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BÉZIER VARIANT OF MODIFIED SRIVASTAVA–GUPTA OPERATORS

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Abstract. Srivastava and Gupta proposed in 2003 a general family of linear positive operators which include several well known operators as its special cases and investigated the rate of convergence of these operators for functions of bounded variation by using the decomposition techniques. Subsequently, researchers proposed several modifications of these operators and studied their various approximation properties. Yadav, in 2014, proposed a modification of these operators and studied a Voronovskaya-type approximation theorem and statistical convergence. In this paper, we introduce the Bézier variant of the operators defined by Yadav and give a direct approximation theorem by means of the Ditzian–Totik modulus of smoothness and the rate of convergence for absolutely continuous functions having a derivative equivalent to a function of bounded variation. Furthermore, we show the comparisons of the rate of convergence of the Srivastava–Gupta operators vis-à-vis its Bézier variant to a certain function by illustrative graphics using Maple algorithms.

1. Introduction

In order to approximate Lebesgue integrable functions on $[0,\infty)$, Gupta and Srivastava [14] introduced a general family of summation-integral type operators as

$$L_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c)f(t)\,dt + p_{n,0}(x,c)f(0), \tag{1.1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x)$$

and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c=0, \\ (1+cx)^{-n/c}, & c\in\mathbb{N} := \{1,2,3,\dots\}. \end{cases}$$

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Ispir and Yuksel [11] introduced the Bézier variant of the operators (1.1) and studied the estimate of the rate of convergence of these operators for functions of bounded variation. Verma and Agrawal [15] introduced the generalized form of the operators (1.1) and studied some of its approximation properties. Deo [6] gave a modification of these operators and established the rate of convergence and Voronovskaya-type result. Recently, Acar et al. [1] introduced a Stancu-type generalization of the operators (1.1) and obtained an estimate of the rate of convergence for functions having derivatives of bounded variation and also studied the simultaneous approximation for these operators.

Yadav [16] introduced a modification of the operators (1.1) as

$$\begin{aligned} G_{n,c}(f,x) = n \sum_{k=1}^{\infty} p_{n,k}(x,c) \int_0^{\infty} p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t}{n}\right) dt \\ + p_{n,0}(x,c) f(0). \end{aligned} \tag{1.2}$$

and studied its moment estimates, direct estimate, asymptotic formula and statistical convergence. Very recently, Maheshwari [13] studied the rate of approximation for the functions having derivative of bounded variation on every finite subinterval of $[0,\infty)$ for the operators (1.2).

It is well known that Bézier curves are the parametric curves used in computer graphics and designs. In vector graphics they are used to model smooth curves and also used in animation designs. Zeng and Piriou [18] pioneered the study of Bézier variants of Bernstein operators. The papers by other researchers (e.g., [3, 5, 8, 13, 17]) motivate us to study further in this direction.

The present paper is an attempt to continue the study of Bézier variants of different sequences of operators. We propose a Bézier variant of the operators given by (1.2) as

$$\begin{aligned} G^{\alpha}_{n,c}(f,x) = n \sum_{k=1}^{\infty} Q^{(\alpha)}_{n,k}(x,c) \int_0^{\infty} p_{n+c,k-1}(t,c) f\left(\frac{(n-c)t}{n}\right) dt \\ + Q^{(\alpha)}_{n,0}(x,c) f(0), \end{aligned} \tag{1.3}$$

where $Q^{(\alpha)}_{n,k}(x,c) = [J_{n,k}(x,c)]^{\alpha} - [J_{n,k+1}(x,c)]^{\alpha}$, $\alpha \geq 1$, with $J_{n,k}(x,c) = \sum_{j=k}^{\infty} p_{n,j}(x,c)$ when $k < \infty$ and 0 otherwise. Clearly, $G^{\alpha}_{n,c}(f,x)$ is a linear positive operator. If $\alpha = 1$, then the operators $G^{\alpha}_{n,c}(f,x)$ reduce to the operators $G_{n,c}(f,x)$.

In the last decade, the study of the rate of convergence for functions with a derivative of bounded variation has become an active area of research in approximation theory. Recently, Ispir et al. [10] considered the Kantorovich modification of Lupas operators based on Polya distribution and studied the rate of approximation of the functions having derivative of bounded variation. Very recently, Maheswari [13] considered the operators (1.2) and studied the rate of approximation of the function having derivative of bounded variation on every finite subinterval of $[0,\infty)$. The order of approximation of the summation-integral type operators for functions with derivatives of bounded variation is estimated in [2, 4, 9, 12].

The aim of this paper is to investigate a direct approximation result and the rate of convergence for functions having a derivative equivalent with a function of bounded variation on every finite subinterval of $[0, \infty)$ for the operators (1.2). Lastly, a comparison of the rate of convergence of the operators (1.2) vis-à-vis operators (1.3) to a certain function is illustrated by some graphics.

2. Auxiliary results

Lemma 2.1 ([16])**.** *For $G_{n,c}(t^m, x)$, $m = 0, 1, 2$, one has*

(1) $G_{n,c}(1, x) = 1$
(2) $G_{n,c}(t, x) = x$
(3) $G_{n,c}(t^2, x) = \frac{(n-c)(x^2(n+c)+2x)}{n(n-2c)}$.

Consequently,

$$G_{n,c}((t-x)^2, x) = \frac{x^2 c(2n-c) + 2(n-c)x}{n(n-2c)}$$

and

$$G_{n,c}((t-x)^2, x) \leq \frac{\lambda x(1+cx)}{n}$$

for sufficiently large n and $\lambda > 1$. From [13], one has

$$G_{n,c}((t-x)^{2r}, x) = O(n^{-r}). \tag{2.1}$$

Remark 2.2. We have

$$\begin{aligned} G^{\alpha}_{n,c}(1; x) &= \sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x, c) = [J_{n,0}(x, c)]^{\alpha} \\ &= \left[\sum_{j=0}^{\infty} p_{n,k}(x, c)\right]^{\alpha} = 1, \end{aligned}$$

since $\sum_{j=0}^{\infty} p_{n,k}(x, c) = 1$.

Let $C_B[0, \infty)$ denote the space of all bounded and continuous functions on $[0, \infty)$ endowed with the norm

$$\|f\| = \sup_{[0,\infty)} |f(x)|.$$

Lemma 2.3. *For every $f \in C_B[0, \infty)$, we have*

$$\|G^{\alpha}_{n,c}(f; .)\| \leq \|f\|.$$

Applying Remark 2.2, the proof of this lemma easily follows. Hence the details are omitted.

Remark 2.4. For $0 \leq a, b \leq 1$, $\alpha \geq 1$, using the inequality

$$|a^{\alpha} - b^{\alpha}| \leq \alpha |a - b|$$

and from the definition of $Q^{(\alpha)}_{n,k}(x, c)$, for all $k = 0, 1, 2, \ldots$, we have

$$0 < [J_{n,k}(x, c)]^{\alpha} - [J_{n,k+1}(x, c)]^{\alpha} \leq \alpha(J_{n,k}(x, c) - J_{n,k+1}(x, c)) = \alpha p_{n,k}(x, c).$$

Hence from the definition of $G^{\alpha}_{n,c}(f;x)$ we get

$$|G^{\alpha}_{n,c}(f,x)| \leq \alpha G_{n,c}(|f|,x).$$

3. Main results

To describe our first result, we recall the definitions of the Ditzian–Totik first order modulus of smoothness and the K-functional [7]. Let $\phi(x)=\sqrt{x(1+cx)}$, $f \in C[0,\infty)$. The first order modulus of smoothness is given by

$$\omega_{\phi}(f,t) = \sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|,\ x\pm\frac{h\phi(x)}{2}\in[0,\infty)\right\},$$

and the appropriate Petree's K-functional is defined by

$$\overline{K}_{\phi}(f,t) = \inf_{g\in W_{\phi}}\{\|f-g\|+t\|\phi g'\|+t^2\|g'\|,\ t>0\}, \tag{3.1}$$

where $W_{\phi}=\{g : g\in AC_{loc}, \|\phi g'\|<\infty, \|g'\|<\infty\}$ and $g\in AC_{loc}$ means that g is an absolutely continuous function on every $[a,b]\subset[0,\infty)$. It is well known ([7] Theorem 3.1.2) that $\overline{K}_{\phi}(f,t)\sim\omega_{\phi}(f,t)$, which means that there exists a constant $C>0$ such that

$$C^{-1}\omega_{\phi}(f,t)\leq\overline{K}_{\phi}(f,t)\leq C\omega_{\phi}(f,t). \tag{3.2}$$

Now, we establish a direct approximation theorem by means of the Ditzian–Totik modulus of smoothness.

Theorem 3.1. *Let $f\in C[0,\infty)$ and $\phi(x)=\sqrt{x(1+cx)}$, then for every $x\in[0,\infty)$, we have*

$$|G^{\alpha}_{n,c}(f;x)-f(x)|\leq C\omega_{\phi}\left(f;\frac{1}{\sqrt{n}}\right), \tag{3.3}$$

where C is a constant independent of n and x.

Proof. For fixed n and x, choosing $g=g_{n,x}\in W_{\phi}$ and using the representation

$$g(t)=g(x)+\int_x^t g'(u)\,du,$$

we get

$$|G^{\alpha}_{n,c}(g;x)-g(x)| = \left|G^{\alpha}_{n,c}\left(\int_x^t g'(u)\,du;x\right)\right|. \tag{3.4}$$

Now to find the estimate we split the domain into two parts: $F_n^c=[0,\frac{1}{n}]$ and $F_n=(\frac{1}{n},\infty)$. First, if $x\in(\frac{1}{n},\infty)$ then $G^{\alpha}_{n,c}((t-x)^2;x)\sim\frac{2\alpha}{n}\phi^2(x)$.

We have

$$\left|\int_x^t g'(u)\,du\right|\leq\|\phi g'\|\left|\int_x^t\frac{1}{\phi(u)}\,du\right|. \tag{3.5}$$

For any $x, t \in (0, \infty)$, we find that

$$\begin{aligned}
\left| \int_x^t \frac{1}{\phi(u)}\, du \right| &= \left| \int_x^t \frac{1}{\sqrt{u(1+cu)}}\, du \right| \\
&\leq \left| \int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1+cu}} \right) du \right| \\
&\leq 2\left(\sqrt{t} - \sqrt{x} + \frac{\sqrt{1+ct} - \sqrt{1+cx}}{c} \right) \\
&= 2|t-x| \left(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1+ct}+\sqrt{1+cx}} \right) \\
&< 2|t-x| \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1+cx}} \right) \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} \frac{|t-x|}{\phi(x)}.
\end{aligned} \tag{3.6}$$

Combining (3.4)-(3.6) and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
|G_{n,c}^{\alpha}(g;x) - g(x)| &< \frac{2(c+1)}{\sqrt{c(c-1)}} \|\phi g'\| \phi^{-1}(x) G_{n,c}^{\alpha}(|t-x|;x) \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} \|\phi g'\| \phi^{-1}(x) \Big(G_{n,c}^{\alpha}((t-x)^2;x) \Big)^{1/2} \\
&\leq \frac{2(c+1)}{\sqrt{c(c-1)}} \|\phi g'\| \left(\frac{2\alpha}{n} \right)^{1/2} \\
&\leq C \|\phi g'\| \frac{1}{\sqrt{n}}.
\end{aligned} \tag{3.7}$$

For $x \in F_n^c = [0, 1/n]$, $G_{n,c}^{\alpha}((t-x)^2;x) \sim \frac{2\alpha}{n^2}$ and

$$\left| \int_x^t g'(u)\, du \right| \leq \|g'\| \ |t-x|.$$

Therefore, using the Cauchy–Schwarz inequality we have

$$\begin{aligned}
|G_{n,c}^{\alpha}(g;x) - g(x)| &\leq \|g'\| G_{n,c}^{\alpha}(|t-x|;x) \\
&\leq \|g'\| \Big(G_{n,c}^{\alpha}((t-x)^2;x) \Big)^{1/2} \\
&\leq \|g'\| \frac{\sqrt{2\alpha}}{n} \leq C \|g'\| \frac{1}{n}.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$|G_{n,c}^{\alpha}(g;x) - g(x)| < C \left(\|\phi g'\| \frac{1}{\sqrt{n}} + \|g'\| \frac{1}{n} \right). \tag{3.9}$$

Using Lemma 2.3 and (3.9), we can write

$$\begin{aligned}|G^{\alpha}_{n,c}(f;x)-f(x)| &\leq |G^{\alpha}_{n,c}(f-g;x)|+|f(x)-g(x)|+|G^{\alpha}_{n,c}(g;x)-g(x)| \\ &\leq C\left(\|f-g\|+\|\phi g'\|\frac{1}{\sqrt{n}}+\|g'\|\frac{1}{n}\right).\end{aligned} \tag{3.10}$$

Taking the infimum on the right hand side of the above inequality over all $g \in W_{\phi}$, we get

$$|G^{\alpha}_{n,c}(f;x)-f(x)| = C\overline{K}_{\phi}\left(f;\frac{1}{\sqrt{n}}\right).$$

Using $\overline{K_{\phi}}(f,t) \sim \omega_{\phi}(f,t)$ and (3.2), we get the desired relation (3.3). This completes the proof of the theorem. □

Lastly, we shall discuss the rate of approximation of functions with a derivative of bounded variation on $[0,\infty)$. Let $DBV_{\gamma}[0,\infty)$, $\gamma \geq 0$, denote the class of all absolutely continuous functions f defined on $[0,\infty)$, having a derivative f' equivalent with a function of bounded variation on every finite subinterval of $[0,\infty)$ and $|f(t)| \leq Mt^{\gamma}$.

We observe that the functions $f \in DBV_{\gamma}[0,\infty)$ possess a representation

$$f(x)=\int_0^x g(t)\,dt+f(0),$$

where $g \in BV[0,\infty)$, i.e., g is a function of bounded variation on every finite subinterval of $[0,\infty)$.

In order to discuss the approximation of functions with derivatives of bounded variation, we express the operators $G^{\alpha}_{n,c}$ in an integral form as follows:

$$G^{\alpha}_{n,c}(f;x)=\int_0^{\infty} K^{\alpha}_{n,c}(x,t)f\left(\frac{(n-c)t}{n}\right)dt, \tag{3.11}$$

where the kernel $K^{\alpha}_{n,c}(x,t)$ is given by

$$K^{\alpha}_{n,c}(x,t)=\sum_{k=1}^{\infty} Q^{(\alpha)}_{n,k}(x,c)p_{n+c,k-1}(t,c)+Q^{(\alpha)}_{n,0}(x,c)\delta(t),$$

$\delta(u)$ being the Dirac delta function.

Lemma 3.2. *For a fixed $x \in (0,\infty)$ and sufficiently large n, we have*

(1) $\xi^{\alpha}_{n,c}(x,y)=\int_0^y K^{\alpha}_{n,c}(x,t)\,dt \leq \alpha\frac{\lambda x(1+cx)}{n}\frac{1}{(x-y)^2}, \quad 0 \leq y < x,$
(2) $1-\xi^{\alpha}_{n,c}(x,z)=\int_z^{\infty} K^{\alpha}_{n,c}(x,t)\,dt \leq \alpha\frac{\lambda x(1+cx)}{n}\frac{1}{(z-x)^2}, \quad x < z < \infty.$

Proof. (1) Using Lemma 2.1 and Remark 2.4, we get

$$\begin{aligned}\xi^{\alpha}_{n,c}(x,y) &= \int_0^y K^{\alpha}_{n,c}(x,t)\,dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K^{\alpha}_{n,c}(x,t)\,dt \\ &\leq G^{\alpha}_{n,c}((t-x)^2;x)(x-y)^{-2} \\ &\leq \alpha G_{n,c}((t-x)^2;x)(x-y)^{-2} \\ &\leq \frac{\lambda\alpha x(1+cx)}{n}\frac{1}{(x-y)^2}.\end{aligned}$$

The proof of (2) is similar, hence the details are omitted. □

Theorem 3.3. *Let $f \in DBV_\gamma[0,\infty)$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}|G^{\alpha}_{n,c}(f;x)-f(x)| &\leq \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\frac{2\alpha x(1+cx)}{n}} \\ &\quad + \frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|\sqrt{\frac{2\alpha x(1+cx)}{n}} \\ &\quad + \frac{\lambda\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k} f'_x + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} f'_x \\ &\quad + \frac{\lambda\alpha(1+cx)}{nx}|f(2x)-f(x)-xf'(x+)| \\ &\quad + \frac{\alpha C(n,c,r,x)}{n^r} + \frac{|f(x)|}{x}\frac{\lambda\alpha(1+cx)}{n},\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of $f(x)$ on $[a,b]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t)-f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t)-f'(x+), & x < t < \infty. \end{cases} \tag{3.12}$$

Proof. Since $G^{\alpha}_{n,c}(1;x)=1$, using (3.11), for every $x \in (0,\infty)$ we get

$$\begin{aligned}G^{\alpha}_{n,c}(t;x)-f(x) &= \int_0^\infty K^{\alpha}_{n,c}(x,t)(f(t)-f(x))\,dt \\ &= \int_0^\infty K^{\alpha}_{n,c}(x,t)\int_x^t f'(u)\,du\,dt.\end{aligned} \tag{3.13}$$

For any $f \in DBV_\gamma[0,\infty)$, from (3.12) we may write

$$\begin{aligned}f'(u) &= f'_x(u) + \frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-)) \\ &\quad + \frac{1}{2}(f'(x+)-f'(x-))\left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right) \\ &\quad \times \delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))],\end{aligned} \tag{3.14}$$

where

$$\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x.\end{cases}$$

From equations (3.13) and (3.14), we get

$$\begin{aligned}G_{n,c}^{\alpha}(t;x)-f(x)=\int_0^{\infty}K_{n,c}^{\alpha}(x,t)\int_x^t\Big[f_x'(u)+\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\\ +\frac{1}{2}(f'(x+)-f'(x-))\left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right)\\ +\delta_x(u)\big[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\big]\,du\Big]\,dt\\ =A_1+A_2+A_3+A_{n,c}^{\alpha}(f_x',x)+B_{n,c}^{\alpha}(f_x',x),\end{aligned}$$

where

$$\begin{aligned}A_1&=\int_0^{\infty}\left(\int_x^t\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\,du\right)K_{n,c}^{\alpha}(x,t)\,dt,\\ A_2&=\int_0^{\infty}K_{n,c}^{\alpha}(x,t)\left(\int_x^t\frac{1}{2}(f'(x+)-f'(x-))\left(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)dt,\\ A_3&=\int_0^{\infty}\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)\,du\right)K_{n,c}^{\alpha}(x,t)\,dt,\\ A_{n,c}^{\alpha}(f_x',x)&=\int_0^{x}\left(\int_x^t f_x'(u)\,du\right)K_{n,c}^{\alpha}(x,t)\,dt,\\ B_{n,c}^{\alpha}(f_x',x)&=\int_x^{\infty}\left(\int_x^t f_x'(u)\,du\right)K_{n,c}^{\alpha}(x,t)\,dt.\end{aligned}$$

Obviously,

$$\begin{aligned}A_3&=\int_0^{\infty}\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)\,du\right)K_{n,c}^{\alpha}(x,t)\,dt\\ &=0.\end{aligned}\tag{3.15}$$

We get

$$\begin{aligned}A_1&=\int_0^{\infty}\left(\int_x^t\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\,du\right)K_{n,c}^{\alpha}(x,t)\,dt\\ &=\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\int_0^{\infty}(t-x)K_{n,c}^{\alpha}(x,t)\,dt\\ &=\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))G_{n,c}^{\alpha}((t-x);x)\end{aligned}\tag{3.16}$$

and

$$\begin{aligned} A_2 &= \int_0^\infty K_{n,c}^\alpha(x,t)\bigg(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))\bigg(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\bigg)\,du\bigg)\,dt \\ &= \frac{1}{2}\bigg(f'(x+)-f'(x-)\bigg)\bigg[-\int_0^x\bigg(\int_t^x\bigg(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\bigg)\,du\bigg)K_{n,c}^\alpha\,dt \\ &\quad+\int_x^\infty\bigg(\int_x^t\bigg(\operatorname{sgn}(u-x)+\frac{\alpha-1}{\alpha+1}\bigg)\,du\bigg)K_{n,c}^\alpha(x,t)\,dt\bigg] \qquad (3.17) \\ &\le \frac{\alpha}{\alpha+1}\bigg(f'(x+)-f'(x-)\bigg)\int_0^\infty |t-x|\,K_{n,c}^\alpha(x,t)\,dt \\ &= \frac{\alpha}{\alpha+1}\bigg(f'(x+)-f'(x-)\bigg)G_{n,c}^\alpha\bigg(|t-x|\,;x\bigg). \end{aligned}$$

Using Lemma 2.4 and equations (3.13–3.17) and applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} |G_{n,c}^\alpha(f;x)-f(x)| &\le \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|(\alpha G_{n,c}((t-x)^2;x))^{1/2} \\ &\quad+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|(\alpha G_{n,c}((t-x)^2;x))^{1/2} \\ &\quad+|A_{n,c}^\alpha(f'_x,x)|+|B_{n,c}^\alpha(f'_x,x)| \\ &\le \frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\frac{\lambda\alpha x(1+cx)}{n}} \qquad (3.18) \\ &\quad+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|\sqrt{\frac{\lambda\alpha x(1+cx)}{n}} \\ &\quad+|A_{n,c}^\alpha(f'_x,x)|+|B_{n,c}^\alpha(f'_x,x)|. \end{aligned}$$

Thus our problem is reduced to calculate the estimates of the terms $A_{n,c}^\alpha(f'_x,x)$ and $B_{n,c}^\alpha(f'_x,x)$. Since $\int_a^b d_t\xi_{n,c}^\alpha(x,t)\le 1$ for all $[a,b]\subseteq[0,\infty)$, using integration by parts and applying Lemma 3.2 with $y=x-x/\sqrt{n}$, we have

$$\begin{aligned} |A_{n,c}^\alpha(f'_x,x)| &= \bigg|\int_0^x\bigg(\int_x^t f'_x(u)\,du\bigg)d_t\xi_{n,c}^\alpha(x,t)\bigg| = \bigg|\int_0^x \xi_{n,c}^\alpha(x,t)f'_x(t)\,dt\bigg| \\ &\le \int_0^y |f'_x(t)|\;|\xi_{n,c}^\alpha(x,t)|\,dt+\int_y^x |f'_x(t)|\;|\xi_{n,c}^\alpha(x,t)|\,dt \\ &\le \frac{\lambda\alpha x(1+cx)}{n}\int_0^y \bigvee_t^x f'_x)(x-t)^{-2}\,dt+\int_y^x\bigvee_t^x f'_x\,dt \\ &\le \frac{\lambda\alpha x(1+cx)}{n}\int_0^y \bigvee_t^x f'_x(x-t)^{-2}\,dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x f'_x \\ &= \frac{\lambda\alpha x(1+cx)}{n}\int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x f'_x(x-t)^{-2}\,dt+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^x f'_x. \end{aligned}$$

Substituting $u = x/(x-t)$, we get

$$\begin{aligned}\frac{\lambda\alpha x(1+cx)}{n}\int_0^{x-x/\sqrt{n}}(x-t)^{-2}\bigvee_t^x f'_x\,dt &= \frac{\lambda\alpha x(1+cx)}{n}x^{-1}\int_1^{\sqrt{n}}\bigvee_{x-x/u}^{x} f'_x\,du\\ &\leq \frac{\lambda\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-x/k}^{x} f'_x\,du\\ &\leq \frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x} f'_x.\end{aligned}$$

Thus,

$$|A^{\alpha}_{n,c}(f'_x,x)| \leq \frac{\lambda(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x} f'_x + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x} f'_x. \tag{3.19}$$

Again, using integration by parts in $B^{\rho}_n(f'_x,x)$ and applying Lemma 3.2 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|B^{\alpha}_{n,c}(f'_x,x)| &\leq \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(u)\,du\right)d_tK^{\alpha}_{n,c}(x,t)\right|\\ &\quad+\left|\int_x^{2x}\left(\int_x^t f'_x(u)\,du\right)d_t(1-\xi^{\alpha}_{n,c}(x,t))\right|\\ &\leq \left|\int_{2x}^{\infty}(f(t)-f(x))K^{\alpha}_{n,c}(x,t)\right| + |f'(x+)|\left|\int_{2x}^{\infty}(t-x)K^{\alpha}_{n,c}(x;t)\,dt\right|\\ &\quad+\left|\int_x^{2x}f'_x(u)\,du\right||1-\xi^{\alpha}_{n,c}(x,2x)| + \left|\int_x^{2x}f'_x(t)(1-\xi^{\alpha}_{n,c}(x,t))\,dt\right|\\ &\leq \left|\int_{2x}^{\infty}f(t)K^{\alpha}_{n,c}(x,t)\right| + |f(x)|\left|\int_{2x}^{\infty}K^{\alpha}_{n,c}(x,t)\right|\\ &\quad+|f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2K^{\alpha}_{n,c}(x;t)\,dt\right)^{1/2}\\ &\quad+\frac{\lambda\alpha(1+cx)}{nx}\left|\int_x^{2x}((f'(u)-f'(x+))\,du\right| + \left|\int_x^{x+x/\sqrt{n}}f'_x(t)\,dt\right|\\ &\quad+\frac{\lambda\alpha x(1+cx)}{n}\left|\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}f'_x(t)\,dt\right|.\end{aligned}$$

We see that there exists an integer r $(2r \geq \gamma)$, such that $f(t) = O(t^{2r})$, as $t \to \infty$. Now proceeding in a manner similar to the estimate of $A^{\alpha}_{n,c}(f'_x;x)$, on substituting

$t = x + \frac{x}{u}$ we get

$$\begin{aligned}|B^{\alpha}_{n,c}(f',x)| &\leq M\int_{2x}^{\infty} t^{2r}K^{\alpha}_{n,c}(x,t)\,dt + |f(x)|\int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)\,dt \\ &\quad + |f'(x+)|\sqrt{\frac{\lambda\alpha x(1+cx)}{n}} + \frac{\lambda\alpha(1+cx)}{nx}|f(2x)-f(x)-xf'(x+)| \\ &\quad + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}(f'_x) + \frac{\lambda\alpha x(1+cx)}{n}\left|\int_{x+x/\sqrt{n}}^{2x}(t-x)^{-2}f'_x(t)\,dt\right| \\ &\leq M\int_{2x}^{\infty} t^{2r}K^{\alpha}_{n,c}(x,t)\,dt + |f(x)|\int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)\,dt \\ &\quad + |f'(x+)|\sqrt{\frac{\lambda\alpha x(1+cx)}{n}} + \frac{\lambda\alpha(1+cx)}{nx}|f(2x)-f(x)-xf'(x+)| \\ &\quad + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}} f'_x + \frac{\lambda\alpha(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+x/\sqrt{n}} f'_x. \end{aligned} \tag{3.20}$$

For $t \geq 2x$, we get $t \leq 2(t-x)$ and $x \leq t-x$. Now using equation (2.1) and Lemma 3.2, we obtain

$$\begin{aligned}&\int_{2x}^{\infty} t^{2r}K^{\alpha}_{n,c}(x,t)\,dt + |f(x)|\int_{2x}^{\infty} K^{\alpha}_{n,c}(x,t)\,dt \\ &\quad \leq 2^{2r}\int_{2x}^{\infty}(t-x)^{2r}K^{\alpha}_{n,c}(x,t)\,dt + \frac{|f(x)|}{x^2}\int_{2x}^{\infty}(t-x)^{2}K^{\alpha}_{n,c}(x,t)\,dt \qquad (3.21) \\ &\quad \leq \frac{\alpha C(n,c,r,x)}{n^r} + \frac{|f(x)|}{x^2}\frac{\lambda\alpha x(1+cx)}{n}. \end{aligned}$$

Collecting the estimates (3.18)–(3.21), we get the required result. This completes the proof. □

Next, we illustrate the comparison of the rate of convergence of the operators (1.2) and (1.3) to a certain function by some graphics using Maple algorithms.

Let us consider the function

$$f(x) = \begin{cases} 0, & x = 0 \\ x^{1/3}\sin(\pi/x), & x \neq 0. \end{cases} \tag{3.22}$$

Then, f is of bounded variation on $[0,1]$.

Example 1. For $c = 150$, $\alpha = 10$, $n = 160$ and $n = 200$, the convergence of the Bézier–Gupta–Srivastava (named as BzGS in figures) operators given by (1.3) (green and red) to $f(x)$ (blue), for $x \in [0,1]$, $x \in [0,2/\pi]$, $x \in [0,5]$ and $x \in [0,10]$

is given in Figures 1, 2, 3 and 4.

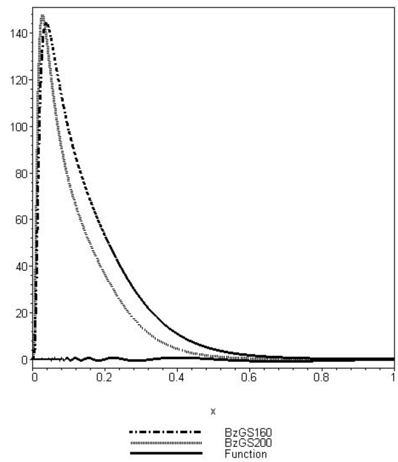

Figure 1

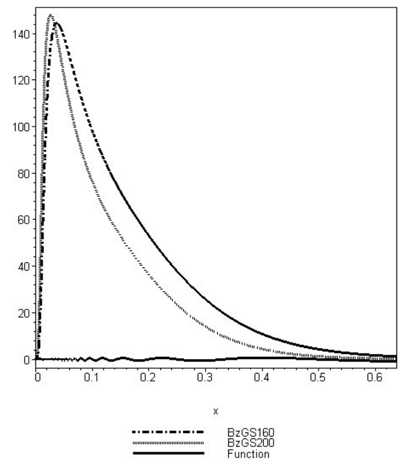

Figure 2

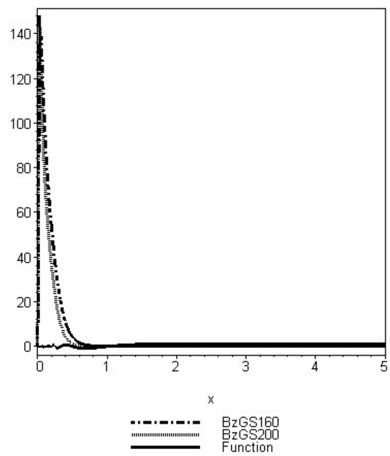

Figure 3

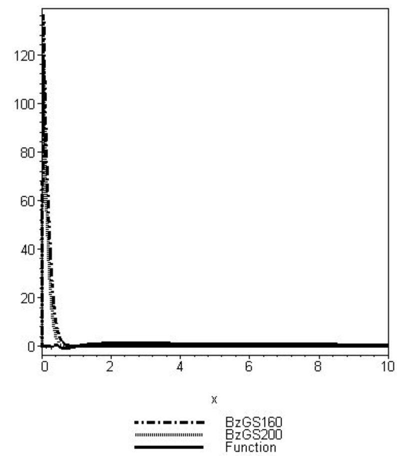

Figure 4

Example 2. For $c = 150$, $\alpha = 10$ and $n = 160$, the convergence of the Gupta–Srivastava (named as GS in Figures) operators given by (1.1) and the Bézier–Gupta–Srivastava operators given by (1.3) to the function $f(x)$ given by (3.22), for

$x \in [0, 2/\pi]$ and $x \in [0, 10]$ is shown in Figures 5 and 6 respectively.

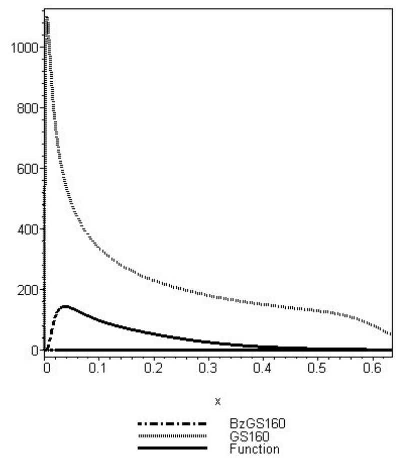

Figure 5

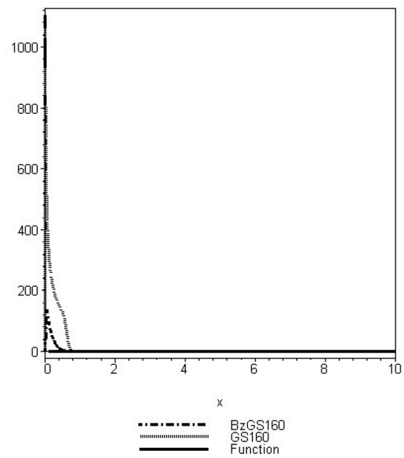

Figure 6

Example 3. In case $c = 0$, $\alpha = 10$ for $n = 160$, $n = 200$ and $n = 50$, $n = 100$, the convergence of Bézier–Gupta–Srivastava operators to the function $f(x)$ given by (3.22) for $x \in [0, 1]$ is shown in Figures 7 and 8, respectively.

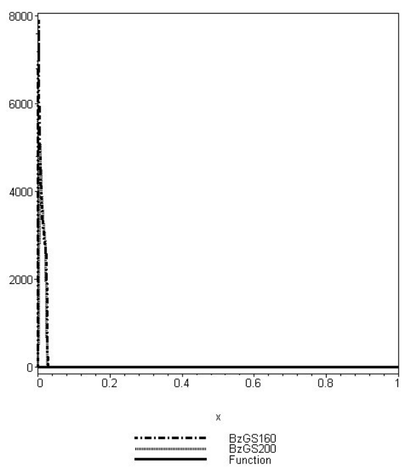

Figure 7

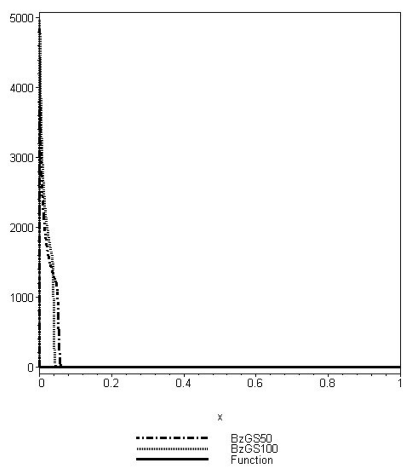

Figure 8

Example 4. In Figure 9, for $c = 1$, $n = 160, 200$ and $\alpha = 10$, the convergence of the the Bézier–Gupta–Srivastava operators given by (1.3) to the function $f(x)$

given by (3.22) is illustrated.

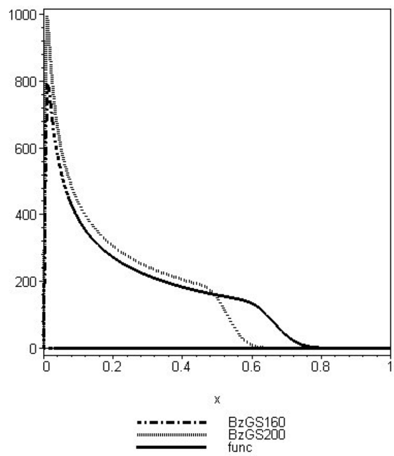

Figure 9

Now let us compare the convergence of the operators $G_{n,c}$ given by (1.2) and $G^{\alpha}_{n,c}$ given by (1.3)

In case $c = 0$, for $n = 50$ and $\alpha = 10$, the comparison of the convergence of the operators (1.2) (named as Ydv in Figures) and the operators (1.3) to the function (3.22) is shown in Figure 10.

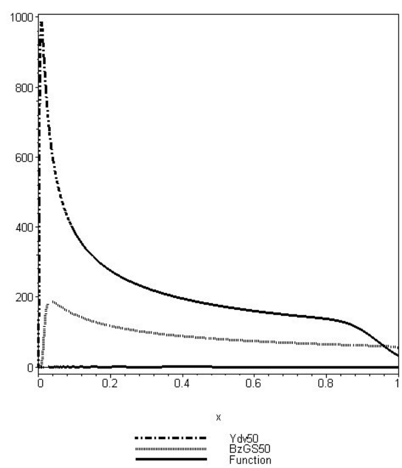

Figure 10

In Figures 11 and 12, in cases $c = 1$ and $c = 150$, for $\alpha = 10$ and $n = 160$, the comparison of the convergence of the operators (1.2) and (1.3) to the function $f(x)$

given by (3.22) is illustrated.

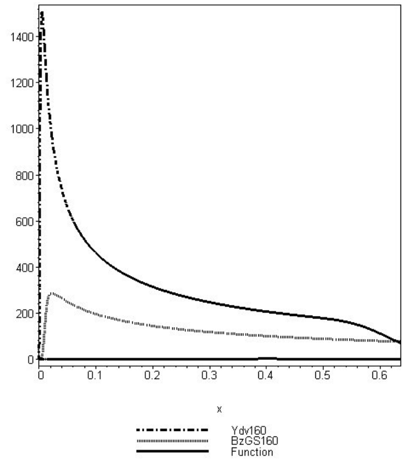

Figure 11

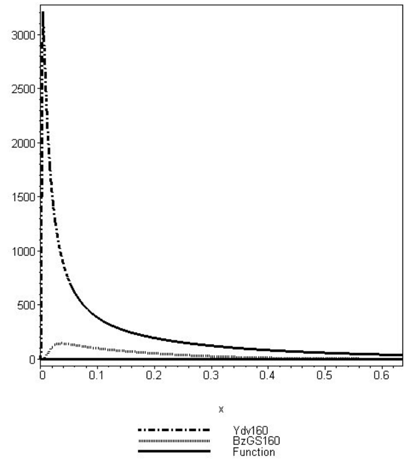

Figure 12

From the above comparisons, it turns out that the operator given by (1.3) yields a better rate of convergence than the operator given by (1.2) in certain cases.

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