

Theorem 6.1. *Let $u\in\mathcal{S}'(\mathbb{R})$. Then the support of $\mathcal{H}_\Lambda(u)$ is included in $[-M,M]$, $M>0$, if and only if for all $R>M$ we have*

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0 \quad \textit{in } \mathcal{S}'(\mathbb{R}).$$

Proof. Let $u\in\mathcal{S}'(\mathbb{R})$ and $M>0$ such that

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u=0,\quad \text{for all } R>M.$$

Let $\varphi\in D(\mathbb{R})$ satisfy $\operatorname{supp}(\varphi)\subset[-M,M]^c$. We have to prove that

$$\langle\mathcal{H}_\Lambda(u),\varphi\rangle=0.$$

Let $r > M$ satisfy $\varphi(x) = 0$ for all $x \in [-r, r]$ and $R \in (M, r)$. Then for all $n \in \mathbb{N}$ the function $x^{-2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = \langle (-x)^n R^{-2n} \mathcal{H}_\Lambda(u), (-x)^{-n} R^{2n}\varphi\rangle,$$

and by formula (29) we have

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = \langle \mathcal{H}_\Lambda(R^{-2n} \triangle_A^n(u)), (-x)^{-n} R^{2n}\varphi\rangle.$$

The hypothesis implies that $\mathcal{H}_\Lambda(R^{-2n}\triangle_A^n(u)) \to 0$ in $\mathcal{S}'(\mathbb{R})$. Moreover from the Leibniz formula we deduce that $(-x)^{-n}R^{2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'(\mathbb{R})$ and $M > 0$ such that $\operatorname{supp} \mathcal{H}_\Lambda(u) \subset [-M, M]$. We are going to prove that for all $R > M$

$$\lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Let $M < R$ and choose $t \in (M, R)$ and $\psi \in D(\mathbb{R})$ satisfying $\psi \equiv 1$ on a neighborhood of $[-M, M]$ and $\psi(x) = 0$ for all $x \notin [-t, t]$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = \langle \mathcal{H}_\Lambda(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{H}_\Lambda(R^{-2n}\triangle_A^n(u)), \varphi\rangle = \langle \mathcal{H}_\Lambda(u), (-x)^n R^{-2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-x)^n R^{-2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. □

Corollary 6.2. *From the previous theorem we obtain*

$$\Gamma_u = \inf\left\{R > 0 : \lim_{n\to\infty} R^{-2n}\triangle_A^n u = 0 \ \textit{in } \mathcal{S}'(\mathbb{R})\right\}.$$

Let $u \in \mathcal{S}'(\mathbb{R})$. We put $\gamma_u := \sup\{r \in [0, \infty) : \operatorname{supp}(\mathcal{H}_\Lambda(u)) \subset (-r, r)^c\}$.

Theorem 6.3. *Let $u \in \mathcal{S}'(\mathbb{R})$ such that $(-x)^{-n}\mathcal{H}_\Lambda(u) \in \mathcal{S}'(\mathbb{R})$ for all $n \in \mathbb{N}$. Let $u_n = \mathcal{H}_\Lambda^{-1}((-x)^{-n}\mathcal{H}_\Lambda(u))$. Then the support of $\mathcal{H}_\Lambda(u)$ is included in $(-M, M)^c$, $M > 0$, if and only if for all $R < M$ we have*

$$\lim_{n\to\infty} R^{2n} u_n = 0 \quad \textit{in } \mathcal{S}'(\mathbb{R}).$$

Proof. Let $u \in \mathcal{S}'(\mathbb{R})$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{2n} u_n = 0, \quad \text{for all } R < M.$$

Let $\varphi \in D(\mathbb{R})$ satisfy $\operatorname{supp}(\varphi) \subset (-M, M)$. We want to prove that

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = 0.$$

Let $r \in (0, M)$ such that $\operatorname{supp}\varphi \subset (-r, r)$ and $R \in (r, M)$. Then for all $n \in \mathbb{N}$ the function $x^{2n}\varphi$ is in $D(\mathbb{R})$ and we can write

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = \left\langle (-x)^{-n} R^{2n}\mathcal{H}_\Lambda(u), (-x)^n R^{-2n}\varphi\right\rangle = \left\langle \mathcal{H}_\Lambda(R^{2n}u_n), (-x)^n R^{-2n}\varphi\right\rangle.$$

The hypothesis implies that $\mathcal{H}_\Lambda(R^{2n}u_n) \to 0$ in $\mathcal{S}'(\mathbb{R})$. Moreover from the Leibniz formula we deduce that $(-x)^n R^{-2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'(\mathbb{R})$ and $M > 0$ such that $\operatorname{supp} \mathcal{H}_\Lambda(u) \subset (-M, M)^c$. We are going to prove that for all $R < M$

$$\lim_{n\to\infty} R^{2n}u_n = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Let $M > R$ and choose $t \in (R, M)$ and $\psi \in D(\mathbb{R})$ satisfying $\psi(x) \equiv 1$ for $|x| \geq \frac{M+t}{2}$ and $\psi(x) = 0$ for all $|x| \leq t$. Then for all $\varphi \in D(\mathbb{R})$ we have

$$\langle \mathcal{H}_\Lambda(u), \varphi\rangle = \langle \mathcal{H}_\Lambda(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{H}_\Lambda(R^n u_n), \varphi\rangle = \langle \mathcal{H}_\Lambda(u), (-x)^{-n}R^{2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-x)^{-n}R^{2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R})$. $\square$

Corollary 6.4. *From the previous theorem we obtain*

$$\gamma_u = \sup\left\{R > 0, \lim_{n\to\infty} R^{2n}u_n = 0 \ \textit{in } \mathcal{S}'(\mathbb{R})\right\}.$$

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