REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 2, 2017, Pages 253–258
Published online: May 11, 2017

A NOTE ON WEIGHTED INEQUALITIES FOR A ONE-SIDED MAXIMAL OPERATOR IN $\mathbb{R}^n$

MARÍA LORENTE AND FRANCISCO J. MARTÍN-REYES

Abstract. We introduce a new dyadic one-sided maximal operator $M_d^{+\cdots+}$ in $\mathbb{R}^n$ that allows us to obtain good weights for the L^p-boundedness of a one-sided maximal operator $N^{+\cdots+}$ in $\mathbb{R}^n$, which is equivalent to the classical one-sided Hardy–Littlewood maximal operator in the case $n=1$, but not in the case $n>1$. In order to do this, we characterize the good pairs of weights for the weak and strong type inequalities for $M_d^{+\cdots+}$ and we use a Fefferman–Stein type inequality which gives that, in a certain sense, $M_d^{+\cdots+}$ controls $N^{+\cdots+}$.

1. Introduction

For f locally integrable on $\mathbb{R}$, the one-sided Hardy–Littlewood maximal functions are

$$M^+f(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|\,dy \quad \text{and} \quad M^-f(x)=\sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f(y)|\,dy.$$

These operators are interesting because they control some one-sided operators such as singular integrals with kernels supported in $(-\infty,0)$ or $(0,\infty)$. Sawyer characterized the L^p-weighted inequalities for M^+ in [10] (see other proofs in [1], [7], [8] and [5]).

In [5] we introduced an appropriate dyadic one-sided maximal operator M_d^+ in $\mathbb{R}$, which allows us to give a new proof of Sawyer's results, proving a Feffermann–Stein type inequality in means as in the classical case.

The natural generalization of M^+ in $\mathbb{R}^n$ is the following: given $x=(x_1,x_2,\dots,x_n)$ we define

$$M^{+\cdots+}f(x)=\sup_{h>0}\frac{1}{h^n}\int_{Q_x(h)}|f(y)|\,dy,$$

where $Q_x(h)=[x_1,x_1+h)\times[x_2,x_2+h)\times\cdots\times[x_n,x_n+h)$. In $\mathbb{R}$ we have two one-sided operators. In $\mathbb{R}^n$ we obviously have 2^n one-sided operators that we do not

---

2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 26D15.
Key words and phrases. Dyadic operators, one-sided maximal operators, weighted inequalities.
The authors are supported by grants MTM2011-28149-C02-02 and MTM2015-66157-C2-2-P of the Ministerio de Economía y Competitividad (Spain) and grant FQM-354 of the Junta de Andalucía.

write explicitly. Quite surprisingly, the weighted inequalities for $M^{+\cdots+}$ in $\mathbb{R}^n$ have not been characterized. The characterization of the weak type (p,p) inequality in $\mathbb{R}^2$ has been obtained in [3]. It is not difficult to prove the following:

Theorem 1.1. *Let $1 \le p < \infty$ and let u, v be nonnegative measurable functions. If $M^{+\cdots+}$ is of weak type (p,p) with respect to (u,v), that is, there exists $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(v)$*

$$\int_{\{x\in\mathbb{R}^n : M^{+\cdots+}f(x)>\lambda\}} u(x)\,dx \le \frac{C}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx, \tag{1}$$

then the pair of weights (u,v) satisfies A_p^+, that is, there exists $C > 0$ such that for all $h > 0$,

$$\frac{1}{h^n}\left(\int_{Q_x^-(h)} u\right)^{1/p}\left(\int_{Q_x(h)} v^{1-p'}\right)^{1/p'} < C, \qquad p > 1,$$

$$\frac{1}{h^n}\int_{Q_x^-(h)} u \le Cv(x), \quad a.e.\ x = (x_1,\dots,x_n), \qquad p = 1,$$

where $Q_x^-(h) = [x_1 - h, x_1) \times \cdots \times [x_n - h, x_n)$.

In [3] it was proved that the previous conditions are equivalent in the case $n = 2$. But, apparently, the arguments used are valid only for $n = 2$. Consequently, the problem of characterizing the weighted L^p inequalities for $M^{+\cdots+}$ remains open in dimension greater than one.

We have tried to extend the results of [5] to higher dimensions. We have not achieved our main goal, that is, characterizing the good weights for $M^{+\cdots+}$, but we have got new interesting results for the following one-sided maximal operator in $\mathbb{R}^n$ which has appeared in several previous works (see [9], [4] and [2]): Given $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let us define

$$N^{+\cdots+}f(x) = \sup_{h>0}\frac{1}{h^n}\int_{Q_x^+(h)} |f(y)|\,dy,$$

where $Q_x^+(h) = [x_1 + h, x_1 + 2h) \times [x_2 + h, x_2 + 2h) \times \cdots \times [x_n + h, x_n + 2h)$. For $n = 1$, M^+ and N^+ are equivalent. It is clear that $N^{+\cdots+}f(x) \le 2^n M^{+\cdots+}f(x)$ but there is no constant $C > 0$ satisfying $M^{+\cdots+}f(x) \le CN^{+\cdots+}f(x)$ for $n > 1$. In [9] Ombrosi proved that $(u,v) \in A_p^+$ implies the weak type inequality (1) for $N^{+\cdots+}$ (instead of $M^{+\cdots+}$). In [4] Lerner and Ombrosi proved that in the case of equal weights, $u = v$, and $n = 2$, the A_p^+ condition implies that N^{++} is bounded from $L^p(u)$ into $L^p(u)$. In [2] Berkovits extends this result for $n \ge 3$. In this note we generalize the result in [4] to any dimension giving a proof completely different from the one proposed by Berkovits, and we also give a sufficient condition in a pair of weights (u,v) for the boundedness of $N^{+\cdots+}$ from $L^p(v)$ into $L^p(u)$.

2. One-sided dyadic maximal function in $\mathbb{R}^n$

We are going to consider a one-sided dyadic maximal function in $\mathbb{R}^n$ that generalizes the one defined in [5]. In order to do this, let us establish the following notation: if $Q = [x_1, x_1 + h) \times \cdots \times [x_n, x_n + h)$ is a cube, denote $Q^- = [x_1, x_1+h/2)\times\cdots\times[x_n, x_n+h/2)$ and $Q^+ = [x_1+h/2, x_1+h)\times\cdots\times[x_n+h/2, x_n+h)$. Let $A_x^+ = \{Q \text{ dyadic} : x \in Q^-\}$. The one-sided dyadic maximal function is defined by

$$M_d^{+\cdots+}f(x) = \sup_{Q\in A_x^+} \frac{1}{|Q^+|}\int_{Q^+} |f(y)|\,dy.$$

Observe the difference between this operator and the corresponding one in [9]. Our operator satisfies that $M_d^{+\cdots+}f \leq CM_d f$, where M_d is the classical dyadic maximal operator $M_d f(x) = \displaystyle\sup_{Q \text{ dyadic}\,:\, x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy$, while the one-sided dyadic operator in [9] does not satisfy this condition.

It is interesting to note the following properties. Let Q and R be dyadic cubes.

(i) If $Q^+ \subset R^+$ then $Q^- \subset R^- \cup R^+$.
(ii) If $Q^- \cap R^- \neq \emptyset$ and $Q^+ \cap R^+ \neq \emptyset$ then $R = Q$.
(iii) If $Q^+ \cap R^+ \neq \emptyset$ and $Q^+ \cap R^- \neq \emptyset$ then $R \subset Q^+$.

These properties allow us to get a Fefferman–Stein type inequality as in [5]. Let $h = (h_1, \dots, h_n) \in \mathbb{R}^n$. Denote $\tau_h f(x) = \tau_h f(x_1, \dots, x_n) = f(x_1-h_1, \dots, x_n-h_n)$.

Proposition 2.1. *There exists a constant $C > 0$ such that*

$$N_k^{+\cdots+}f(x) \leq \frac{C}{(2^{k+4})^n}\int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]} (\tau_{-t}\circ M_d^+ \circ \tau_t)f(x)\,dt,$$

for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and f bounded with compact support, where $N_k^{+\cdots+}$ is the truncated operator of $N^{+\cdots+}$ taking supremum on $0 < h < 2^k$.

The proof of this result follows the same pattern as in the case $n = 1$ and it is left to the interested reader.

As we said before, Proposition 2.1 will allow us to get some results for $N^{+\cdots+}$ by studying the good weights for the dyadic operator $M_d^{+\cdots+}$.

Definition 2.2. We shall say that a pair of weights (u, v) belongs to the class $A_{p,d}^+$ if there exists a constant $C > 0$ such that for all dyadic cubes Q,

$$\frac{1}{|Q|}\left(\int_{Q^-} u\right)^{1/p}\left(\int_{Q^+} v^{1-p'}\right)^{1/p'} \leq C, \quad p > 1, \tag{$A_{p,d}^+$}$$

$$\frac{1}{|Q^-|}\int_{Q^-} u \leq Cv(x), \text{ a.e. } x \in Q^+, \quad p = 1. \tag{$A_{1,d}^+$}$$

In the case $u = v$ we simply write $u \in A_{p,d}^+$ instead of $(u, u) \in A_{p,d}^+$.

In the same way that for $n = 1$ we have the following characterization.

Theorem 2.3. *Let $1 < p < \infty$. Then the following assertions are equivalent.*

(1) *$M_d^{+\cdots+}$ is bounded from $L^p(u)$ into $L^p(u)$.*

(2) $u \in A^+_{p,d}$.

Theorem 2.3 and Proposition 2.1 provide us with a sufficient condition in a weight u for the boundedness of $N^{+\cdots+}$ in $L^p(u)$, $1 < p < \infty$. This extends Ombrosi's result to dimension $n \geq 3$ with a proof different from the one proposed by Berkovits. In [3] it was proved that this condition is necessary for the boundedness of $M^{+\cdots+}$ in $L^p(u)$.

Theorem 2.4. *Let $1 < p < \infty$. If u is a weight satisfying*

$$\sup_{x\in\mathbb{R}^n}\sup_{h>0}\frac{1}{h^n}\left(\int_{Q_x^-(h)} u\right)^{1/p}\left(\int_{Q_x(h)} u^{1-p'}\right)^{1/p'} < \infty, \qquad (A_p^+)$$

then $N^{+\cdots+}$ is bounded from $L^p(u)$ into $L^p(u)$.

Proof. Observe first that for $t = (t_1, \dots, t_n)$, $t_1, \dots, t_n > 0$, the weights $\tau_t u$ satisfy $(A^+_{p,d})$ uniformly on t. Then, by Jensen's and Fubini's theorems

$$\begin{aligned}
&\int_{\mathbb{R}^n} (N_k^{+\cdots+} f(x))^p u(x)\,dx \\
&\quad \leq C\int_{\mathbb{R}^n}\left(\frac{1}{(2^{k+4})^n}\int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]}(\tau_{-t}\circ M_d^{+\cdots+}\circ\tau_t)f(x)\,dt\right)^p u(x)\,dx \\
&\quad \leq C\frac{1}{(2^{k+4})^n}\int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]}\left(\int_{\mathbb{R}^n}(M_d^{+\cdots+}(\tau_t f)(x+t))^p u(x)\,dx\right)dt \\
&\quad = C\frac{1}{(2^{k+4})^n}\int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]}\left(\int_{\mathbb{R}^n}(M_d^{+\cdots+}(\tau_t f)(x))^p \tau_t u(x)\,dx\right)dt \\
&\quad \leq C\frac{1}{(2^{k+4})^n}\int_{(0,2^{k+4}]\times\cdots\times(0,2^{k+4}]}\left(\int_{\mathbb{R}^n}|\tau_t f(x)|^p \tau_t u(x)\,dx\right)dt \\
&\quad = C\int_{\mathbb{R}^n}|f(x)|^p u(x)\,dx.
\end{aligned}$$

Letting k tend to infinity we get the desired result. □

For different weights we have the next results.

Theorem 2.5. *Let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(1) *$M_d^{+\cdots+}$ is of weak type (p,p) with respect to (u,v).*
(2) *$(u,v) \in A^+_{p,d}$.*

Theorem 2.6. *Let $1 < p < \infty$. Let u and v be nonnegative measurable functions and $\sigma = v^{1-p'}$. The following assertions are equivalent.*

(1) *$M_d^{+\cdots+}$ is bounded from $L^p(v)$ into $L^p(u)$.*
(2) *There exists $C > 0$ such that*

$$\int_{Q^-\cup Q^+}(M_d^{+\cdots+}(\sigma\chi_{Q^+})(x))^p u(x)\,dx \leq C\int_{Q^+}\sigma(x)\,dx < \infty, \qquad (S^+_{p,d})$$

for all dyadic cubes Q with $\int_{Q^-} u > 0$.

We also omit the proof of these results because it follows again the steps of the corresponding results in [5]. We only want to point out that the n dimensional version of Lemma 2.1 in [9] holds, with the dyadic cubes Q.

Using again Proposition 2.1 and Theorem 2.6 we obtain the following.

Theorem 2.7. *Let $1 < p < \infty$. If u, v are two weights satisfying that there exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and all $h > 0$,*

$$\int_{Q^-_{x,h} \cup Q_{x,h}} (M^{+\cdots+}(\sigma \chi_{Q_{x,h}}))^p u \leq C \int_{Q_{x,h}} \sigma < \infty \tag{S^+_p}$$

whenever $\int_{Q^-_{x,h}} u > 0$, where $\sigma = v^{1-p'}$, then $N^{+\cdots+}$ is bounded from $L^p(v)$ into $L^p(u)$.

Proof. The proof follows the same steps that the proof of Theorem 2.4, once we prove that $(\tau_t u, \tau_t v)$ satisfies $S^+_{p,d}$ uniformly on t. Then, we have to prove that there exists $C > 0$ such that for all dyadic cubes and all t,

$$\int_{Q^- \cup Q^+} (M^{+\cdots+}_d(\tau_t \sigma \chi_{Q^+})(x))^p \tau_t u(x)\, dx \leq C \int_{Q^+} \tau_t \sigma(x)\, dx < \infty,$$

for all dyadic cubes Q with $\int_{Q^-} \tau_t u > 0$.

By a change of variable,

$$\begin{aligned}\int_{Q^- \cup Q^+} &(M^{+\cdots+}_d(\tau_t \sigma \chi_{Q^+})(x))^p \tau_t u(x)\, dx \\ &= \int_{(Q^- - t) \cup (Q^+ - t)} (M^{+\cdots+}_d(\tau_t \sigma \chi_{Q^+})(x+t))^p u(x)\, dx.\end{aligned}$$

Using again a change of variable we get

$$\begin{aligned} M^{+\cdots+}_d(\tau_t \sigma \chi_{Q^+})(x+t) &= \sup_{A^+_{x+t}} \frac{1}{|R^+|} \int_{R^+} \tau_t \sigma \chi_{Q^+}(s)\, ds \\ &= \sup_{A^+_{x+t}} \frac{1}{|R^+|} \int_{(R^+ - t) \cap (Q^+ - t)} \sigma(z)\, dz \\ &= \sup_{\{R \text{ dyadic}: x \in R^- - t\}} \frac{1}{|R^+ - t|} \int_{(R^+ - t)} \sigma(z) \chi_{Q^+ - t}\, dz \\ &\leq C M^{+\cdots+}(\sigma \chi_{Q^+ - t})(x). \end{aligned}$$

Then, using (S^+_p),

$$\begin{aligned}\int_{Q^- \cup Q^+} (M^{+\cdots+}_d(\tau_t \sigma \chi_{Q^+}))^p \tau_t u &\leq C \int_{(Q^- - t) \cup (Q^+ - t)} M^{+\cdots+}(\sigma \chi_{Q^+ - t}) \\ &\leq C \int_{(Q^+ - t)} \sigma = C \int_{Q^+} \tau_t \sigma < \infty. \qquad \square\end{aligned}$$

Remark 2.8. In [6] we studied another one-sided dyadic maximal operator,

$$\tilde{M}_d^{+\cdots+} f(x) = \sup_{Q\in A_x^+} \frac{1}{|Q\setminus Q^-|}\int_{Q\setminus Q^-} |f(y)|\,dy,$$

that allowed us to obtain a sufficient condition for the boundedness of $M^{+\cdots+}$ (Th. 4.3). Unluckily, we have to say that the condition $(A)_p^+$ appearing in [6] is, actually, equivalent to the classical A_p Muckenhoupt condition.

References

[1] K. F. Andersen, *Weighted inequalities for maximal functions associated with general measures*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 907–920. MR 1038012.

[2] L. Berkovits, *Parabolic Muckenhoupt weights in the Euclidean space*, J. Math. Anal. Appl. **379** (2011), 524–537. MR 2784340.

[3] L. Forzani, F. J. Martín-Reyes and S. Ombrosi, *Weighted inequalities for the two-dimensional one-sided Hardy–Littlewood maximal function*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1699–1719. MR 2746661.

[4] A. Lerner and S. Ombrosi, *A boundedness criterion for general maximal operators*, Publ. Mat. **54** (2010), 53–71. MR 2603588.

[5] M. Lorente and F. J. Martín-Reyes, *The one-sided dyadic Hardy–Littlewood maximal operator*, Proc. Roy. Soc. Edinburgh Sect. A. **144** (2014), no. 5, 991–1006. MR 3265541.

[6] M. Lorente and F. J. Martín-Reyes, *The one-sided dyadic Hardy–Littlewood maximal function*, Proceedings of the XIIth "Dr. Antonio A. R. Monteiro" Congress, 83–91, Univ. Nac. Sur Dep. Mat. Inst. Mat., Bahía Blanca, 2014. MR 3307004.

[7] F. J. Martín-Reyes, *New proofs of weighted inequalities for the one-sided Hardy–Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), 691–698. MR 1111435.

[8] F. J. Martín-Reyes, P. Ortega Salvador and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 517–534. MR 0986694.

[9] S. Ombrosi, *Weak weighted inequalities for a dyadic one-sided maximal function in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **133** (2005), no. 6, 1769–1775. MR 2120277.

[10] E. Sawyer, *Weighted inequalities for the one-sided Hardy–Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61. MR 0849466.

M. Lorente ✉
Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga,
29071 Málaga, Spain
m_lorente@uma.es

F. J. Martín Reyes
Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga,
29071 Málaga, Spain
martin_reyes@uma.es

Received: October 1, 2016
Accepted: March 10, 2017