

## GEOMETRIC INEQUALITIES FOR EINSTEIN TOTALLY REAL SUBMANIFOLDS IN A COMPLEX SPACE FORM

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ABSTRACT. Two geometric inequalities are established for Einstein totally real submanifolds in a complex space form. As immediate applications of these inequalities, some non-existence results are obtained.

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### 1. INTRODUCTION

According to Chen's cornerstone work [1], the following problem is fundamental: *to establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of Riemannian submanifolds*. The basic relationships discovered until now are inequalities and the study of this topic has attracted a lot of attention during the last two decades. Roughly speaking, there are three main aspects of the study of this topic, one looking at the new Riemannian invariants introduced by Chen [2, 3, 4, 6, 10, 11, 17, 18, 20, 21, 23], the other looking at the DDVV inequalities [7, 9, 14, 15, 16], and the last looking at the Casorati curvatures [8, 12, 13, 19, 22]. In this paper, we are interested in obtaining characterizations of the relationships by Chen's invariants.

Let  $M$  be a Riemannian  $n$ -manifold and  $p$  a point in  $M$ . Suppose that  $K(\pi)$  is the sectional curvature of  $M$  with respect to a plane section  $\pi \subset T_p M$ . For each unit tangent vector  $X$  of  $M$  at  $p$ , the Ricci curvature  $\text{Ric}(X)$  is defined by

$$\text{Ric}(X) = \sum_{j=2}^n K(X \wedge e_j),$$

where  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal basis of  $T_p M$  with  $e_1 = X$ .

In general, an  $n$ -dimensional manifold  $M$  whose Ricci tensor has an eigenvalue of multiplicity at least  $n - 1$  is called quasi-Einstein. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Further, we say that  $M$  is an Einstein manifold if  $\text{Ric}(X)$  is independent of the choice of the unit vector  $X$ .

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Then for any unit tangent vector  $X$  of  $M$  at  $p$ , one has

$$\text{Ric}(X) = \frac{2}{n} \tau(p),$$

where  $\tau(p)$  is the scalar curvature at  $p$  defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

For a given point  $p$  in  $M$ , let  $\pi_1, \dots, \pi_q$  be  $q$  mutually orthogonal plane sections in  $T_pM$ , where  $q$  is a positive integer  $\leq \frac{n}{2}$ . Following [2], we define

$$K_q^{\text{inf}}(p) = \inf_{\pi_1 \perp \dots \perp \pi_q} \frac{K(\pi_1) + \dots + K(\pi_q)}{q},$$

where  $\pi_1, \dots, \pi_q$  run over all mutually orthogonal  $q$  plane sections in  $T_pM$ . For each positive integer  $q \leq \frac{n}{2}$ , define the invariant  $\delta_q^{\text{Ric}}$  on  $M$  by

$$\delta_q^{\text{Ric}} = \sup_{X \in T_p^1 M} \text{Ric}(X) - \frac{2q}{n} K_q^{\text{inf}}(p),$$

where  $X$  runs over all unit vectors in  $T_p^1 M := \{X \in T_p M : \|X\| = 1\}$ .

In [2], Chen established two inequalities in terms of the Riemannian invariant  $\delta_q^{\text{Ric}}$  for Einstein submanifolds in a real space form. As a natural prolongation, in this paper, we obtain two inequalities for Einstein totally real submanifolds in a complex space form. Unlike [2], we do not need the algebraic lemma from [3]. Our algebraic techniques also provide new approaches to establish inequalities obtained in [2].

## 2. PRELIMINARIES

Let  $N^m$  be a complex  $m$ -dimensional Kähler manifold, i.e.  $N^m$  is endowed with an almost complex structure  $J$  and with a  $J$ -Hermitian metric  $\tilde{g}$ . By a complex space form  $N^m(4c)$  we mean an  $m$ -dimensional Kähler manifold with constant holomorphic sectional curvature  $4c$ . A complete simply connected complex space form  $N^m(4c)$  is holomorphically isometric to the complex Euclidean  $m$ -plane  $\mathbb{C}^m$ , the complex projective  $m$ -space  $\mathbb{C}P^m(4c)$ , or a complex hyperbolic  $m$ -space  $\mathbb{C}H^m(4c)$  according to  $c = 0, c > 0$  or  $c < 0$ , respectively. Denote by  $\tilde{\nabla}$  its Levi-Civita connection. The Riemannian curvature tensor field  $\tilde{R}$  with respect to  $\tilde{\nabla}$  has the expression

$$\begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= c(\langle \tilde{X}, \tilde{Z} \rangle \langle \tilde{Y}, \tilde{W} \rangle - \langle \tilde{X}, \tilde{W} \rangle \langle \tilde{Y}, \tilde{Z} \rangle + \langle J\tilde{X}, \tilde{Z} \rangle \langle J\tilde{Y}, \tilde{W} \rangle \\ &\quad - \langle J\tilde{X}, \tilde{W} \rangle \langle J\tilde{Y}, \tilde{Z} \rangle + 2\langle \tilde{X}, J\tilde{Y} \rangle \langle \tilde{Z}, J\tilde{W} \rangle), \end{aligned}$$

for any vector fields  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$  on  $N^m(4c)$ .

Let  $M$  be a totally real submanifold in  $N^m(4c)$ . According to the behavior of the tangent spaces under the action of  $J$ , a submanifold  $M$  in  $N^m(4c)$  is called totally real if the complex structure  $J$  of  $N^m(4c)$  carries each tangent space  $T_pM$  of  $M$  into its corresponding normal space  $T_p^\perp M$  [5]. We denote the Levi-Civita connection of  $M$  by  $\nabla$  and by  $R$  the curvature tensor on  $M$  with respect to  $\nabla$ .

The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

for tangent vector fields  $X$  and  $Y$  and normal vector field  $\xi$ , where  $\nabla^\perp$  is the normal connection and  $A$  is the shape operator. The second fundamental form  $h$  is related to  $A_\xi$  by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector  $\vec{H}$  of  $M$  is defined by

$$\vec{H} = \frac{1}{n} \text{trace } h,$$

and we set  $H = \|\vec{H}\|$  for convenience.

A submanifold  $M$  is called pseudo-umbilical if  $\vec{H}$  is nonzero and the shape operator  $A_{\vec{H}}$  at  $\vec{H}$  is proportional to the identity map. If  $\vec{H} = 0$ , we say  $M$  is minimal. Besides,  $M$  is called totally geodesic if  $h = 0$ .

For totally real submanifolds, we have [5]

$$\nabla_X^\perp JY = J\nabla_X Y, \quad A_{JX} Y = -Jh(X, Y) = A_{JY} X.$$

The above formulas immediately imply that  $\langle h(X, Y), JZ \rangle$  is totally symmetric. Moreover, the Gauss equation is given by [5]

$$\begin{aligned} R(X, Y, Z, W) = & c(\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle) \\ & + \langle h(X, Z), h(Y, W) \rangle - \langle h(X, W), h(Y, Z) \rangle \end{aligned}$$

for all vector fields  $X, Y, Z, W$  on  $M$ .

Choosing a local frame

$$e_1, \dots, e_n, e_{n+1}, \dots, e_m,$$

$$e_{m+1} = J(e_1), \dots, e_{m+n} = J(e_n), e_{m+n+1} = J(e_{n+1}), \dots, e_{2m} = J(e_m)$$

in  $N^m(4c)$  in such a way that, restricted to  $M$ ,  $e_1, e_2, \dots, e_n$  are tangent to  $M$ . With respect to the local frame of  $N^m(4c)$  chosen above, we denote the coefficients of the second fundamental form  $h$  by  $\{h_{ij}^r\}$ ,  $1 \leq i < j \leq n$ ;  $n + 1 \leq r \leq 2m$ .

### 3. THE FIRST INEQUALITY

**Theorem 3.1.** *For any integer  $k \geq 2$ , let  $M$  be a  $2k$ -dimensional Einstein totally real submanifold of an  $m$ -dimensional complex space form  $N^m(4c)$  of constant holomorphic sectional curvature  $4c$ . Then we have*

$$\delta_k^{\text{Ric}} \leq 2(k - 1)(c + H^2). \tag{3.1}$$

The equality case of (3.1) holds if and only if one of the following two cases occurs:

- (i)  $M$  is a minimal and Einstein totally real submanifold, such that, with respect to suitable orthonormal frames  $\{e_1, \dots, e_{2k}, e_{2k+1}, \dots, e_{2m}\}$ , the shape operators of

$M$  take the following form:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k^r \end{pmatrix}, \quad r = 2k + 1, \dots, 2m,$$

where  $A_i^r, i = 1, \dots, k$ , are symmetric  $2 \times 2$  submatrices satisfying  $\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = 0$ .

(ii)  $M$  is a pseudo-umbilical and Einstein totally real submanifold, such that, with respect to suitable orthonormal frames  $\{e_1, \dots, e_{2k}, e_{2k+1}, \dots, e_{2m}\}$ , the shape operators of  $M$  take the following form:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k^r \end{pmatrix}, \quad r = 2k + 2, \dots, 2m,$$

where  $A_i^r, i = 1, \dots, k$ , are symmetric  $2 \times 2$  submatrices satisfying  $\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = 0$ .

*Proof.* For a given point  $p$  in  $M$ , let  $\pi_1, \dots, \pi_k$  be  $k$  mutually orthogonal plane sections at  $p$ . We choose an orthonormal basis  $\{e_1, \dots, e_{2k}\}$  of  $T_pM$  such that

$$\pi_1 = \text{Span}\{e_1, e_2\}, \dots, \pi_k = \text{Span}\{e_{2k-1}, e_{2k}\}.$$

Since  $M$  is a  $2k$ -dimensional Einstein manifold, we have  $\tau = k \text{ Ric}(X)$ . From the definition of  $\delta_k^{\text{Ric}}$  and the Gauss equation, we have

$$\begin{aligned} k\delta_k^{\text{Ric}} &= \tau - [K(\pi_1) + K(\pi_2) + \dots + K(\pi_k)] \\ &= k(2k - 1)c + \sum_r \sum_{1 \leq i < j \leq 2k} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] - \left\{ c + \sum_r [h_{11}^r h_{22}^r - (h_{12}^r)^2] \right. \\ &\quad \left. + \dots + c + \sum_r [h_{2k-1, 2k-1}^r h_{2k, 2k}^r - (h_{2k-1, 2k}^r)^2] \right\} \\ &\leq 2k(k - 1)c + \sum_r \left[ \sum_{1 \leq i < j \leq 2k} h_{ii}^r h_{jj}^r - (h_{11}^r h_{22}^r + \dots + h_{2k-1, 2k-1}^r h_{2k, 2k}^r) \right] \\ &= 2k(k - 1)c + \frac{1}{2} \sum_r \left\{ \left( \sum_{i=1}^{2k} h_{ii}^r \right)^2 - [(h_{11}^r + h_{22}^r)^2 + \dots \right. \\ &\quad \left. + (h_{2k-1, 2k-1}^r + h_{2k, 2k}^r)^2] \right\} \end{aligned} \tag{3.2}$$

Using the Cauchy inequality, we obtain that

$$(h_{11}^r + h_{22}^r)^2 + \dots + (h_{2k-1, 2k-1}^r + h_{2k, 2k}^r)^2 \geq \frac{1}{k} \left( \sum_{i=1}^{2k} h_{ii}^r \right)^2, \tag{3.3}$$

with the equality case of (3.3) holds if and only if

$$h_{11}^r + h_{22}^r = \dots = h_{2k-1, 2k-1}^r + h_{2k, 2k}^r.$$

Plunging (3.3) into (3.2), we have

$$\begin{aligned} k\delta_k^{\text{Ric}} &\leq 2k(k-1)c + \frac{1}{2} \sum_r \left\{ \left( \sum_{i=1}^{2k} h_{ii}^r \right)^2 - \frac{1}{k} \left( \sum_{i=1}^{2k} h_{ii}^r \right)^2 \right\} \\ &= 2k(k-1)c + \frac{k-1}{2k} \left( \sum_{i=1}^{2k} h_{ii}^r \right)^2 \\ &= 2k(k-1)c + \frac{k-1}{2k} 4k^2 H^2 \\ &= 2k(k-1)(c + H^2), \end{aligned}$$

which implies

$$\delta_k^{\text{Ric}} \leq 2(k-1)(c + H^2).$$

Next, we will discuss the equality case. The equality case of (3.1) at a point  $p \in M$  holds if and only if we have the equality in (3.2) and (3.3), i.e. with respect to suitable orthonormal frames, the shape operators take the following form:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k^r \end{pmatrix}, \quad r = 2k + 1, \dots, 2m,$$

where  $A_i^r, i = 1, \dots, k$ , are symmetric  $2 \times 2$  submatrices satisfying

$$\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r).$$

The rest of the discussion is similar to that of the proof of Theorem 1 in [2]. □

#### 4. THE SECOND INEQUALITY

**Theorem 4.1.** *Let  $M$  be an  $n$ -dimensional Einstein totally real submanifold of an  $m$ -dimensional complex space form  $N^m(4c)$ . Then for every positive integer  $q < \frac{n}{2}$ , we have*

$$\delta_q^{\text{Ric}} \leq \left( n - 1 - \frac{2q}{n} \right) c + \frac{n(n - q - 1)}{n - q} H^2. \tag{4.1}$$

The equality case of (4.1) holds if and only if  $M$  is a totally geodesic submanifold.

*Proof.* Given a point  $p$  in  $M$  and a positive integer  $q < \frac{n}{2}$ , let  $\pi_1, \dots, \pi_q$  be  $q$  mutually orthogonal plane sections of  $M$  at  $p$ . We choose an orthonormal basis of  $T_p M$  such that

$$\pi_1 = \text{Span}\{e_1, e_2\}, \dots, \pi_q = \text{Span}\{e_{2q-1}, e_{2q}\}.$$

Then from the definition of  $\delta_q^{\text{Ric}}$  we have

$$\begin{aligned} n\delta_q^{\text{Ric}}(p) &= n \text{Ric}(X) - 2[K(\pi_1) + \dots + K(\pi_q)] \\ &= 2q \text{Ric}(X) - 2[K(\pi_1) + \dots + K(\pi_q)] + (n - 2q) \text{Ric}(X). \end{aligned} \tag{4.2}$$

For convenience, we set

$$I = 2q \text{Ric}(X) - 2[K(\pi_1) + \dots + K(\pi_q)], \quad II = (n - 2q) \text{Ric}(X).$$

Now we compute I and II separately. First, we rewrite I as

$$I = \sum_{l=1}^q [\text{Ric}(e_{2l-1}, e_{2l-1}) + \text{Ric}(e_{2l}, e_{2l}) - 2K(\pi_l)],$$

which together with the Gauss equation gives

$$\begin{aligned} I &\leq 2q(n-2)c + \sum_r [(\sum_{j \neq 1} h_{11}^r h_{jj}^r + \sum_{j \neq 2} h_{22}^r h_{jj}^r + \cdots + \sum_{j \neq 2q} h_{2q,2q}^r h_{jj}^r) \\ &\quad - 2(h_{11}^r h_{22}^r + h_{33}^r h_{44}^r + \cdots + h_{2q-1,2q-1}^r h_{2q,2q}^r)] \\ &= 2q(n-2)c + \sum_r [ \sum_{1 \leq i \leq 2q, 2q+1 \leq j \leq n} h_{ii}^r h_{jj}^r + 2 \sum_{1 \leq i < j \leq 2q} h_{ii}^r h_{jj}^r \\ &\quad - 2(h_{11}^r h_{22}^r + h_{33}^r h_{44}^r + \cdots + h_{2q-1,2q-1}^r h_{2q,2q}^r)] \\ &= 2q(n-2)c + \sum_r \left\{ \sum_{1 \leq i \leq 2q, 2q+1 \leq j \leq n} h_{ii}^r h_{jj}^r + (h_{11}^r + \cdots + h_{2q,2q}^r)^2 \right. \\ &\quad \left. - [(h_{11}^r + h_{22}^r)^2 + \cdots + (h_{2q-1,2q-1}^r + h_{2q,2q}^r)^2] \right\}. \end{aligned} \tag{4.3}$$

On the other hand, we can rewrite II as

$$II = \text{Ric}(e_{2q+1}, e_{2q+1}) + \text{Ric}(e_{2q+2}, e_{2q+2}) + \cdots + \text{Ric}(e_n, e_n),$$

which together with the Gauss equation gives

$$\begin{aligned} II &= (n-2q)(n-1)c + \sum_r \sum_{j \neq 2q+1} [h_{2q+1,2q+1}^r h_{jj}^r - (h_{2q+1,j}^r)^2] \\ &\quad + \cdots + \sum_r \sum_{j \neq n} [h_{nn}^r h_{jj}^r - (h_{nj}^r)^2] \\ &\leq (n-2q)(n-1)c + \sum_r (\sum_{j \neq 2q+1} h_{2q+1,2q+1}^r h_{jj}^r + \cdots + \sum_{j \neq n} h_{nn}^r h_{jj}^r) \\ &= (n-2q)(n-1)c + \sum_r (2 \sum_{2q+1 \leq i < j \leq n} h_{ii}^r h_{jj}^r + \sum_{1 \leq i \leq 2q, 2q+1 \leq j \leq n} h_{ii}^r h_{jj}^r). \end{aligned} \tag{4.4}$$

Plunging (4.3) and (4.4) into (4.2), we obtain that

$$\begin{aligned}
 n\delta_q^{\text{Ric}}(p) &\leq (n^2 - n - 2q)c + \sum_r (h_{11}^r + \dots + h_{2q,2q}^r)^2 - \sum_r [(h_{11}^r + h_{22}^r)^2 \\
 &\quad + \dots + (h_{2q-1,2q-1}^r + h_{2q,2q}^r)^2] + 2 \sum_r \sum_{2q+1 \leq i < j \leq n} h_{ii}^r h_{jj}^r \\
 &\quad + 2 \sum_r \sum_{1 \leq i \leq 2q, 2q+1 \leq j \leq n} h_{ii}^r h_{jj}^r \\
 &= (n^2 - n - 2q)c + \sum_r (h_{11}^r + \dots + h_{2q,2q}^r)^2 - \sum_r [(h_{11}^r + h_{22}^r)^2 \\
 &\quad + \dots + (h_{2q-1,2q-1}^r + h_{2q,2q}^r)^2] + 2 \sum_r \sum_{2q+1 \leq i < j \leq n} h_{ii}^r h_{jj}^r \tag{4.5} \\
 &\quad + [n^2 H^2 - \sum_r (h_{11}^r + \dots + h_{2q,2q}^r)^2 - \sum_r (h_{2q+1,2q+1}^r + \dots + h_{nn}^r)^2] \\
 &= (n^2 - n - 2q)c + n^2 H^2 - \sum_r [(h_{11}^r + h_{22}^r)^2 + (h_{33}^r + h_{44}^r)^2 \\
 &\quad + \dots + (h_{2q-1,2q-1}^r + h_{2q,2q}^r)^2 + (h_{2q+1,2q+1}^r)^2 + \dots + (h_{nn}^r)^2].
 \end{aligned}$$

From the Cauchy inequality, we know that

$$\begin{aligned}
 (h_{11}^r + h_{22}^r)^2 + \dots + (h_{2q-1,2q-1}^r + h_{2q,2q}^r)^2 + (h_{2q+1,2q+1}^r)^2 + \dots + (h_{nn}^r)^2 \\
 \geq \frac{1}{n-q} (h_{11}^r + h_{22}^r + \dots + h_{nn}^r)^2, \tag{4.6}
 \end{aligned}$$

with the equality case of (4.6) holds if and only if

$$h_{11}^r + h_{22}^r = \dots = h_{2q-1,2q-1}^r + h_{2q,2q}^r = h_{2q+1,2q+1}^r = \dots = h_{nn}^r.$$

Then we plunge (4.6) into (4.5), namely,

$$\begin{aligned}
 n\delta_q^{\text{Ric}}(p) &\leq (n^2 - n - 2q)c + n^2 H^2 - \frac{1}{n-q} \sum_r (h_{11}^r + h_{22}^r + \dots + h_{nn}^r)^2 \\
 &= (n^2 - n - 2q)c + \frac{n^2(n-q-1)}{n-q} H^2,
 \end{aligned}$$

which means

$$\delta_q^{\text{Ric}} \leq (n-1 - \frac{2q}{n})c + \frac{n(n-q-1)}{n-q} H^2.$$

Next, we will discuss the equality case. The equality case of (4.1) at a point  $p \in M$  holds if and only if we have the equality in (4.3), (4.4) and (4.6), i.e. with respect to suitable orthonormal frames, the shape operators take the following form:

$$A_r = \begin{pmatrix} A_1^r & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & A_k^r & 0 \\ 0 & \dots & 0 & \mu_r E \end{pmatrix}, \quad r = n+1, \dots, 2m,$$

where  $E$  is the  $(n-2q) \times (n-2q)$  identity matrix and  $A_i^r, i = 1, \dots, k$ , are symmetric  $2 \times 2$  submatrices satisfying

$$\text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r.$$

The rest of the discussion is similar to that of the proof of Theorem 2 in [2].  $\square$

## 5. IMMEDIATE APPLICATIONS

From Theorems 3.1 and 4.1 we obtain immediately the following.

**Corollary 5.1.** *If a Riemannian  $n$ -manifold  $M$  admits a totally real isometric immersion into a complex Euclidean space which satisfies*

$$\delta_q^{\text{Ric}} > \frac{n(n-q-1)}{n-q} H^2,$$

for some positive integer  $q \leq \frac{n}{2}$  at some point, then  $M$  is not an Einstein manifold.

Theorems 3.1 and 4.1 also imply the following.

**Corollary 5.2.** *If an Einstein  $n$ -manifold satisfies*

$$\delta_q^{\text{Ric}} > (n-1 - \frac{2q}{n})c,$$

for some positive integer  $q \leq \frac{n}{2}$  at some point, then it admits no totally real minimal isometric immersion into a complex space form of constant holomorphic sectional curvature  $4c$  regardless of codimension.

Besides, from Theorems 3.1 and 4.1, we can also get Corollary 3 in [2].

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