REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 1, 2017, Pages 163–188
Published online: June 13, 2017

ELEMENTARY PROOF OF THE CONTINUITY OF THE TOPOLOGICAL ENTROPY AT $\theta = \underline{1001}$ IN THE MILNOR–THURSTON WORLD

ANDRÉS JABLONSKI AND RAFAEL LABARCA

Abstract. In 1965, Adler, Konheim and McAndrew introduced the topological entropy of a given dynamical system, which consists of a real number that explains part of the complexity of the dynamics of the system. In this context, a good question could be if the topological entropy $H_{\text{top}}(f)$ changes continuously with f. For continuous maps this problem was studied by Misisurewicz, Slenk and Urbański. Recently, and related with the lexicographic and the Milnor–Thurston worlds, this problem was studied by Labarca and others. In this paper we will prove, by elementary methods, the continuity of the topological entropy in a maximal periodic orbit ($\theta = \underline{1001}$) in the Milnor–Thurston world. Moreover, by using dynamical methods, we obtain interesting relations and results concerning the largest eigenvalue of a sequence of square matrices whose lengths grow up to infinity.

Introduction

It is well know that one of the purposes of the topological theory of dynamical systems is to find universal models describing the topological dynamics of a large class of systems [16]. One of these models is the shift on n-symbols (Σ_n, τ, σ), where Σ_n is the set of sequences $\{\theta : \mathbb{N}_0 \to \{0, 1, 2, \dots, n-1\}\}$ endowed with a certain topology, τ, and $\sigma : \Sigma_n \to \Sigma_n$ is the shift map defined by $(\sigma(\theta))(i) = \theta(i+1)$ (see [2, 13, 7, 12]). These models have been extensively used to obtain a great amount of information about maps defined in an interval, vector fields on three dimensional manifolds, and other kinds of dynamical systems (see, for instance, [3, 6, 9, 10]).

By using a result in [8] —which is a slight modification of a result proved by Block, Guckenheimer, Misiurewicz and Young ([5])— and elementary methods we compute the topological entropy of some periodic orbits, and using matrix (linear) algebra we are able to prove the continuity of the topological entropy at the periodic orbit $\theta = \underline{1001}$ in the Milnor–Thurston world (the continuity was proved in [11] but

2010 *Mathematics Subject Classification.* 15B36, 37B10.

Key words and phrases. Topological entropy, Incidence matrix, A-graph, Milnor–Thurston world, Periodic orbit, Shift map.

Part of this paper is an outgrowth of research during a visit of the second author to the Collège de France. We thank Collège de France and USACH for their support while preparing the present paper.

here we prove it by elementary methods). To do so, we consider a surjective two to one linear map defined in two subintervals of $[0,1]$ which preserve orientation at the first interval and reverse orientation at the second one ($T : I_0 \cup I_1 \to I = [0,1]$). The maximal invariant set of this map can be modeled whit a succession of "zeros" and "ones", i.e., *full-shift* of two symbols. The dynamics of the restriction of the linear map to this maximal invariant set endowed with the order and the topology induced by the interval is called the Milnor–Thurston world (MTW) and it is a representation of the shift of two symbols with the order defined by Milnor and Thurston in [14].

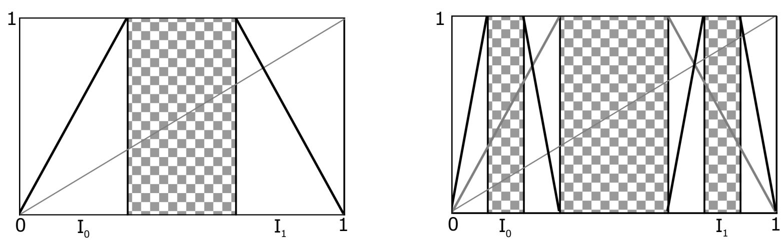
While in [11] the continuity of the topological entropy is proved by using the Hausdorff dimension, the novelty of the present paper is related with the techniques implemented: we build a graph associated with the maximal invariant set corresponding to the orbit $\theta = \underline{1001}$, that describes the dynamics of the system. Then we associate to this graph an incidence matrix; we compute its characteristic polynomial and we compute the largest real eigenvalue. Finally, the entropy h_0 associated with the sub-shift generated by the periodic orbit corresponds to the log of the largest real eigenvalue of the matrix ([8, 5]). To prove the continuity we compute the entropy at a family of periodic orbits $h_n := \underline{(1001)^n 1}$, and we prove that $h_n \to h_0$ as $n \to \infty$.

1. Preliminaries

1.1. **The Milnor–Thurston world.** Let $T : I_0 \cup I_1 \to I = [0,1]$ be a linear map defined as follows: $T|_{I_0}$ preserves orientation and $T|_{I_1}$ reverses orientation, and both restrictions are surjective. For this map we can consider the set $\Lambda_T = \bigcap_{k=0}^{\infty} T^{-k}(I_0 \cup I_1)$, and for any $x \in \Lambda_T$ we have:

$$x \in \Lambda_T \Longleftrightarrow x \in T^{-k}(I_0 \cup I_1), \quad \forall\, k \in \mathbb{N}$$

or, equivalently, $T^k(x) \in (I_0 \cup I_1)$, $\forall\, k \in \mathbb{N}$. Figure 1 represents this map.



(a) Representation of T (b) Representation of T^2

Figure 1. Graph of the map T.

Clearly, we can associate to each $x \in \Lambda_T$ a succession of 0's and 1's using the itinerary function $I_T : \Lambda_T \longrightarrow \Sigma_2$, $I_T(x) = (I_T(x)(i))$, where $I_T(x)(i) = j \Longleftrightarrow T^i(x) \in I_j$.

In Λ_T we consider the Euclidean topology and we use the bijection I_T to induce a topology in Σ_2: A set $A \subset \Sigma_2$ is open $\Leftrightarrow I_T^{-1}(A)$ is open in Λ_T. In the same way, we can define an order induced by the map I_T: $\theta \leq_T \alpha$ in $\Sigma_2 \Longleftrightarrow I_T^{-1}(\theta) \leq I_T^{-1}(\alpha)$ in $\mathbb{R}$. For being bijective and by inducing the topology and the order (in Σ_2) the map I_T is an homeomorphism which preserves the order. Using this property and the fact that the shift function $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ satisfies $\sigma(\theta_0\theta_1\theta_2\ldots) = \theta_1\theta_2\ldots$, it is easy to check that the diagram in Figure 2 is commutative.

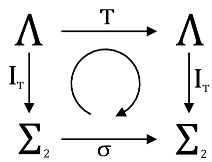


FIGURE 2.

Now we can formally define the Milnor–Thurston world. A sequence $\alpha \in \Sigma_0 = \{\theta \in \Sigma_2;\ \theta_0 = 0\}$ is called **minimal** if $\sigma^i(\alpha) \geq_T \alpha$ for all $i \in \mathbb{N}_0$. A sequence $\beta \in \Sigma_1 = \{\theta \in \Sigma_2;\ \theta_0 = 1\}$ is called **maximal** if $\sigma^i(\beta) \leq_T \beta$ for all $i \in \mathbb{N}_0$. Let *Min* (resp. *Max*) denote the set of minimal (resp. maximal) sequences. For $\alpha \in \Sigma_0$ and $\beta \in \Sigma_1$ let $\Sigma[\alpha,\beta] = \{\theta \in \Sigma_2; \alpha \leq_T \sigma^i(\theta) \leq_T \beta \text{ for any } i \in \mathbb{N}_0\} = \bigcap_{j=0}^{\infty} \sigma^{-j}([\alpha,\beta])$, where $[\alpha,\beta] = \{\theta \in \Sigma_2; \alpha \leq_T \theta \leq_T \beta\}$. Finally, the Milnor–Thurston world (MTW) is the set of pairs $(\alpha,\beta) \in Min \times Max$ such that $\{\alpha,\beta\} \subset \Sigma[\alpha,\beta]$ (see [12, 11]).

Remark 1.1. $\Sigma[\alpha,\beta]$ is a sub-shift of Σ_2.

1.2. **Definitions of topological entropy.**

Definition 1.2. Let $M \neq \emptyset$ and τ a topology such that (M,τ) is a compact space. Let $\mathcal{A} \subset \tau$ be a covering by open sets of M. We define the entropy of the covering $\mathcal{A}$ as $H(\mathcal{A}) = \log(N(\mathcal{A}))$, where $N(\mathcal{A}) = \min\{\mathrm{Card}(\mathcal{A}') : \mathcal{A}' \subset \mathcal{A}$ is a finite subcovering of $M\}$.

Definition 1.3. Let $\mathcal{A},\mathcal{B}$ be two coverings of the space M. We define $\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \subset \mathcal{A}, B \subset \mathcal{B}\}$.

Now, let $\phi : M \longrightarrow M$ be a continuous map, and $\phi^{-1}(\mathcal{A}) = \{\phi^{-1}(U);\ U \in \mathcal{A}\}$, the inverse image covering.

Definition 1.4. The topological entropy of ϕ with respect to the covering $\mathcal{A}$ is $h(\phi,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{k=0}^{n-1} \phi^{-k}(\mathcal{A}))$.

Definition 1.5. The topological entropy of ϕ is $h_{\mathrm{top}}(\phi) = \sup_{\mathcal{A}}\{h(\phi,\mathcal{A})\}$, where the sup is taken over all the open coverings $\mathcal{A}$ of M.

Finally, in our case, we define the entropy map H:

Definition 1.6. $H : \mathrm{MTW} \longrightarrow [0,\ln(2)]$, $H(\alpha,\beta) = h_{\mathrm{top}}(\sigma|_{\Lambda_T([\alpha,\beta])})$.

1.3. **Some notation and results.** Let $I \subset \mathbb{R}$ be an interval, $f : I \longrightarrow \mathbb{R}$ a continuous map, and $A = \{I_1, I_2, \ldots, I_s\}$ a partition of I such that $\bigcup_{i=1}^{s} I_i = I$, with $i \neq j$ and each pairwise intersection is at most one point.

Definition 1.7. We say that $I - f$ covers n times J if there are subintervals $K_1, K_2, \ldots, K_n$ on I (with disjoint interior) such that $f(K_i) = J$, $i = 1, 2, \ldots, n$.

Definition 1.8 (A-graph). Associated to the partition A of the interval I we define a A-graph oriented with vertices $I_1, I_2, \ldots, I_s$ such that if $I_i f$ covers n times I_j then there are n arrows from I_i to I_j.

Definition 1.9 (Incidence matrix). Associated to the A-graph, we define the incidence matrix $M = (m_{ij})_{s\times s}$ where m_{ij} is the number of arrows from I_i to I_j.

Now we define the entropy associated to the A-graph.

Definition 1.10 (Perron's eigenvalue). The entropy of the A-graph is defined as $\log(r(M))$, where $r(M) = \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$.

Definition 1.11. Let $M = (m_{ij})$ be an $n \times n$ matrix. Associated to a finite sequence $p = (p_j)_{j=0}^{k}$ of elements from $\{1, 2, \ldots, n\}$ we define its **width** as $w(p) = \prod_{j=1}^{k} m_{p_{j-1}p_j}$. We will say that p is a **path** in M if $w(p) \neq 0$. In this case, the number $k = l(p)$ is called the **length** of the path p. A **loop** is a path $p = (p_0, p_1, \ldots, p_k)$ for which $p_i \neq p_{i+1}$, $i = 0, 1, \ldots, k-1$ and $p_k = p_0$. A **rome** in M is a subset $R \subset \{1, \ldots, n\}$ for which there is no loop outside of R, i.e., there is no loop $p = (p_0, \ldots, p_k)$ with $\{p_0, p_1, \ldots, p_k\} \cap R = \emptyset$. Given a rome R and a path $p = (p_0, \ldots, p_k)$, we will say the path p is **simple** regarding R if $\{p_0, p_k\} \subset R$ and $\{p_1, \ldots, p_{k-1}\} \bigcap R = \emptyset$. Finally, given a rome $R = \{r_1, \ldots, r_k\}$ with $r_i \neq r_j$ for $i \neq j$ the matrix A_R is defined by $A_R = A_R(x) = (a_{ij})_{i,j=1}^{k} = (a_{ij}(x))_{i,j=1}^{k}$, where $a_{ij}(x) = \sum_p w(p) x^{-l(p)}$ and the sum is over all the simple paths that originate in r_i and end at r_j.

2. The main result

The following proposition has an important role in proving the main result of this article, because we use it to compute the characteristic polynomial for a given matrix M.

Proposition 2.1. *Given a matrix $M = (m_{ij})_{n\times n}$ and a rome $R = \{I_{r_0}, I_{r_1}, \ldots, I_{r_k}\}$ (where $r_i \neq r_j$ if $i \neq j$) over M, then*

$$p_M(x) = \det(M - x\mathbb{I}_n) = (-1)^{n-k} x^n \det(A_R(x) - \mathbb{I}_k).$$

For the proof see [8].

Using this result and Perron's definition 1.10, we prove in the present paper the following theorem.

Theorem 2.2. *Let h_{top} be the topological entropy and σ the shift map. Then $h_{\text{top}}(\theta) = H(\sigma(\theta), \theta) = h_{\text{top}}(\sigma|_{\Lambda_\theta})$ is continuous at $\theta = \underline{1001}$, where*

$$\Lambda_\theta = \bigcap_{n=0}^{\infty} \sigma^{-n}([\sigma(\theta), \theta]).$$

In this paper we follow the line of research started in [4].

3. Proof of the main result

To prove the main result we will prove that the map h_{top} is continuous from the right and from the left at the value $\theta = \underline{1001}$.

Lemma 3.1. *h_{top} is continuous from the right at $\theta = \underline{1001}$.*

Proof. In the MTW we have $\underline{1001} < \underline{1000}$, so first we will prove $h_{\text{top}}(\underline{1001}) = h_{\text{top}}(\underline{1000})$. To do that we will compute the topological entropy for both periodic orbits.

In order to do that we can use the following methodology: Given a maximal periodical orbit $\theta \in \Sigma_2$ of period n, we have its maximal invariant set $\Lambda_\theta = \bigcap_{n=0}^{\infty} \sigma^{-n}([\sigma(\theta), \theta])$ (which is a finite sub shift; see Figure 3). Initially we produce a partition of the interval $[\sigma(\theta), \theta]$ by using the induced order of the iterations $\sigma^i(\theta),\ i \in \{0, 1, 2, \ldots, n\}$.

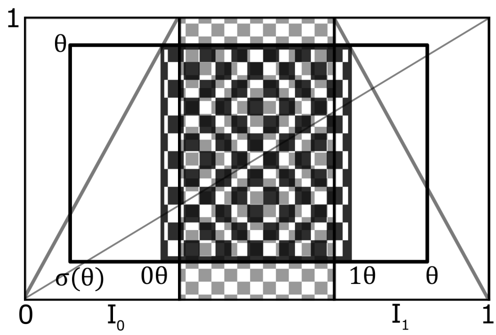


Figure 3. Graph of Λ_θ.

In the interior of this partition we can find the sequences 0θ and 1θ $(= \sigma^{n-1}(\theta))$. We include both values as part of the partition (one of these two values is part of the orbit of θ) and we compute the associated A-graph. With this we have associated the incidence matrix $M = (m_{ij})_{n\times n}$. Finally, we choose a rome to compute the polynomial matrix $A_R(x)$ and we calculate the maximal real root of the characteristic polynomial.

The topological entropy at $\theta = \underline{1001}$:

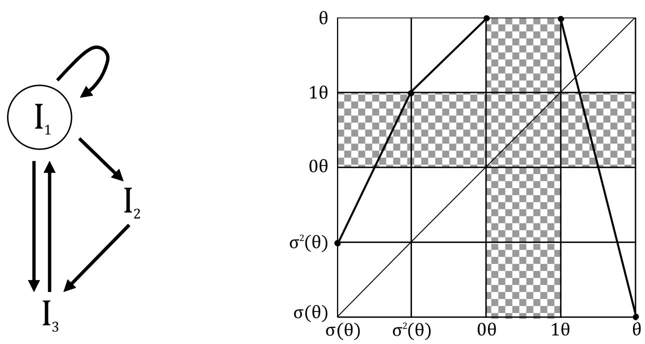
First, we apply the shift map to the periodic orbit:

$$0\colon \theta = \underline{1001} \tag{3.1}$$
$$1\colon \sigma(\theta) = \underline{0011} \tag{3.2}$$
$$2\colon \sigma^2(\theta) = \underline{0110} \tag{3.3}$$
$$3\colon \sigma^3(\theta) = \underline{1100} \tag{3.4}$$

From (3.1) and (3.4) we have $\theta > \sigma^3(\theta)$; (3.2) and (3.3) imply $\sigma^2(\theta) > \sigma(\theta)$; finally, $\sigma^3(\theta) > \sigma^2(\theta)$, then we have $\theta > \sigma^3(\theta) > \sigma^2(\theta) > \sigma(\theta)$.

Defining $0\theta = 0\underline{1001}$ and $1\theta = 1\underline{1001}$ we have $1\theta = 1\underline{1001} = 1100110011001\ldots = \underline{1100} = \sigma^3(\theta)$; also, $0\theta > \sigma^2(\theta)$. Expanding the inequality: $\theta > 1\theta = \sigma^3(\theta) > 0\theta > \sigma^2(\theta) > \sigma(\theta)$. Figure 4 shows the relationship between the iterations of the periodical orbit and the associated A-graph. Its incidence matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$



(a) A-graph associated to $\underline{1001}$

(b) Graph of the shift map associated to $\underline{1001}$

FIGURE 4. Topological entropy at $\theta = \underline{1001}$.

We consider a rome with a unique element, I_1, so the closed paths are[1]: $1 \to 1$, $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 1$, so the $A_R(x)$ matrix is given by $x^{-1} + x^{-2} + x^{-3}$. Applying Proposition 2.1 we have:

$$(-1)^{3-1} x^3 \det(A_R(x) - \mathbb{I}) = x^3(x^{-1} + x^{-2} + x^{-3} - 1) = 1 + x + x^2 - x^3.$$

Finally, we have $p(\underline{1001})(x) = 1 + x + x^2 - x^3$, whose largest real root is 1.8392867. We conclude that $h_{\text{top}}(\underline{1001}) \approx \log(1.8392867)$.

[1] From now on, the vector (x, y, z, u, v, w) corresponds to $x \to y \to z \to u \to v \to w$.

The topological entropy at $\theta = \underline{1000}$:

For the iterations of the sequence $\underline{1000}$ we have:

$$0\colon \theta = \underline{1000} \tag{3.5}$$

$$1\colon \sigma(\theta) = \underline{0001} \tag{3.6}$$

$$2\colon \sigma^2(\theta) = \underline{0010} \tag{3.7}$$

$$3\colon \sigma^3(\theta) = \underline{0100} \tag{3.8}$$

Defining $0\theta = 0\underline{1000}$ and $1\theta = 1\underline{1000}$ we have $0\theta = 0\underline{1000} = 0100010001000\ldots = \underline{0100} = \sigma^3(\theta)$, also $0\theta > \sigma^2(\theta)$. Expanding the inequalities: $\theta > 1\theta > 0\theta = \sigma^3(\theta) > \sigma^2(\theta) > \sigma(\theta)$. The graph associated to the shift map associated to $\underline{1000}$ is the same as the previous one. This implies that we obtain the same value for the topological entropy.

Hence, we have $h_{\text{top}}(\underline{1001}) = h_{\text{top}}(\underline{1000})$, so for any subsequence $\eta_n \subset \Sigma_2$ such that $\underline{1001} \le \eta_n \le \underline{1000}$ we have $h_{\text{top}}(\underline{1001}) \le h_{\text{top}}(\eta_n) \le h_{\text{top}}(\underline{1000})$. Applying the limit over n we conclude:

$$\lim_{n\to\infty} h_{\text{top}}(\eta_n) = h_{\text{top}}(\underline{1001}) = h_{\text{top}}(\underline{1000}). \qquad \square$$

Remark 3.2. To complete the proof of Proposition 2.2 we have to prove the continuity of topological entropy at $\underline{1001}$ from the left. This is —essentially— the main difficulty of this paper.

Lemma 3.3. *h_{top} is continuous from the left at $\underline{1001}$.*

To prove this lemma we will use a subsequence θ_n that converges from the left to $\underline{1001}$, then we calculate h_{top} for θ_n, with $n = 1, 2, 3, 4, 5$, to find a pattern on the subsequence of the roots of the characteristic polynomials. Finally, we will show that for any sequence between a subsequence $\theta_{k(n)}$ and $\theta_{k(n)+1}$ its topological entropy converges to the topological entropy of $\underline{1001}$. However, this procedure is not direct and we have to prove it by induction.

3.1. **Induction procedure.**

3.1.1. *Step 1. Deduction of the general procedure.* For $\theta_1 = \underline{10011}$ we can order its iterations like $0 > 3 > 4 = 1\theta_1 > 0\theta_1 > 2 > 1$. This allows us to define the following intervals:

$$I_1 = [1\theta_1 = \sigma^4(\theta_1), \sigma^3(\theta_1)], \qquad I_2 = [\sigma^3(\theta_1), \theta_1],$$
$$I_3 = [\sigma(\theta_1), \sigma^2(\theta_1)], \qquad I_4 = [\sigma^3(\theta_1), 0\theta_1].$$

By ordering them we get Figure 5.

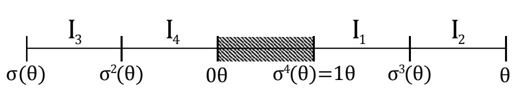
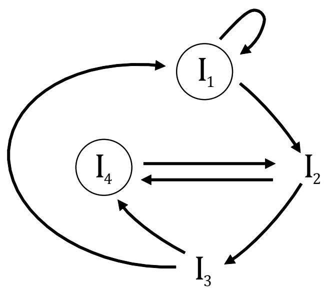
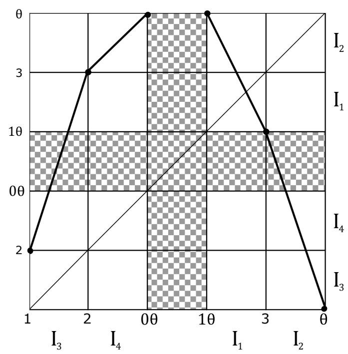


FIGURE 5. Intervals associated to θ_1.

Also, we can check that $\underline{1} \in I_1$ and obtain the A-graph of θ_1 by the associated graph of $\sigma|_{\theta_1}$ (Figure 6).



(a) A-graph associated to θ_1



(b) shift map associated to θ_1

FIGURE 6. Representation of the dynamics associated to θ_1.

The restriction $\sigma|_{\theta_1}$ has the following dynamics: I_1 σ-cover I_1 and I_2; I_2 σ-cover I_3 and I_4; I_3 σ-cover I_4 and I_1; and I_4 σ-cover I_2 (Figure 7).

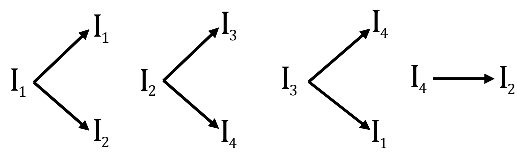


FIGURE 7. Representation of the σ-covering associated to θ_1.

We summarize the loops and paths associated to the rome $R = \{1, 4\}$:

$$a_{11}\colon I_1 \to I_1 : (1,1) \text{ and } (1,2,3,1),$$
$$a_{14}\colon I_1 \to I_4 : (1,2,4) \text{ and } (1,2,3,4),$$
$$a_{41}\colon I_4 \to I_1 : (4,2,3,1),$$
$$a_{44}\colon I_4 \to I_4 : (4,2,4) \text{ and } (4,2,3,4).$$

Finally, the transition matrix $A_R(x)$ is

$$\begin{pmatrix} x^{-1}+x^{-3} & x^{-2}+x^{-3} \\ x^{-3} & x^{-2}+x^{-3} \end{pmatrix}.$$

Then

$$p_1(x) = (-1)^{4-2}x^4 \det(A_R(x) - \mathbb{I}) = -xp(\underline{1001})(x) + 1,$$

whose maximal real root is $\lambda = 1.7220838$. So we have that $h_{\text{top}} \approx \log(1.7220838)$.

For $\theta_2 = \overline{(1001)^2 1} = \underline{100110011}$ we can order its iterations like $0 > 4 > 7 > 3 > 8 = 1\theta_2 > 0\theta_2 > 6 > 2 > 5 > 1$. This order allows us to define the following intervals:

$$I_1 = [\sigma^3(\theta_2), \sigma^7(\theta_2)], \qquad I_2 = [\sigma^7(\theta_2), \sigma^4(\theta_2)], \qquad I_3 = [\sigma^5(\theta_2), \sigma^2(\theta_2)],$$
$$I_4 = [\sigma^6(\theta_2), 0\theta_2], \qquad I_5 = [\sigma^4(\theta_2), \theta_2], \qquad I_6 = [\sigma(\theta_2), \sigma^5(\theta_2)],$$
$$I_7 = [\sigma^2(\theta_2), \sigma^6(\theta_2)], \qquad I_8 = [1\theta_2 = \sigma^8(\theta_2), \sigma^3(\theta_2)].$$

By ordering them we get Figure 8.

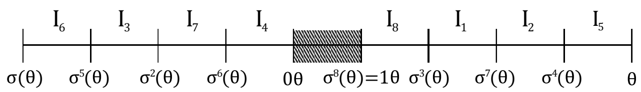
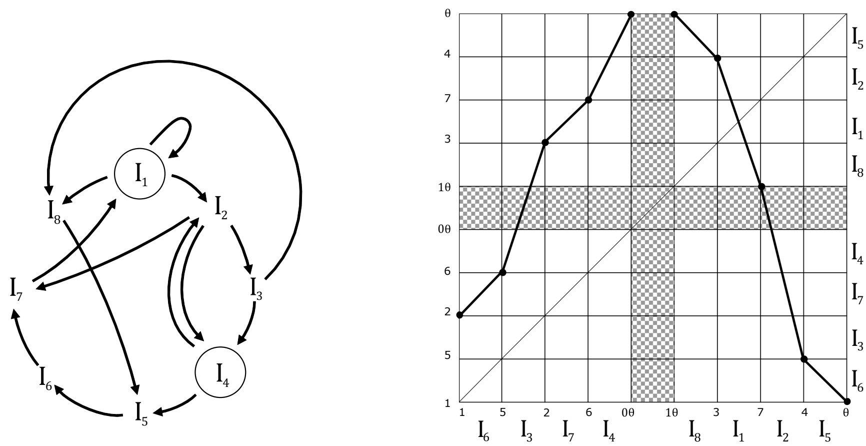


FIGURE 8. Intervals associated to θ_2.

Also, we can check that $\underline{1} \in I_1$ and obtain the A-graph of $\sigma|_{\theta_2}$ (Figure 9).



(a) A-graph associated to θ_2 (b) shift map associated to θ_2

FIGURE 9. Representation of the dynamics associated to θ_2.

The restriction $\sigma|_{\theta_2}$ has the following dynamics: I_1 σ-cover I_1, I_2 and I_8; I_2 σ-cover I_3, I_4 and I_7; I_3 σ-cover I_4 and I_8; I_4 σ-cover I_2 and I_5; I_5 σ-cover I_6; I_6 σ-cover I_7; I_7 σ-cover I_1; and I_8 σ-cover I_5 (Figure 10).

From this representation we can obtain the loops and paths of the rome $R = \{1, 4\}$:

$$a_{11}\colon (1,1); (1,2,7,1); (1,8,5,6,7,1); (1,2,3,8,5,6,7,1),$$
$$a_{14}\colon (1,2,4) \text{ and } (1,2,3,4),$$
$$a_{41}\colon (4,2,7,1); (4,5,6,7,1); (4,2,3,8,5,6,7,1),$$
$$a_{44}\colon (4,2,4) \text{ and } (4,2,3,4).$$

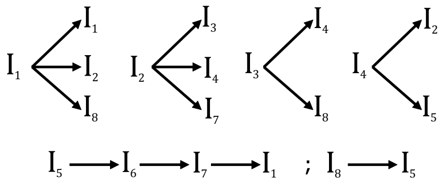


FIGURE 10. Representation of the σ-covering associated to θ_2.

Finally, the transition matrix $A_R(x)$ is:

$$\begin{pmatrix} x^{-1}+x^{-3}+x^{-5}+x^{-7} & x^{-2}+x^{-3} \\ x^{-3}+x^{-4}+x^{-7} & x^{-2}+x^{-3} \end{pmatrix}.$$

Then,

$$p_2(x) = (-1)^{8-2}x^8 \det(A_R(x) - \mathbb{I}) = x^4 p_1(x) - (x^3 + x^2 + x - 1).$$

For the next steps (3, 4 and 5) we just show the main stage of each iteration.

Intervals associated to θ_i, $i = 3, 4, 5$ (Figures 11–13):

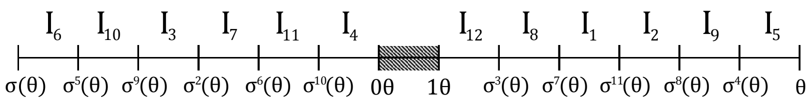


FIGURE 11. Intervals associated to θ_3.

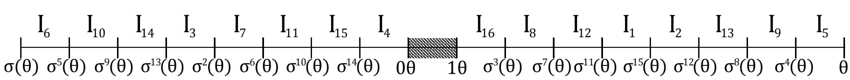


FIGURE 12. Intervals associated to θ_4.

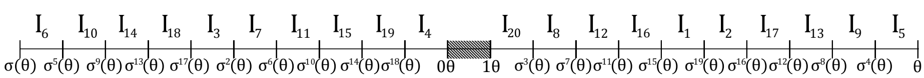
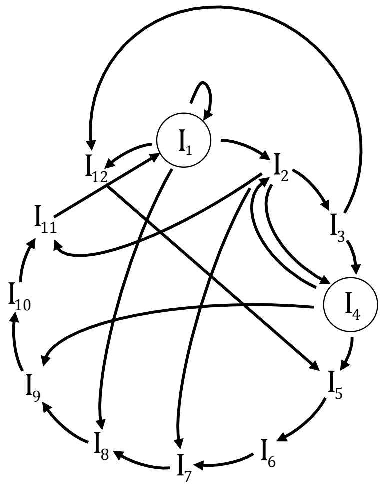
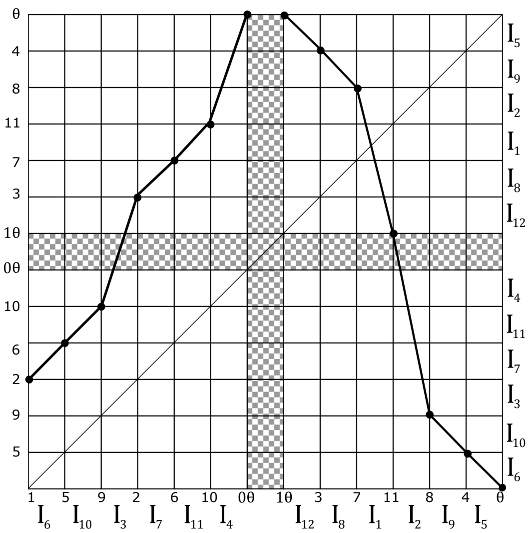


FIGURE 13. Intervals associated to θ_5.

Representation of the A-graphs and shift maps associated to θ_i, $i = 3, 4, 5$ (Figures 14–16):

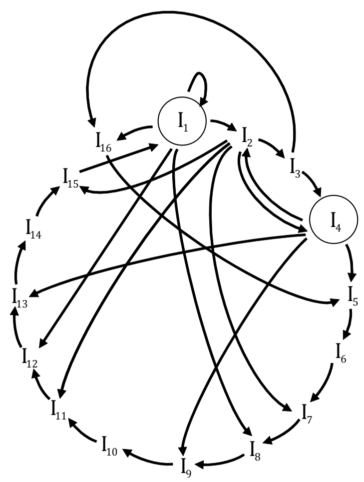


(a) A-graph associated to θ_3

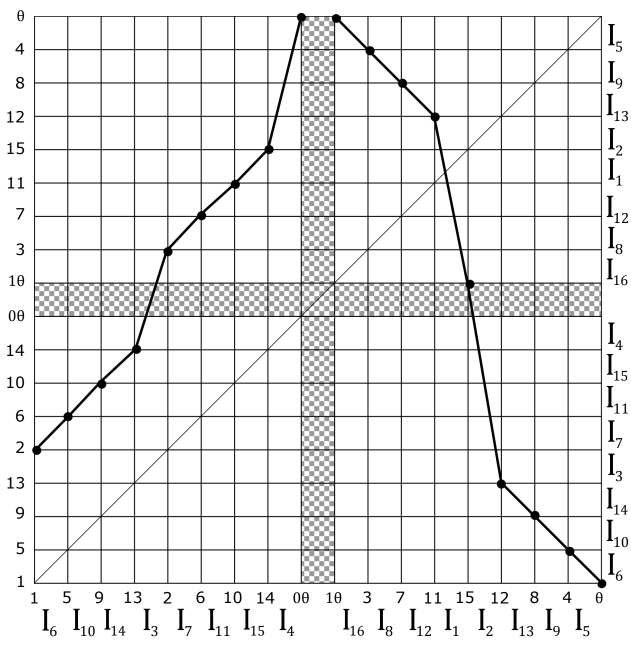


(b) shift map associated to θ_3

FIGURE 14. Representation of the dynamics asssociated to θ_3.

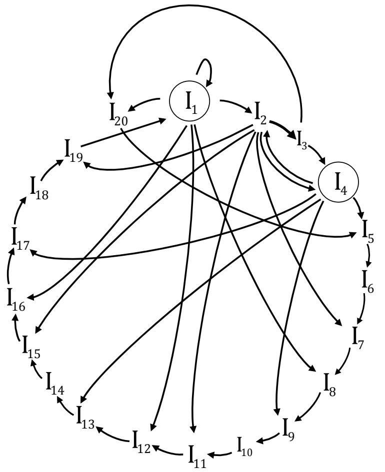


(a) A-graph associated to θ_4

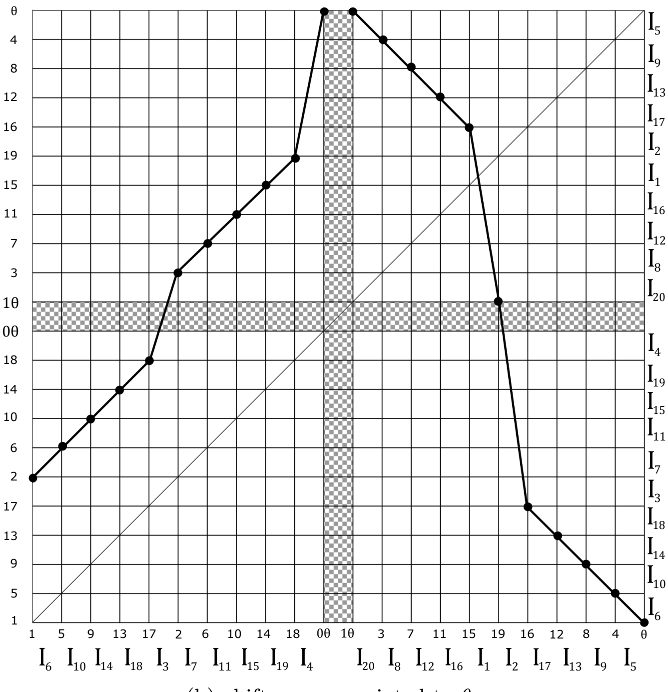


(b) shift map associated to θ_4

FIGURE 15. Representation of θ_4.



(a) A-graph associated to θ_5



(b) shift map associated to θ_5

FIGURE 16. Representation of θ_5.

σ-coverings associated to θ_i, $i = 3, 4, 5$ (Figures 17–19):

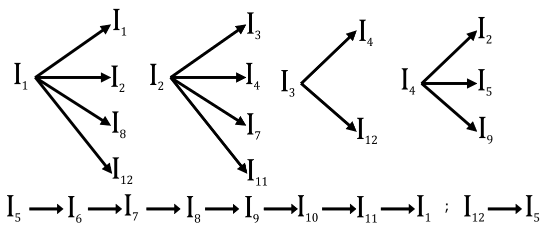


FIGURE 17. σ-covering associated to θ_3.

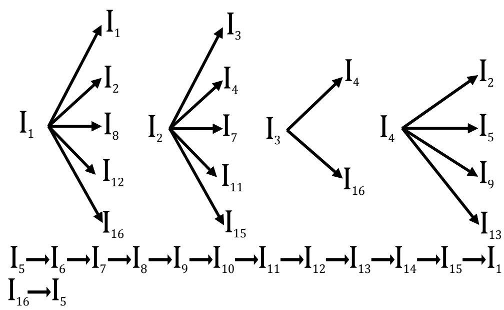


FIGURE 18. σ-covering associated to θ_4.

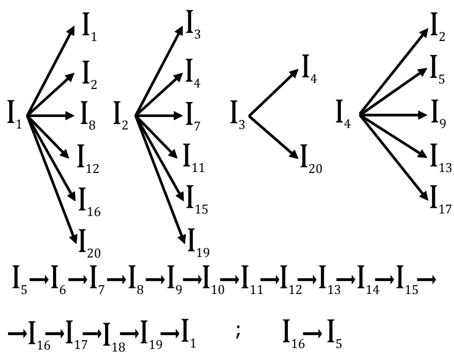


FIGURE 19. σ-covering associated to θ_5.

$A_R(x)$ matrices associated to θ_i, $i = 3, 4, 5$:

For θ_3:

$$\begin{pmatrix} x^{-1}+x^{-3}+x^{-5}+x^{-7}+x^{-9}+x^{-11} & x^{-2}+x^{-3} \\ x^{-3}+x^{-4}+x^{-7}+x^{-8}+x^{-11} & x^{-2}+x^{-3} \end{pmatrix},$$

$$p_3(x) = (-1)^{12-2}x^{12}\det(A_R(x)-\mathbb{I}) = x^4 p_2(x) - (x^3+x^2+x-1).$$

For θ_4:

$$\begin{pmatrix} x^{-1}+x^{-3}+x^{-5}+x^{-7}+x^{-9}+x^{-11}+x^{-13}+x^{-15} & x^{-2}+x^{-3} \\ x^{-3}+x^{-4}+x^{-7}+x^{-8}+x^{-11}+x^{-12}+x^{-15} & x^{-2}+x^{-3} \end{pmatrix},$$

$$p_4(x) = (-1)^{16-2}x^{16}\det(A_R(x)-\mathbb{I}) = x^4 p_3(x) - (x^3+x^2+x-1).$$

For θ_5:

$$\begin{pmatrix} x^{-1}+x^{-3}+x^{-5}+x^{-7}+x^{-9}+x^{-11}+x^{-13}+x^{-15}+x^{-17}+x^{-19} & x^{-2}+x^{-3} \\ x^{-3}+x^{-4}+x^{-7}+x^{-8}+x^{-11}+x^{-12}+x^{-15}+x^{-16}+x^{-19} & x^{-2}+x^{-3} \end{pmatrix},$$

$$p_5(x) = (-1)^{20-2}x^{20}\det(A_R(x)-\mathbb{I}) = x^4 p_5(x) - (x^3+x^2+x-1).$$

3.1.2. *Step 2. Generalization.* Considering the previous constructions, we can assume (inductively) the following order for $\theta = \theta_k = \underline{(1001)^k 1}$ and its iterations: $\theta > 4 > 8 > 12 > 16 > 20 > \cdots > 4(k-1) > 4k-1 > 4(k-1)-1 > 4(k-2)-1 > \cdots > 23 > 19 > 15 > 11 > 7 > 3 > 4k = 1\theta_k > 0\theta_k > 4k-2 > 4(k-1)-2 > 4(k-2)-2 > \cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k-3 > 4(k-1)-3 > 4(k-2)-3 > \cdots > 21 > 17 > 13 > 9 > 5 > 1$.

This order of the iterations allows us to define the intervals

$$\begin{aligned}
I_1 &= [\sigma^{4k-5}(\theta_k), \sigma^{4k-1}(\theta_k)] & I_2 &= [\sigma^{4k-1}(\theta_k), \sigma^{4(k-1)}(\theta_k)] \\
I_3 &= [\sigma^{4k-3}(\theta_k), \sigma^{2}(\theta_k)] & I_4 &= [\sigma^{4k-2}(\theta_k), 0\theta_k] \\
I_5 &= [\sigma^{4}(\theta_k), \theta_k] & I_6 &= [\sigma(\theta_k), \sigma^{5}(\theta_k)] \\
I_7 &= [\sigma^{2}(\theta_k), \sigma^{6}(\theta_k)] & I_8 &= [\sigma^{3}(\theta_k), \sigma^{7}(\theta_k)] \\
I_9 &= [\sigma^{8}(\theta_k), \sigma^{4}(\theta_k)] & I_{10} &= [\sigma^{5}(\theta_k), \sigma^{9}(\theta_k)] \\
I_{11} &= [\sigma^{6}(\theta_k), \sigma^{1}0(\theta_k)] & I_{12} &= [\sigma^{7}(\theta_k), \sigma^{1}1(\theta_k)] \\
&\vdots & &\vdots \\
I_{4k-2} &= [\sigma^{4k-7}(\theta_k), \sigma^{4k-3}(\theta_k)] & I_{4k-1} &= [\sigma^{4k-6}(\theta_k), \sigma^{4k-2}(\theta_k)] \\
I_{4k} &= [1\theta_k = \sigma^{4k-1}(\theta_k), \sigma^{3}(\theta_k)],
\end{aligned}$$

whose representation, over the Milnor–Thurston world, is given by Figure 20:

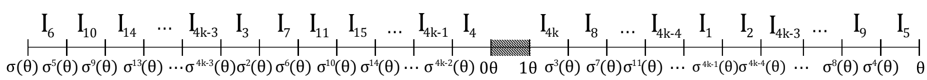


FIGURE 20. Order of the intervals associated to θ_k.

Also, we can check that $\underline{1} \in I_1$ and that the associated graph of $\sigma|_{\theta_k}$ is as shown in Figure 21; its diagram of σ-covering is given by Figure 22.

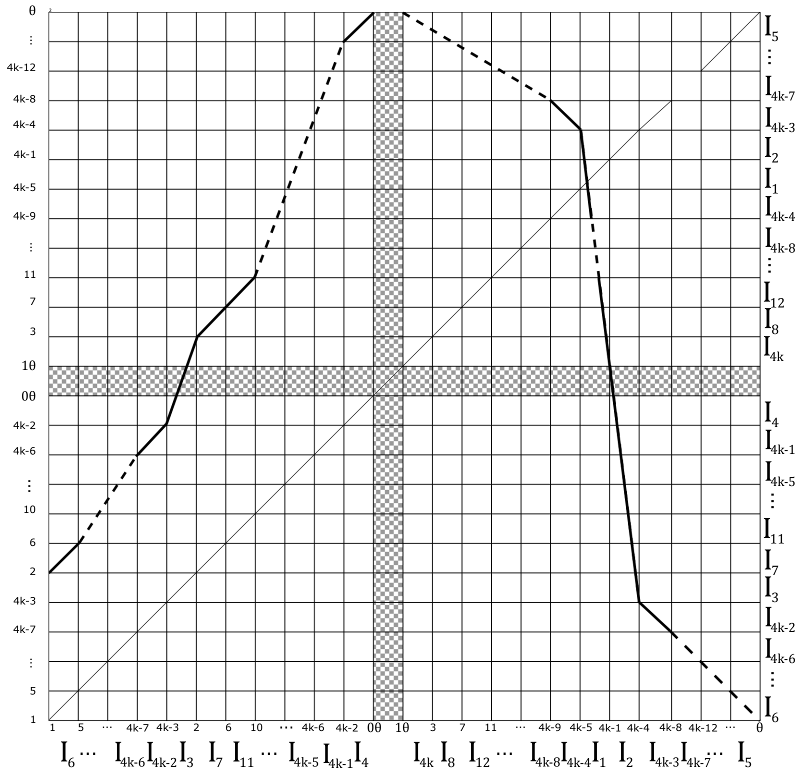


FIGURE 21. Shift map associated to θ_k.

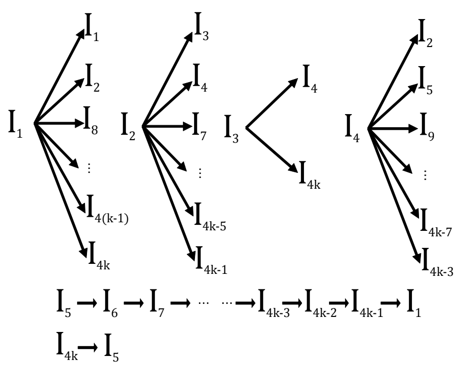


FIGURE 22. Diagram of σ-covering associated to θ_k.

From that diagram we can deduce the loops and paths associated to the rome $R = \{1, 4\}$. With this we can obtain its associated $A_R(x)$ matrix

$$A_R^k(x) = \begin{pmatrix} a_{11}(x) & x^{-2} + x^{-3} \\ a_{21}(x) & x^{-2} + x^{-3} \end{pmatrix},$$

where

$$a_{11}(x) = x^{-1} + x^{-3} + \cdots + x^{-(4(k-1)-3)} + x^{-(4(k-1)-1)} + x^{-(4k-3)} + x^{-(4k-1)}$$
$$a_{21}(x) = x^{-3} + x^{-4} + \cdots + x^{-(4(k-1)-4)} + x^{-(4(k-1)-1)} + x^{-(4k-4)} + x^{-(4k-1)}.$$

Finally, for the characteristic polinomial associated to the matrix defined by the A-graph, we have:

$$p_k(x) = (-1)^{4k-2} x^{4k} \det(A_R^k(x) - \mathbb{I}) = x^4 p_{k-1}(x) - (x^3 + x^2 + x - 1).$$

3.1.3. *Step 3. Proof for the case $n = k+1$.* Now, let us take $\theta_{k+1} = \underline{(1001)^{k+1}1}$. It is not hard to see that θ_{k+1} and its iterations satisfy: $(\sigma^j(\theta_{k+1}),\ j = 0, 1, 2, \ldots, 4k+4)$: $\theta > 4 > 8 > 12 > 16 > \cdots > 4(k-2) > 4(k-1) > 4k > 4k+3 > 4k-1 > 4(k-1)-1 > \cdots > 19 > 15 > 11 > 7 > 3 > 4k = 1\theta_{k+1} > 0\theta_{k+1} > 4k+2 > 4k-2 > 4(k-1)-2 > \cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k+1 > 4k-3 > 4(k-1)-3 > \cdots > 21 > 17 > 13 > 9 > 5 > 1$. (A proof of this fact will be given later in this section).

This order allows us to define the following intervals:

$$\begin{aligned}
I_1 &= [\sigma^{4k-1}(\theta_{k+1}), \sigma^{4k+3}(\theta_{k+1})] & I_2 &= [\sigma^{4k+3}(\theta_{k+1}), \sigma^{4k}(\theta_{k+1})] \\
I_3 &= [\sigma^{4k+1}(\theta_{k+1}), \sigma^{2}(\theta_{k+1})] & I_4 &= [\sigma^{4k+2}(\theta_{k+1}), 0\theta_{k+1}] \\
I_5 &= [\sigma^{4}(\theta_{k+1}), \theta_{k+1}] & I_6 &= [\sigma(\theta_{k+1}), \sigma^{5}(\theta_{k+1})] \\
I_7 &= [\sigma^{2}(\theta_{k+1}), \sigma^{6}(\theta_{k+1})] & I_8 &= [\sigma^{3}(\theta_{k+1}), \sigma^{7}(\theta_{k+1})] \\
I_9 &= [\sigma^{8}(\theta_{k+1}), \sigma^{4}(\theta_{k+1})] & I_{10} &= [\sigma^{5}(\theta_{k+1}), \sigma^{9}(\theta_{k+1})] \\
&\vdots & &\vdots \\
I_{4k+1} &= [\sigma^{4k-4}(\theta_{k+1}), \sigma^{4k}(\theta_{k+1})] & I_{4k+2} &= [\sigma^{4k-3}(\theta_{k+1}), \sigma^{4k+1}(\theta_{k+1})] \\
I_{4k+3} &= [\sigma^{4k-2}(\theta_{k+1}), \sigma^{4k+2}(\theta_{k+1})] & I_{4k+4} &= [1\theta_{k+1} = \sigma^{4k+3}(\theta_{k+1}), \sigma^{3}(\theta_{k+a})],
\end{aligned}$$

whose representation, over the Milnor–Thurston world, is given by Figure 23:

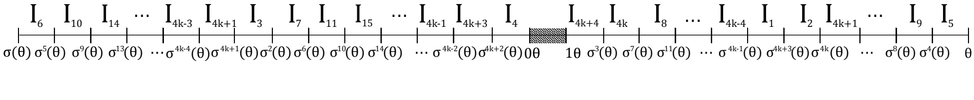


FIGURE 23. Order of the intervals associated to θ_{k+1}.

Also, we can check that $\underline{1} \in I_1$ and that the graph associated to $\sigma|_{\theta_{k+1}}$ is as shown in Figure 24; its diagram of σ-covering is given by Figure 25.

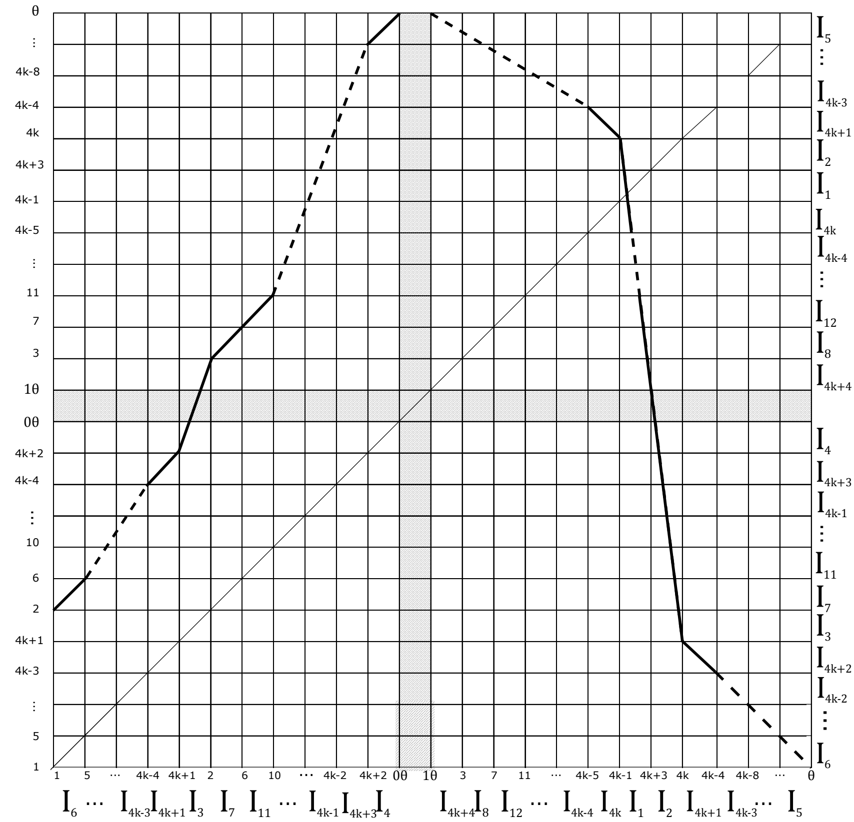


FIGURE 24. Shift map associated to θ_{k+1}.

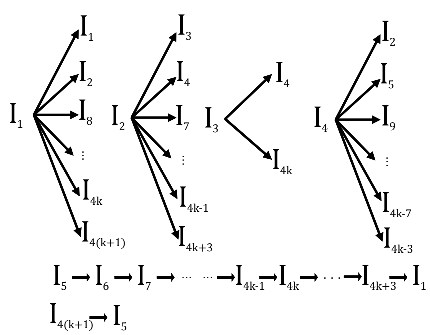


FIGURE 25. Diagram of σ-covering associated to θ_{k+1}.

We can obtain the loops and paths associated to the rome $R = \{1,4\}$ and their associated $A_R(x)$ matrix,

$$A_R^k(x) = \begin{pmatrix} a_{11}(x) & x^{-2}+x^{-3} \\ a_{21}(x) & x^{-2}+x^{-3} \end{pmatrix},$$

where

$$\begin{aligned} a_{11}(x) &= x^{-1}+x^{-3}+\cdots+x^{-(4k-3)}+x^{-(4k-1)}+x^{-(4k+1)}+x^{-(4k+3)} \\ a_{21}(x) &= x^{-3}+x^{-4}+\cdots+x^{-(4k-4)}+x^{-(4k-1)}+x^{-4k}+x^{-(4k+3)}. \end{aligned}$$

Then, if we write

$$A_R^k(x) = \begin{pmatrix} a_k^{11} & a_k^{14} \\ a_k^{41} & a_k^{44} \end{pmatrix}, \qquad A_R^{k+1}(x) = \begin{pmatrix} a_{k+1}^{11} & a_{k+1}^{14} \\ a_{k+1}^{41} & a_{k+1}^{44} \end{pmatrix},$$

we can see that

$$\begin{aligned} a_{k+1}^{11} &= a_k^{11} + x^{-(4k+1)} + x^{-(4k+3)}, \quad & a_{k+1}^{14} &= a_k^{14}, \\ a_{k+1}^{41} &= a_k^{41} + x^{-(4k+3)} + x^{-4k}, & a_{k+1}^{44} &= a_k^{44}. \end{aligned}$$

Then,

$$\begin{aligned} p_{k+1}(x) &= (-1)^{4(k+1)-2} x^{4(k+1)} \det(A_R^{k+1}(x) - \mathbb{I}) \\ &= x^{4k+4} \begin{vmatrix} a_{11}^{k+1} - 1 & a_{14}^{k+1} \\ a_{41}^{k+1} & a_{44}^{k+1} - 1 \end{vmatrix} \\ &= x^{4k+4} \begin{vmatrix} a_{11}^{k} + x^{-(4k+1)} + x^{-(4k+3)} - 1 & a_{14}^{k} \\ a_{41}^{k+1} + x^{-(4k+3)} + x^{-4k} & a_{44}^{k} - 1 \end{vmatrix} \\ &= x^{4(k+1)} \big[((a_{11}^k - 1) + x^{-(4k+1)} + x^{-(4k+3)})(a_{44}^k - 1) \\ &\quad - (a_{41}^k + x^{-(4k+3)} + x^{-4k})(a_{14}^k) \big] \\ &= x^{4(k+1)} \big[\{(a_{11}^k - 1)(a_{44}^k - 1) - a_{41}^k a_{14}^k\} \\ &\quad + (a_{44}^k - 1)(x^{-(4k+1)} + x^{-(4k+3)}) - a14^k(x^{-(4k+3)} + x^{-4k}) \big]. \end{aligned}$$

Since $a_{44}^k = a_{14}^k$ we have:

$$\begin{aligned} &= x^4 \cdot p_k(x) + x^{4k+4}[a_{44}^k x^{-(4k+1)} - x^{-(4k+1)} - x^{-(4k+3)} - a_{14}^k x^{-4k}] \\ &= x^4 \cdot p_k(x) + x^{4k+4}[x^{-(4k+3)} + x^{-(4k+4)} - x^{-(4k+1)} \\ &\quad - x^{-(4k+3)} - x^{-(4k+2)} - x^{-(4k+3)}] \\ &= x^4 \cdot p_k(x) + x^{4k+4}[x^{-(4k+4)} - x^{-(4k+3)} - x^{-(4k+2)} - x^{-(4k+1)}] \\ &= x^4 \cdot p_k(x) + (1 - x - x^2 - x^3) \\ &= x^4 \cdot p_k(x) - (x^3 + x^2 + x - 1). \end{aligned}$$

Therefore $p_{k+1}(x) = x^4 \cdot p_k(x) - (x^3 + x^2 + x - 1)$.

So, let us prove that iterations of the sequence θ_{k+1} appear in the given order. For that we consider:

$$\begin{aligned}
1 &\equiv \sigma(\theta_{k+1}) = \sigma(\underline{1(0011)^{k+1}}) = \underline{(0011)^{k+1}1}\\
2 &\equiv \sigma(1) = \sigma(\underline{0011\ldots001100111}) = \sigma(\underline{0(0110)^{k}0111}) = \underline{(0110)^{k}01110}\\
3 &\equiv \sigma(2) = \sigma(\underline{0(1100)^{k}1110}) = \underline{(1100)^{k}11100}\\
4 &\equiv \sigma(3) = \sigma(\underline{1(1001)^{k}1100}) = \underline{(1001)^{k}1(1001)}\\
5 &\equiv \sigma(4) = \sigma(\underline{1(0011)^{k}(1001)}) = \underline{(0011)^{k}(1001)}\\
6 &\equiv \sigma(5) = \sigma(\underline{0(0110)^{k-1}01110011}) = \underline{(0110)^{k-1}011(1001)10}\\
7 &\equiv \sigma(6) = \sigma(\underline{0(1100)^{k-1}11(1001)100}) = \underline{(1100)^{k-1}11(1001)100}\\
8 &\equiv \sigma(7) = \sigma(\underline{1(1001)^{k-1}1(1001)100}) = \underline{(1001)^{k-1}1(1001)^{2}}\\
9 &\equiv \sigma(8) = \sigma(\underline{1(0011)^{k-1}(1001)^{2}}) = \underline{(0011)^{k-1}(1001)^{2}1}\\
10 &\equiv \sigma(9) = \sigma(\underline{0(0110)^{k-2}011(1001)^{2}1}) = \underline{(0110)^{k-2}011(1001)^{2}10}\\
11 &\equiv \sigma(10) = \sigma(\underline{0(1100)^{k-2}11(1001)^{2}10}) = \underline{(1100)^{k-2}11(1001)^{2}100}\\
12 &\equiv \sigma(11) = \sigma(\underline{1(1001)^{k-2}1(1001)^{2}100}) = \underline{(1001)^{k-2}1(1001)^{3}}\\
13 &\equiv \sigma(12) = \sigma(\underline{1(0011)^{k-2}(1001)^{3}}) = \underline{(0011)^{k-2}(1001)^{3}1}\\
14 &\equiv \sigma(13) = \sigma(\underline{0(0110)^{k-3}011(1001)^{3}1}) = \underline{(0110)^{k-3}011(1001)^{3}10}\\
15 &\equiv \sigma(14) = \sigma(\underline{0(1100)^{k-3}11(1001)^{3}10}) = \underline{(1100)^{k-3}11(1001)^{3}100}\\
16 &\equiv \sigma(15) = \sigma(\underline{1(1001)^{k-3}1(1001)^{3}100}) = \underline{(1001)^{k-3}1(1001)^{4}}
\end{aligned}$$

So, $0 = \underline{(1001)^{k+1}1}$; $4 = \underline{(1001)^{k}1(1001)}$; $8 = \underline{(1001)^{k-1}1(1001)^{2}}$; $12 = \underline{(1001)^{k-2}1(1001)^{3}}$; $16 = \underline{(1001)^{k-3}1(1001)^{4}}$.

$$\Longrightarrow 4j-4 = \underline{(1001)^{k-(j-2)}1(1001)^{j-1}}, \quad j \geq 1.$$

On the other hand, $1 = \underline{(0011)^{k+1}1}$; $5 = \underline{(0011)^{k}(1001)1}$; $9 = \underline{(0011)^{k-1}(1001)^{2}1}$; $13 = \underline{(0011)^{k-2}(1001)^{3}1}$

$$\Longrightarrow 4j-3 = \underline{(0011)^{k-(j-2)}(1001)^{j-1}1}, \quad j \geq 1.$$

Also, $2 = \underline{(0110)^{k}01110}$; $6 = \underline{(0110)^{k-1}011(1001)10}$; $10 = \underline{(0110)^{k-2}011(1001)^{2}10}$; $14 = \underline{(0110)^{k-3}011(1001)^{3}10}$

$$\Longrightarrow 4j-2 = \underline{(0110)^{k-(j-1)}011(1001)^{j-1}10}, \quad j \geq 1.$$

Finally, $3 = \underline{(1001)^{k}11100}$; $7 = \underline{(1100)^{k-1}11(1001)100}$; $11 = \underline{(1100)^{k-2}11(1001)^{2}100}$; $15 = \underline{(1100)^{k-3}11(1001)^{3}100}$

$$\Longrightarrow 4j-1 = \underline{(1100)^{k-(j-1)}11(1001)^{j-1}100}, \quad j \geq 1.$$

Taking $j = k$ we have

$$\begin{aligned} 4k-4 &= \underline{(1001)^2 1 (1001)^{k-1}} \\ 4k-3 &= \underline{(0011)^2 (1001)^{k-1} 1} \\ 4k-2 &= \underline{(0110) 011 (1001)^{k-1} 10} \\ 4k-1 &= \underline{(1100) 11 (1001)^{k-1} 100}. \end{aligned}$$

Iterating the previous part by σ^4, we get:

$$\begin{aligned} 4(k+1)-4 &= \underline{(1001) 1 (1001)^{k}} \\ 4(k+1)-3 &= \underline{(0011)(1001)^{k} 1} \\ 4(k+1)-2 &= \underline{011 (1001)^{k} 10} \\ 4(k+1)-1 &= \underline{11 (1001)^{k} 100}. \end{aligned}$$

So, from the expressions of the iterations of $\theta_{k+1} = \underline{(1001)^{k+1}1}$ we conclude that $\theta > 4 > 8 > 12 > 16 > \cdots > 4(k-1) > 4k > 4k+3 > 4k-1 > 4(k-1)-1 > \cdots > \cdots > 19 > 15 > 11 > 7 > 3 > 4k = 1\theta_k > 0\theta_k > 4k+2 > 4k-2 > 4(k-1)-2 > \cdots > 22 > 18 > 14 > 10 > 6 > 2 > 4k+1 > 4k-3 > 4(k-1)-3 > \cdots > 21 > 17 > 13 > 9 > 5 > 1$. Which completes the inductive procedure and we have the characteristic polynomials.

Let us now show the existence of the largest real root of the polynomials $p(\underline{(1001)^k 1})(x)$, $k = 1, 2, 3, \ldots$; later we will see that this sequence of roots is increasing and tends to the largest root of the polynomial $p(\underline{1001})(x)$.

Let $\overline{x} = 1.8393\ldots$ be the largest real root of the polynomial $p(\underline{1001})(x)$. We have:

$$p(\underline{(1001)1})(x) = -xp(\underline{1001})(x) + 1 \quad \Rightarrow \quad p(\underline{(1001)1})(\overline{x}) = 1 > 0$$

and

$$p(\underline{(1001)1})(1) = -p(\underline{1001})(1) + 1 = -(-1+1+1+1) + 1 < 0.$$

So, $\exists\, x_1 \in]1, \overline{x}[$ such that $p(\underline{(1001)1})(x_1) = 0$. We have $1 < x_1 < \overline{x}$.

From $p(\underline{(1001)1})(x) = -x(1+x+x^2-x^3)+1 = x^4 - x^3 - x^2 - x + 1$, we obtain for $x \geq \overline{x}$, $p(\underline{(1001)1})(x) \geq p(\underline{(1001)1})(\overline{x}) = 1$, i.e., x_1 is the largest real root of $p(\underline{(1001)1})(x)$.

For $k = 2$,

$$\begin{aligned} p(\underline{(1001)^2 1})(x) &= x^4 p(\underline{(1001)1})(x) - (x^3 + x^2 + x - 1) \\ &= x^4(-xp(\underline{1001})(x) + 1) - (x^3 + x^2 + x - 1) \\ \Longrightarrow p(\underline{(1001)^2 1})(\overline{x}) &= \overline{x}^4(1) - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1) \\ &= -\overline{x}\underbrace{(-\overline{x}^3 + \overline{x}^2 + \overline{x} + 1)}_{0} + 1. \end{aligned}$$

Therefore,

$$p(\underline{(1001)^2 1})(\overline{x}) = 1 > 0.$$

Also, $p(\underline{(1001)^2 1})(x_1) = -(x_1^3 + x_1^2 + x_1 - 1) < 0$. Therefore $\exists\ x_2 \in]x_1, \overline{x}[$ such that $p(\underline{(1001)^2 1})(x_2) = 0$. We have $1 < x_1 < x_2 < \overline{x}$.

Now,

$$\begin{aligned} p(\underline{(1001)^2 1})(x) &= x^4(x^4 - x^3 - x^2 - x + 1) - (x^3 + x^2 + x - 1) \\ &= x^5(x^3 - x^2 - x - 1) + x(x^3 - x^2 - x - 1) + 1. \end{aligned}$$

Therefore, if $x \geq \overline{x}$, $p(\underline{(1001)^2 1})(x) \geq p(\underline{(1001)^2 1})(\overline{x}) = 1$, i.e., x_2 is the largest real root of $p(\underline{(1001)^2 1})(x)$.

For $k = 3$,

$$\begin{aligned} p(\underline{(1001)^3 1})(x) &= x^4 p(\underline{(1001)^2 1})(x) - (x^3 + x^2 + x - 1) \\ \Longrightarrow p(\underline{(1001)^3 1})(\overline{x}) &= \overline{x}^4 - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1) \\ &= -\overline{x}\underbrace{(-\overline{x}^3 + \overline{x}^2 + \overline{x} + 1)}_{0} + 1 = 1. \end{aligned}$$

Therefore, $p(\underline{(1001)^3 1})(\overline{x}) = 1 > 0$.

Also,

$$p(\underline{(1001)^3 1})(x_2) = -(x_2^3 + x_2^2 + x_2 - 1) < 0.$$

Then, $\exists\ x_3 \in]x_2, \overline{x}[$ such that $p(\underline{(1001)^3 1})(x_3) = 0$. We have $1 < x_1 < x_2 < x_3 < \overline{x}$.

Now,

$$p(\underline{(1001)^3 1})(x) = x^4(x(x^4 + 1)(x^3 + x^2 + x - 1) + 1) - (x^3 + x^2 + x - 1).$$

Therefore, if $x \geq \overline{x}$, $p(\underline{(1001)^3 1})(x) \geq p(\underline{(1001)^3 1})(\overline{x}) = 1$, i.e., x_3 is the largest real root of $p(\underline{(1001)^3 1})(x)$.

Let us now assume that $\exists\, x_k \in]x_{k-1}, \overline{x}[$ such that $p(\underline{(1001)^k 1})(x_k) = 0$. Also, that we have $1 < x_1 < x_2 < x_3 < \cdots < x_k < \overline{x}$.

For $k + 1$ we have

$$\begin{aligned} p(\underline{(1001)^{k+1} 1})(x) &= x^4 p(\underline{(1001)^k 1})(x) - (x^3 + x^2 + x - 1) \\ \Longrightarrow p(\underline{(1001)^{k+1} 1})(\overline{x}) &= \overline{x}^4 - (\overline{x}^3 + \overline{x}^2 + \overline{x} - 1) \\ &= -\overline{x}\underbrace{(-\overline{x}^3 + \overline{x}^2 + \overline{x} + 1)}_{0} + 1 = 1. \end{aligned}$$

Therefore $p(\underline{(1001)^{k+1} 1})(\overline{x}) = 1 > 0$.

Also,

$$p(\underline{(1001)^{k+1} 1})(x_k) = -(x_k^3 + x_k^2 + x_k - 1) < 0.$$

Therefore $\exists\ x_{k+1} \in]x_k, \overline{x}[$ such that $p(\underline{(1001)^{k+1} 1})(x_{k+1}) = 0$. We have $1 < x_1 < x_2 < x_3 < \cdots < x_k < x_{k+1} < \overline{x}$.

In the same way, if $x \geq \overline{x}$, $p(\underline{(1001)^{k+1} 1})(x) \geq p(\underline{(1001)^{k+1} 1})(\overline{x}) = 1$, i.e., x_{k+1} is the largest real root of $p(\underline{(1001)^{k+1} 1})(x)$.

With these arguments, we have shown that there is a sequence of maximal real roots, monotone increasing and bounded above by $\overline{x}$.

Affirmation 1. $\lim_{k\to\infty} x_k = \overline{x}$.

Proof. We have:

$$\begin{aligned}
&p(\underline{(1001)^{k+1}1})(x)\\
&\quad = x^4 p(\underline{(1001)^k 1})(x) - (x^3 + x^2 + x - 1)\\
&\quad = x^4[x^4 p(\underline{(1001)^{k-1}1})(x) - (x^3 + x^2 + x - 1)] - (x^3 + x^2 + x - 1)\\
&\quad = x^8 p(\underline{(1001)^{k-1}1})(x) - (x^3 + x^2 + x - 1)(1 + x^4)\\
&\quad = x^8[x^4 p(\underline{(1001)^{k-2}1})(x) - (x^3 + x^2 + x - 1)] - (x^3 + x^2 + x - 1)(1 + x^4)\\
&\quad = x^{12} p(\underline{(1001)^{k-2}1})(x) - (x^3 + x^2 + x - 1)(1 + x^4 + x^8)\\
&\quad \vdots\\
&\quad = x^{4k} p(\underline{(1001)1})(x) - (x^3 + x^2 + x - 1)\sum_{i=0}^{k-1} x^{4i}\\
&\quad = x^{4k}(-xp(\underline{1001})(x) + 1) - (x^3 + x^2 + x - 1)\sum_{i=0}^{k-1} x^{4i}\\
&\quad = -x^{4k+1} p(\underline{1001})(x) + x^{4k} - \sum_{i=0}^{k-1} x^{4i+3} - \sum_{i=0}^{k-1} x^{4i+2} - \sum_{i=0}^{k-1} x^{4i+1} + \sum_{i=0}^{k-1} x^{4i}.
\end{aligned}$$

Defining $Z_k(x) = x^{4k} - \sum_{i=0}^{k-1} x^{4i+3} - \sum_{i=0}^{k-1} x^{4i+2} - \sum_{i=0}^{k-1} x^{4i+1} + \sum_{i=0}^{k-1} x^{4i}$ we have

$$\begin{aligned}
Z_1(x) &= x^4 + x^3 + x^2 + x - 1 = p(\underline{(1001)1})(x)\\
Z_2(x) &= x^8 - x^3 - x^7 - x^2 - x^6 - x - x^5 + 1 + x^4\\
&= x^8 - x^7 - x^6 - x^5 + x^4 + x^3 + x^2 + x - 1 = p(\underline{(1001)^2 1})(x)\\
Z_3(x) &= x^{12} - x^3 - x^7 - x^{11} - x^2 - x^6 - x^{10} - x - x^5 - x^9 + 1 + x^4 + x^8\\
&= x^{12} - x^{11} - x^{10} - x^9 + x^8 - x^7 - x^6 - x^5 + x^4 + x^3 + x^2 + x - 1\\
&= p(\underline{(1001)^3 1})(x).
\end{aligned}$$

Inductively, we may conclude that $Z_k(x) = p(\underline{(1001)^k 1})(x)$. So,

$$\begin{aligned}
&p(\underline{(1001)^{k+1}1})(x)\\
&\quad = -x^{4k+1} p(\underline{1001})(x) + p(\underline{(1001)^k 1})(x)\\
&\quad = -x^{4k+1} p(\underline{1001})(x) - x^{4(k-1)+1} p(\underline{1001})(x) + p(\underline{(1001)^{k-1}1})(x)\\
&\quad = -p(\underline{1001})(x)(x^{4k+1} + x^{4(k-1)+1}) + p(\underline{(1001)^{k-1}1})(x)\\
&\quad \vdots\\
&\quad = -p(\underline{1001})(x)(x^{4k+1} + x^{4(k-1)+1} \ldots + x^{4j+1}) + p(\underline{(1001)^{k-j}1})(x)
\end{aligned}$$

$$= -xp(\underline{1001})(x)\sum_{i=0}^{j} x^{4i} + p(\underline{(1001)^{k-j}1})(x)$$
$$= -xp(\underline{1001})(x)\sum_{i=0}^{k-1} x^{4i} + p(\underline{(1001)1})(x)$$
$$= -xp(\underline{1001})(x)\sum_{i=0}^{k-1} x^{4i} + -xp(\underline{1001})(x) + 1$$
$$= -xp(\underline{1001})(x)\left(\sum_{i=0}^{k-1} x^{4i} + 1\right) + 1.$$

As for $x = x_{k+1}$ we have $p(\underline{(1001)^{k+1}1})(x_{k+1}) = 0$. Then

$$x_{k+1}p(\underline{1001})(x_{k+1})\left(\sum_{i=0}^{k-1} x^{4i} + 1\right) = 1$$
$$\Longrightarrow x_{k+1}p(\underline{1001})(x_{k+1}) = \frac{1}{\left(\sum_{i=0}^{k-1} x^{4i} + 1\right)}$$
$$\Longrightarrow x_{k+1}p(\underline{1001})(x_{k+1}) = \frac{1}{\frac{x_{k+1}^{4k}-1}{x_{k+1}^{4}-1} + 1}$$
$$\Longleftrightarrow x_{k+1}p(\underline{1001})(x_{k+1}) = \frac{x_{k+1}^{4}-1}{x_{k+1}^{4k} + x_{k+1}^{4} - 2}$$
$$\Longrightarrow p(\underline{1001})(x_{k+1}) = \frac{1}{x_{k+1}}\frac{x_{k+1}^{4}-1}{x_{k+1}^{4k} + x_{k+1}^{4} - 2}.$$

As $x_{k+1} > 1$, letting $k \to \infty$ we have $(x_{k+1}^{4k} + x_{k+1}^{4} - 2) \to \infty$, then $p(\underline{1001})(x_{k+1}) \to 0$. However,

$$\lim_{k\to\infty} p(\underline{1001})(x_{k+1}) = 0 \iff \lim_{k\to\infty} x_{k+1} = \overline{x}.$$

Then, we have shown that this sequence of maximal real roots converges to $\overline{x}$. □

Finally we are in a position to show the continuity of the topological entropy at $\theta = \underline{1001}$.

4. Continuity of the topological entropy at $\theta = \underline{1001}$

Let us see that

$$h_{\text{top}}(\underline{(1001)^n 1}) = \ln x_n.$$

Applying limits:

$$\lim_{n\to\infty} h_{\text{top}}(\underline{(1001)^n 1}) = \lim_{n\to\infty} \ln x_n = \ln(\overline{x}) = h_{\text{top}}(\underline{1001}) = h(\lim_{n\to\infty} \underline{(1001)^n 1}).$$

Proposition 4.1. *The map $h_{\text{top}} = h_{\text{top}}(\sigma|_{\Sigma[\sigma(\theta),\theta]})$ is continuous at $\theta = \underline{1001}$.*

Proof. We have $h_{\text{top}}(\underline{1001}) = h_{\text{top}}(\underline{1000})$, so for each θ such that $\underline{1001} < \theta < \underline{1000}$ we have

$$h_{\text{top}}(\underline{1001}) \leq h_{\text{top}}(\theta) \leq h_{\text{top}}(\underline{1000}).$$

Hence, $h_{\text{top}}(\theta) = h_{\text{top}}(\underline{1001})$. Thus, for any sequence $(\zeta_n)_{n\in\mathbb{N}} \subset \Sigma_1$ such that $\zeta_{n+1} < \zeta_n$ and $\lim_{n\to\infty} \zeta_n = \underline{1001}$ we have $h_{\text{top}}(\zeta_n) = h_{\text{top}}(\underline{1001})$ (as long as $\underline{1001} < \zeta_n < \underline{1000}$).

On the other hand, if $(\zeta_n)_{n\in\mathbb{N}} \subset \Sigma_1$ is a sequence such that $\zeta_n < \zeta_{n+1}$ and $\lim_{n\to\infty} \zeta_n = \underline{1001}$, then for each n big enough, there exists $k(n)$ such that

$$\underline{(1001)^{k(n)}1} \leq \zeta_n \leq \underline{(1001)^{k(n)+1}1}$$

with $\lim_{n\to\infty} k(n) = \infty$. Then

$$x_{k(n)} = h_{\text{top}}(\underline{(1001)^{k(n)}1}) \leq h_{\text{top}}(\zeta_n) \leq h_{\text{top}}(\underline{(1001)^{k(n)+1}1}) = x_{k(n)+1}.$$

As

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \underline{(1001)^{k(n)}1} = \underline{1001},$$

we conclude that $\lim_{n\to\infty} h_{\text{top}}(\zeta_n) = h_{\text{top}}(\underline{1001})$.

So, we have proved that $h_{\text{top}}(\theta)$ is continuous at $\theta = \underline{1001}$, as we announced in our main result. □

References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy.* Trans. Amer. Math. Soc. 114 (1965), 309–319. MR 0175106.

[2] L. Alsedà, J. Llibre, M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one.* Second edition. Advanced Series in Nonlinear Dynamics, 5. World Scientific, River Edge, NJ, 2000. MR 1807264.

[3] R. Bamón, R. Labarca, R. Mañé, M. J. Pacífico, *The explosion of singular cycles.* Inst. Hautes Études Sci. Publ. Math. No. 78 (1993), 207–232 (1994). MR 1259432.

[4] S. Aranzubía, R. Labarca, *Combinatorial dynamics and an elementary proof of the continuity of the topological entropy at $\theta = \underline{101}$, in the Milnor Thurston world.* Progress and challenges in dynamical systems, 25–52. Springer Proc. Math. Stat., 54, Springer, Heidelberg, 2013. MR 3111771.

[5] L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young, *Periodic points and topological entropy of one-dimensional maps.* Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979), pp. 18–34, Lecture Notes in Math., 819, Springer, Berlin, 1980.

[6] J. Guckenheimer, R. F. Williams, *Structural stability of Lorenz attractors.* Inst. Hautes Études Sci. Publ. Math. No. 50 (1979), 59–72. MR 0556582.

[7] R. Labarca, *A note on the topological classification of Lorenz maps on the interval.* Topics in symbolic dynamics and applications (Temuco, 1997), 229–245, London Math. Soc. Lecture Note Ser., 279, Cambridge Univ. Press, Cambridge, 2000. MR 1776761.

[8] R. Labarca, *La entropía topológica, propiedades generales y algunos cálculos en el caso del shift de Milnor–Thurston.* Con la colaboración de S. Aranzubia y E. Inda. Serie de las Escuelas Venezolanas de Matemáticas, Instituto Venezolano de Investigaciones Científicas (IVIC), 2011.

[9] R. Labarca, C. Moreira, *Essential dynamics for Lorenz maps on the real line and the lexicographical world*. Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), 683–694. MR 2259612.

[10] R. Labarca, C. Moreira, *Bifurcation of the essential dynamics of Lorenz maps on the real line and the bifurcation scenario for Lorenz like flows: the contracting case*. Proyecciones 29 (2010), 247–293. MR 2749568.

[11] R. Labarca, C. Moreira, A. Pumariño, J. A. Rodríguez, *On bifurcation set for symbolic dynamics in the Milnor–Thurston world*. Commun. Contemp. Math. 14 (2012) 1250024, 16 pp. MR 2965669.

[12] R. Labarca, A. Pumariño, J. A. Rodríguez, *On the boundary of topological chaos for the Milnor–Thurston world*. Commun. Contemp. Math. 11 (2009), 1049–1066. MR 2589575.

[13] D. Lind, B. Marcus, *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995. MR 1369092.

[14] J. Milnor, W. Thurston, *On iterated maps of the interval*. Dynamical systems (College Park, MD, 1986–87), 465–563, Lectures Notes in Math. 1342, Springer, Berlin, 1988. MR 0970571.

[15] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*. Studia Math. 67 (1980), 45–63. MR 0579440.

[16] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817. MR 0228014.

[17] M. Urbański *On Hausdorff dimension of invariant sets for expanding maps of a circle*. Ergodic Theory Dynam. Systems 6 (1986), 295–309. MR 0857203.

A. Jablonski
Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Santiago, Chile
andres.jablonski@gmail.com

R. Labarca ✉
Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Santiago, Chile
rafael.labarca@usach.cl

Received: May 9, 2015
Accepted: October 12, 2016