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# HIGHER ORDER ELLIPTIC EQUATIONS IN HALF SPACE

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Abstract. We consider $2m^{\text{th}}$ order elliptic equations with Hölder coefficients in half space. We solve the Dirichlet problem with $m$ conditions on the boundary of the upper half space. We first analyze the constant coefficient case, finding the Green function and a representation formula, and then prove Schauder estimates.

## 1. Introduction

In this paper we solve the Dirichlet problem for elliptic operators of order $2m$ in the upper half space, with complex valued Hölder coefficients. Our method is based on finding explicit formulas for the Green function for the constant coefficient equation and proving a priori estimates in the Hölder spaces for solutions having homogeneous boundary data. This work differs in this sense from the well known paper of Agmon, Douglis and Nirenberg in which they solve first the homogeneous problem with non homogeneous boundary data for an elliptic operator with only principal part. A Radon type transformation due to F. John is key to their work [1, 4]. Our approach is based on Fourier transforming on the first variables the non homogeneous equation with homogeneous data. We find a representation formula which is key to the a priori estimates. We first treat the constant coefficient case and then, using the freezing coefficient method, we treat the variable coefficient equation. We remark that throughout the paper we will use the letter $C$ to denote a constant (not always the same) that depends only on structure. In the case that a constant depends on any aditional quantity we will mention this explicitly. See [5] for a treatment of the problem in full space and [8] for the case of smooth coefficients.

## 2. Preliminaries

In this short section we introduce the Hölder spaces that will be relevant later on.

Set $\Omega = \mathbb{R}^{n+1}_+$. Define

$$|u|_0 = \sup\{|u(x)| : x \in \Omega\},$$

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$$[u]_\delta = \sup\{\tfrac{|u(x)-u(y)|}{|x-y|^\delta} : x, y \in \Omega\},$$
$$|D^k u|_0 = \max\{|D^\alpha u|_0 : |\alpha| = k\},$$
$$[D^k u]_\delta = \max\{[D^\alpha u]_\delta : |\alpha| = k\},$$
$$|u|_{m+\delta} = \sum_{k=0}^{m} |D^k u|_0 + [D^m u]_\delta.$$

The space $C^{m+\delta}(\Omega) = \{u : |u|_{m+\delta} < \infty\}$ is a Banach space.

We will need to use the following interpolation result which we state without proof (see [3]).

**Lemma 2.1.** *For every $\epsilon > 0$ and $0 \le k \le m$ there exists $C_{\epsilon,k}$ such that $[D^k u]_\delta \le C_{\epsilon,k}|u|_0 + \epsilon|u|_{m+\delta}$.*

*For every $\epsilon > 0$ and $0 \le k \le m-1$ there exists $C_{\epsilon,k}$ such that $|D^k u|_0 \le C_{\epsilon,k}|u|_0 + \epsilon|D^m u|_0$.*

*For every $\epsilon > 0$ and $0 \le k \le m-1$ there exists $C_{\epsilon,k}$ such that $[D^k u]_\delta \le C_{\epsilon,k}|u|_0 + \epsilon|D^m u|_0$.*

*In all three cases, the inequalities hold for all $u \in C^{m+\delta}(\Omega)$.*

## 3. Constant coefficients

We consider operators of the form $Lu(x,t) = \sum_{|\alpha|+l\le 2m} a_{\alpha,l} D_x^\alpha D_t^l u(x,t)$, where $a_{\alpha,l}$ are complex numbers, $x \in \mathbb{R}^n$ and $t > 0$.

Let $L^\star u(x,t) = \sum_{|\alpha|+l\le 2m} (-1)^{|\alpha|+l} a_{\alpha,l} D_x^\alpha D_t^l u(x,t)$. We will assume that the characteristic polynomial $p(i\xi, it) = \sum_{|\alpha|+l\le 2m} a_{\alpha,l}(i\xi)^\alpha (it)^l$ satisfies the strong ellipticity condition

$$|p(i\xi, it)| \ge \lambda(1 + |\xi|^2 + t^2)^m \tag{1}$$

for all $(\xi, t) \in \mathbb{R}^{n+1}$. Notice that $p(i\xi, z)$ as a polynomial in the $z$ variable has no pure imaginary roots. It follows that we can write

$$p(i\xi, z) = p^+(i\xi, z)p^-(i\xi, z),$$

where $p^-(i\xi, z) = \prod_{j=1}^m (z - \lambda_j^-(\xi))$, $p^+(i\xi, z) = \prod_{j=1}^m (z - \lambda_j^+(\xi))$, $\{\lambda_j^- : j = 1, \ldots, m\}$ are the roots of $p(i\xi, z)$ with negative real part and $\{\lambda_j^+ : j = 1, \ldots, m\}$ are the roots of $p(i\xi, z)$ with positive real part. This is true for all $\xi \in \mathbb{R}^n$.

We will be able to impose $m$ initial conditions on a solution $u$ at $t = 0$.

**Lemma 3.1.** *If $p(i\xi, z) = 0$ then $|\operatorname{Re}(z)| \ge \frac{\lambda}{C}(1 + |\xi|)$ and $|z| \le C(1 + |\xi|)$, for some $C$ depending on $\max\{|a_{\alpha,l}| : \alpha, l\}$.*

*Proof.* The proof is a straightforward application of (1). □

We believe that the next property is also a consequence of the ellipticity condition (1) and some general theorem on the roots of a complex polynomial, but at this point we do not have a proof so we need to assume it.

This is an essential assumption about the roots $\lambda(\xi)$ of $p(i\xi,z)$: We assume that there exists a constant $C$ depending only on $n,\alpha,\lambda$ and $\max\{|a_{\alpha,l}| : \alpha,l\}$ such that if $\lambda(\xi)$ is a root of $p(i\xi,z)$, then for any multiindex $\alpha$ we have

$$|D_\xi^\alpha \lambda(\xi)| \le C(1+|\xi|)^{1-\alpha} \tag{2}$$

for all $\xi \in \mathbb{R}^n$.

We will now construct the Green function. Let us denote by $\gamma^-(\xi)$ any simple closed curve such that $\mathrm{Re}(\gamma^-(\xi)) \le 0$ and $\gamma^-(\xi)$ encloses the $m$ roots of $p^-(i\xi,z)$. Similarly, $\mathrm{Re}(\gamma^+(\xi)) \ge 0$ and $\gamma^+(\xi)$ encloses the $m$ roots of $p^+(i\xi,z)$. In the sequel, we will use different curves $\gamma^-(\xi)$ and $\gamma^+(\xi)$.

Define

$$h(y,s) = \int\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\,dz\,e^{iy\cdot\xi}\,d\xi \qquad \text{for } s>0$$

and

$$h(y,s) = \int\int_{\gamma^+(\xi)} \frac{-e^{sz}}{p(i\xi,z)}\,dz\,e^{iy\cdot\xi}\,d\xi \qquad \text{for } s<0,$$

and define, for any positive integer $l$ such that $2l+m \ge n+2$, the function

$$h^\star(y,s) = \frac{1}{(|y|^2+s^2)^l}\int_{\mathbb{R}^n}\int_{-\infty}^{+\infty} e^{i(y\cdot\xi+s\xi_{n+1})}(-\Delta)^l\left(\frac{1}{p(i\xi,i\xi_{n+1})}\right)d\xi_{n+1}\,d\xi.$$

**Lemma 3.2.** *$h = h^\star$.*

*Proof.* Formally, it follows by enlarging the contour of integration: Say, $s>0$, then $\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\,dz = \int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})}\,d\xi_{n+1}$ and an integration by parts shows that

$$\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} \frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})}\,d\xi\,d\xi_{n+1} = h^\star(y,s).$$

We proceed with a rigorous proof. Let $\beta \in C_0^\infty(B_2(0))$ with $\beta = 1$ on $B_1(0)$, and $\phi \in C_0^\infty([-2,2])$ with $\phi = 1$ in $[-1,1]$. Fix $(y,s)$ with $y\in\mathbb{R}^n$ and $s>0$ and let $\epsilon>0$. We show that $|h(y,s)-h^\star(y,s)| \le \epsilon$. Write $h(y,s)-h^\star(y,s) = A+B+C+D$, where

$$A = h(y,s) - \int \beta(\frac{\xi}{R})\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\,dz\,e^{iy\cdot\xi}\,d\xi,$$

$$B = \int \beta(\frac{\xi}{R})\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\,dz\,e^{iy\cdot\xi}\,d\xi - \int \beta(\frac{\xi}{R})\int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})}\,d\xi_{n+1}\,e^{iy\cdot\xi}\,d\xi,$$

$$C = \int \beta(\frac{\xi}{R})\int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})}\,d\xi_{n+1}\,e^{iy\cdot\xi}\,d\xi - \int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} \beta(\frac{\xi}{R})\phi(\frac{\xi_{n+1}}{\tilde R})\frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})}\,d\xi\,d\xi_{n+1},$$

$$D = \int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} \beta(\frac{\xi}{R})\phi(\frac{\xi_{n+1}}{\tilde R})\frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})}\,d\xi\,d\xi_{n+1} - h^\star(y,s).$$

We will choose $R$ and $\tilde R$ large enough to make each term small.

To estimate $A$ we can take the contour $\gamma^-(\xi)$ so that $\mathrm{Re}(\gamma^-(\xi)) \leq -C(1+|\xi|)$ and $|p(i\xi,z)| \geq C$ for all $z \in \gamma^-(\xi)$. We have

$$\begin{aligned}|A| &\leq \left| \int (1-\beta(\frac{\xi}{R})) \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\, dz\, e^{iy\cdot\xi}\, d\xi \right| \\ &\leq \int_{|\xi|\geq R} \Big| \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)}\, dz \Big|\, d\xi \leq \int_{|\xi|\geq R} e^{-Cs(1+|\xi|)} \int_{\gamma^-(\xi)} \frac{1}{|p(i\xi,z)|}\, dz d\xi \\ &\leq C \int_{|\xi|\geq R} \mathrm{length}(\gamma^-(\xi)) e^{-Cs(1+|\xi|)}\, d\xi \leq C \int_{|\xi|\geq R} (1+|\xi|) e^{-Cs(1+|\xi|)}\, d\xi.\end{aligned}$$

This last expression goes to 0 as $R$ goes to $\infty$.

To estimate $B$, take $\gamma^-(\xi) = I \cup C_M = \{i\xi_{n+1} : -M \leq \xi_{n+1} \leq M\} \cup \{Me^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$, where $M = M(|\xi|) = K(1+|\xi|)^r$ and $r$ and $K$ are chosen large. So,

$$\begin{aligned}B &\leq \left| \int \beta(\frac{\xi}{R})) \int_{C_M} \frac{e^{sz}}{p(i\xi,z)}\, dz\, e^{iy\cdot\xi}\, d\xi \right| \\ &\quad + \left| \int \beta(\frac{\xi}{R}) \int_{|\xi_{n+1}|\geq M} \frac{e^{is\xi_{n+1}}}{p(i\xi, i\xi_{n+1})}\, d\xi_{n+1} e^{iy\cdot\xi}\, d\xi \right| = |B_1| + |B_2|.\end{aligned}$$

Notice that we can write $p(i\xi,z) = \sum_{j=0}^m a_j(\xi) z^j$ with $|a_j(\xi)| \leq C|\xi|^{m-j}$. Hence for $z \in C_M$ we have $|p(i\xi,z)| \geq C|z|^m$ and hence

$$\begin{aligned}|B_1| &\leq C \int_{|\xi|\leq 2R} \frac{\mathrm{length}(C_M)}{M^m}\, d\xi \\ &\leq C \int_{|\xi|\leq 2R} \frac{1}{M^{m-1}}\, d\xi \leq \frac{C}{K^{m-1}} \int \frac{1}{(1+|\xi|)^{r(m-1)}}\, d\xi,\end{aligned}$$

which can be made arbitrarily small by choosing $r$ and $K$ large. And we have

$$|B_2| \leq C \int_{|\xi|\leq 2R} \int_{|\xi_{n+1}|\geq M} \frac{d\xi_{n+1}}{1+|\xi|^m + |\xi_{n+1}|^m}\, d\xi \leq C \int_{|\xi|\leq 2R} \frac{1}{M}\, d\xi,$$

which again can be made arbitrarily small by choosing $r$ and $K$ large.

The term $C$ can be estimated as

$$|C| \leq \int \beta(\frac{\xi}{R}) \int_{|\xi_{n+1}|\geq \tilde{R}} \frac{d\xi_{n+1}}{1+|\xi|^m + |\xi_{n+1}|^m}\, d\xi \leq C\frac{R^n}{\tilde{R}},$$

which we make small by making $\tilde{R}$ large depending on $R$.

To estimate $D$, first notice that

$$\begin{aligned}&\int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} \beta(\frac{\xi}{R}) \phi(\frac{\xi_{n+1}}{\tilde{R}}) \frac{e^{i(y\cdot\xi + s\xi_{n+1})}}{p(i\xi, i\xi_{n+1})}\, d\xi d\xi_{n+1} \\ &\qquad = \frac{1}{(|y|^2+s^2)^l} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} (-\Delta)^l \left( \frac{\beta(\frac{\xi}{R})\phi(\frac{\xi_{n+1}}{\tilde{R}})}{p(i\xi, i\xi_{n+1})} \right) e^{i(y\cdot\xi + s\xi_{n+1})}\, d\xi d\xi_{n+1} \\ &\qquad = \frac{1}{(|y|^2+s^2)^l} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n} (-\Delta)^l \left( \frac{1}{p(i\xi, i\xi_{n+1})} \right) e^{i(y\cdot\xi + s\xi_{n+1})}\, d\xi d\xi_{n+1} + E,\end{aligned}$$

where we have the estimate $|E| \leq \frac{C}{(|y|^2+s^2)^l}\frac{1}{R}$, which holds for $\tilde{R} \geq R \geq 1$. Therefore,

$$|D| \leq \frac{1}{(|y|^2+s^2)^l}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}(1-\beta(\frac{\xi}{R}))\phi(\frac{\xi_{n+1}}{\tilde{R}})(-\Delta)^l\left(\frac{1}{p(i\xi, i\xi_{n+1})}\right)$$
$$\cdot e^{i(y\cdot\xi+s\xi_{n+1})}\,d\xi\,d\xi_{n+1} + \frac{C}{(|y|^2+s^2)^l}\frac{1}{R}$$
$$\leq \frac{1}{(|y|^2+s^2)^l}\int_{|\xi_{n+1}|\geq R}\int_{|\xi|\geq R}\frac{d\xi d\xi_{n+1}}{(1+|\xi|+|\xi_{n+1}|)^{m+2l}} + \frac{C}{(|y|^2+s^2)^l}\frac{1}{R}$$
$$\leq \frac{C}{(|y|^2+s^2)^l}\frac{1}{R}.$$

Choose $R$ large enough to make $|A|$ and $|D|$ smaller than $\epsilon$ and then, for fixed $R$, choose $\tilde{R} \geq R$ to make $|C| \leq \epsilon$. □

We state in a lemma some well known properties of the function $h$; see [5].

**Lemma 3.3.**

- $h \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$.
- $|D_y^\alpha D_s^k h(y,s)| \leq \dfrac{C}{(|y|+|s|)^{n+1-2m+|\alpha|+k}}$ *for all* $|y|^2+s^2 \leq 1$ *and for any* $k$, $|\alpha|$.
- $|D_y^\alpha D_s^k h(y,s)| \leq \dfrac{C}{(|y|^2+s^2)^l}$ *for any* $l$, $|\alpha| \geq 0$, $k \geq 0$ *and* $|y|^2+s^2 \geq 1$.
- $Lh(y,s) = 0$ *for* $(y,s) \neq (0,0)$.

Let us also define

$$h_R(y,s) = \int_{\mathbb{R}^{n+1}}\frac{\varphi(\frac{\xi,\xi_{n+1}}{R})e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})}\,d\xi d\xi_{n+1}.$$

Note, as in the estimation of term $D$ in Lemma 3.2, that $h_R(y,s) \to h(y,s)$ as $R \to \infty$ for any $(y,s) \neq (0,0)$ and uniformly away from $(0,0)$. Here $\varphi \in C_0^\infty(B_2(0,0))$ and $\varphi = 1$ on $B_1(0,0)$.

We next define a function $k$ to account for the boundary values. For $y \in \mathbb{R}^n$ and $t > 0$, $\tau > 0$, define

$$k(y,t,\tau) = \int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{e^{tw}}{p^-(i\xi,w)(w-z)}\,dwdz\,e^{iy\cdot\xi}\,d\xi.$$

Let

$$\Phi(\xi,t,\tau) = \int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{e^{tw}}{p^-(i\xi,w)(w-z)}\,dwdz.$$

To analize the function $k$ we take into account that according to Lemma 3.1 we have that the $m$ roots of $p^-(i\xi,z)$ satisfy $\mathrm{Re}(\lambda^-(\xi)) \leq -\lambda(1+|\xi|)$ and $|\lambda^-(\xi)| \leq \Lambda(1+|\xi|)$, so we take $\gamma^-(\xi)$ a piecewise smooth contour parametrized by an angle $\theta$ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi|)$ joining the points with real part equal to $\frac{-\lambda}{2}(1+|\xi|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $w(\theta,\xi)$ the points on this contour.

Similarly, since the $m$ roots of $p^+(i\xi, z)$ satisfy $\mathrm{Re}(\lambda^+(\xi)) \geq \lambda(1+|\xi|)$ and $|\lambda^+(\xi)| \leq \Lambda(1+|\xi|)$, we take $\gamma^+(\xi)$ a piecewise smooth contour parametrized by an angle $\phi$ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi|)$ joining the points with real part equal to $\frac{\lambda}{2}(1+|\xi|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $z(\phi,\xi)$ the points on this contour.

The following properties follow directly:

For any multiindex $\alpha$ and for all $\xi$, we have $|D_\xi^\alpha w(\theta,\xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, $|D_\xi^\alpha z(\phi,\xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, $|D_\xi^\alpha D_\theta w(\theta,\xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, and $|D_\xi^\alpha D_\phi z(\phi,\xi)| \leq C(1+|\xi|)^{1-|\alpha|}$. And for all $\xi$, we have $\frac{\lambda}{2}(1+|\xi|) \leq |w(\theta,\xi)-\lambda^-(\xi)| \leq 2\Lambda(1+|\xi|)$, $\frac{\lambda}{2}(1+|\xi|) \leq |z(\phi,\xi)-\lambda^+(\xi)| \leq 2\Lambda(1+|\xi|)$, and $\frac{\lambda}{2}(1+|\xi|) \leq |w(\theta,\xi)-z(\phi,\xi)| \leq 4\Lambda(1+|\xi|)$.

Using the above properties together with (2) we can prove the following estimate:

$$|D_\xi^\alpha D_\tau^l \Phi(\xi,t,\tau)| \leq C \frac{e^{\frac{-(t+\tau)(1+|\xi|)}{2}}}{(1+|\xi|)^{2m-1+|\alpha|-l}} \tag{3}$$

for any $\alpha$, $l$, $\xi \in \mathbb{R}^n$, $\tau > 0$, and $t > 0$.

To prove the estimate, we note that we can write $\Phi(\xi,t,\tau)$ as a sum of four terms of the form

$$\int_\theta \int_\phi \frac{e^{-\tau z(\xi,\phi)}}{p^+(i\xi, z(\xi,\phi)} \, \frac{e^{tw(\xi,\theta)}}{p^-(i\xi, w(\xi,\theta)} \, \frac{z_\phi(\xi,\phi) w_\theta(\xi,\theta)}{w(\xi,\theta)-z(\xi,\phi)} \, d\phi \, d\theta$$

for the appropriate limits of integration in $\theta$ and $\phi$ which do not depend on $\xi$.

We write the integrand as $\frac{g_1 g_2}{g_3}$, where $g_1 = e^{-\tau z(\xi,\phi)} e^{tw(\xi,\theta)}$, $g_2 = z_\phi(\xi,\phi) w_\theta(\xi,\theta)$ and $g_3 = p^+(i\xi, z(\xi,\phi)) p^-(i\xi, w(\xi,\theta))(w(\xi,\theta)-z(\xi,\phi))$. The following estimates follow by direct computation:

$$|D_\xi^\gamma g_1| \leq C \frac{e^{\frac{-(t+\tau)(1+|\xi|)}{2}}}{(1+|\xi|)^{|\gamma|}}$$
$$|D_\xi^\gamma g_2| \leq C(1+|\xi|)^{2-|\gamma|}.$$

Also note that we can write $g_3 = \prod_{j=1}^{2m+1} (\phi_j(\xi) - \psi_j(\xi))$, where $|\phi_j(\xi) - \psi_j(\xi)| \approx 1+|\xi|$ and $|D_\xi^\alpha(\phi_j(\xi) - \psi_j(\xi))| \leq C(1+|\xi|)^{1-|\alpha|}$. It follows that $|D_\xi^\gamma g_3| \leq C(1+|\xi|)^{2m+1-|\gamma|}$ and hence $|D_\xi^\gamma (g_3)^{-1}| \leq \frac{C}{(1+|\xi|)^{2m+|\gamma|+1}}$. Putting the estimates together proves the claim.

Now, using the estimate (3), we can prove the following estimates for the function $k$.

**Lemma 3.4.** *The function $k(y,t,\tau)$ is $C^\infty$ in all its variables. The following estimates hold:*

(1) $|D_y^\alpha D_\tau^l k(y,t,\tau)| \leq \dfrac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}}$, *for all $\alpha$ and $l$.*

(2) $\int_0^\infty \int_{\mathbb{R}^{n-1}} |D_y^\alpha D_\tau^k k(y,t,\tau)|_{|x_k-y_k|=R}\, dy'd\tau \to 0$ *as* $R\to\infty$, *for any* $\alpha$ *and any* $k$ *and* $t>0$.

(3) $\int_{\mathbb{R}^n} |D_y^\alpha D_\tau^k k(y,t,R)|\, dy \to 0$ *as* $R\to\infty$, *for any* $\alpha$ *and any* $k$ *and* $t>0$.

(4) $\int_0^\infty \int_{\mathbb{R}^n} |k(y,t,\tau)|\, dyd\tau \le C$ *for any* $t>0$. $C$ *is independent of* $t$.

*Proof.* We have $k(y,t,\tau) = \int e^{iy\cdot\xi}\Phi(\xi,t,\tau)d\xi$ and $D_y^\alpha D_\tau^l k(y,t,\tau) = \int e^{iy\cdot\xi}(i\xi)^\alpha \cdot D_\tau^l\Phi(\xi,t,\tau)d\xi$, and the first assertion follows immediately.

Let $\eta\in C_0^\infty(B_2(0))$, such that $\eta=1$ in $B_1(0)$. To prove the first estimate, write

$$\begin{aligned} D_y^\alpha D_\tau^l k(y,t,\tau) &= \int e^{iy\cdot\xi}(i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau)\, d\xi \\ &= \frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))\, d\xi \\ &= \frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))\eta(|y|\xi)\, d\xi \\ &\quad + \frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))(1-\eta(|y|\xi))\, d\xi \\ &= A+B. \end{aligned}$$

We have

$$A = \frac{1}{|y|^{2k}}\int (-\Delta_\xi)^k(e^{iy\cdot\xi}(i\xi)^\alpha)D_\tau^l\Phi(\xi,t,\tau))\eta(|y|\xi)\, d\xi,$$

and hence

$$|A| \le C\int_{|\xi|\le\frac{2}{|y|}} |\xi^\alpha D_\tau^l\Phi(\xi,t,\tau))|d\xi \le C\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}\, d\xi.$$

Note that

$$\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}\, d\xi \le \int_{|\xi|\le\frac{2}{|y|}} \frac{1}{(1+|\xi|)^{2m-1-|\alpha|-l}}\, d\xi \le \frac{C}{|y|^{n+1-2m+|\alpha|+l}}$$

and also

$$\begin{aligned} &\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}\, d\xi \\ &\quad\le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}\int_{|\xi|\le\frac{2(t+\tau)}{|y|}} \frac{e^{-|\xi|}}{(t+\tau+|\xi|)^{2m-1-|\alpha|-l}}\, d\xi \\ &\quad\le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}\int_{\mathbb{R}^n} \frac{e^{-|\xi|}}{|\xi|^{2m-1-|\alpha|-l}}\, d\xi \le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}. \end{aligned}$$

Therefore,

$$|A| \le \min\{\frac{1}{|y|^{n+1-2m+|\alpha|+l}}, \frac{1}{(t+\tau)^{n+1-2m+|\alpha|+l}}\} \le \frac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}}.$$

To estimate the term $B$, we note that

$$|B| \leq \frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}},$$

and

$$\begin{aligned} \frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}} &\leq \frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{1}{|\xi|^{2m-1-|\alpha|-l+2k}} \\ &\leq \frac{C}{|y|^{n+1-2m+|\alpha|+l}}. \end{aligned}$$

But also, for $|y| \leq t+\tau$, we have

$$\begin{aligned} &\frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}} \\ &\quad \leq \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}} (\frac{t+\tau}{|y|})^{2k} \int_{|\xi| \geq \frac{t+\tau}{|y|}} \frac{e^{-|\xi|}}{(t+\tau+|\xi|)^{2m-1-|\alpha|-l+2k}} \, d\xi \\ &\quad \leq \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}} (\frac{t+\tau}{|y|})^{2k} \int_{|\xi| \geq \frac{t+\tau}{|y|}} \frac{C_r}{|\xi|^r} \, d\xi \leq \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}, \end{aligned}$$

by choosing $r = 2k+n$.

Therefore,

$$|B| \leq \min \left\{ \frac{1}{|y|^{n+1-2m+|\alpha|+l}}, \frac{1}{(t+\tau)^{n+1-2m+|\alpha|+l}} \right\} \leq \frac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}},$$

which finishes the proof of the first estimate.

To prove the other estimates, we note that for any $k$ we have

$$D_y^\alpha D_\tau^l k(x-y,t,\tau) = \frac{1}{|x-y|^{2k}} \int e^{i(x-y)\cdot\xi} (-\Delta_\xi)^k ((i\xi)^\alpha D_\tau^l \Phi(\xi,t,\tau)) \, d\xi,$$

and hence,

$$|D_y^\alpha D_\tau^l k(x-y,t,\tau)| \leq \frac{1}{|x-y|^{2k}} \int \frac{e^{-(t+\tau)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}} \, d\xi.$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \int_0^\infty |D_y^\alpha D_\tau^l k(x-y,t,\tau)||_{|x_k-y_k|=R} \, d\tau dy' \\ &\quad \leq \int_{\mathbb{R}^{n-1}} \frac{dy'}{(|x'-y'|^2+R^2)^{2k}} \int_0^\infty \int \frac{e^{-(t+\tau)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}} \, d\xi d\tau \\ &\quad \leq \int_{\mathbb{R}^{n-1}} \frac{dy'}{(|x'-y'|^2+R^2)^{2k}} \int \frac{d\xi}{(1+|\xi|)^{2m-l-|\alpha|+2k}} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

To prove that $\int_{\mathbb{R}^n} |D_y^\alpha D_\tau^k k(y,t,R)| dy \to 0$, use the first estimate

$$|D_y^\alpha D_\tau^l k(y,t,R)| \leq \frac{C}{(|y|+t+R)^{n+1-2m+|\alpha|+l}}$$

for $|y| \leq 1$, and use

$$|D_y^\alpha D_\tau^l k(y,t,R)| \leq \frac{1}{|y|^{2k}} \int \frac{e^{-(t+R)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}}\, d\xi$$

for $|y| \geq 1$. Similarly for the last estimate. □

We next state a lemma from complex analysis which follows by shifting the contour of integration.

**Lemma 3.5.** *Let $\gamma(\xi)$ be a simple closed contour enclosing all roots of $p(i\xi, z)$, and $\gamma^+(\xi)$, $\gamma^-(\xi)$ as before. Then*

$$\frac{1}{2\pi i}\int_{\gamma(\xi)} \frac{z^k}{p(i\xi,z)}\, dz = \begin{cases} 0 & \text{for } 0 \leq k \leq 2m-2 \\ 1 & \text{for } k = 2m-1; \end{cases}$$

$$\frac{1}{2\pi i}\int_{\gamma^+(\xi)} \frac{z^j}{p^+(i\xi,z)(w-z)}\, dz = \frac{w^j}{p^+(i\xi,w)} \quad \text{for all } w \in \gamma^-(\xi) \text{ and } 0 \leq j \leq m-1;$$

$$\frac{1}{2\pi i}\int_{\gamma^-(\xi)} \frac{w^j}{p^-(i\xi,w)(w-z)}\, dw = \frac{-z^j}{p^-(i\xi,z)} \quad \text{for all } z \in \gamma^+(\xi) \text{ and } 0 \leq j \leq m-1.$$

We define the Green function for the upper half space $\mathbb{R}^{n+1}_+$ to be

$$g(x-y,t,\tau) = h(x-y,t-\tau) - k(x-y,t,\tau).$$

We state in a theorem the properties of $g$.

**Theorem 3.6.** *$L_{x,t}(g(x-y,t,\tau)) = 0$ for $(x,t) \neq (y,\tau)$ and $L^\star_{y,\tau}(g(x-y,t,\tau)) = 0$ for $(y,\tau) \neq (x,t)$. In addition, $g$ satisfies the following $m$ boundary conditions at $t = 0$, for any $\tau > 0$:*

$$g(x-y,0,\tau) = 0,\ \frac{\partial g}{\partial t}(x-y,0,\tau) = 0,\ \ldots,\ \frac{\partial^{m-1} g}{\partial t^{m-1}}(x-y,0,\tau) = 0,$$

*and the following $m$ boundary conditions at $\tau = 0$, for any $t > 0$:*

$$g(x-y,t,0) = 0,\ \frac{\partial g}{\partial \tau}(x-y,t,0) = 0,\ \ldots,\ \frac{\partial^{m-1} g}{\partial \tau^{m-1}}(x-y,t,0) = 0.$$

*Proof.* The proof is a straightforward computation using Lemma 3.5. To prove $L_{x,t}(g(x-y,t,\tau)) = 0$, we prove $L_{x,t}(h(x-y,t-\tau)) = 0$ and $L_{x,t}(k(x-y,t,\tau)) = 0$. We have, say for $t > \tau$,

$$D_x^\alpha D_t^l h(x-y,t-\tau) = \int\int_{\gamma^-(\xi)} \frac{z^l (i\xi)^\alpha}{p(i\xi,z)} e^{(t-\tau)z}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi,$$

which implies that

$$L_{x,t}(h(x-y,t-\tau)) = \int\int_{\gamma^-(\xi)} e^{(t-\tau)z}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi = 0.$$

Also,

$$D_x^\alpha D_t^l k(x-y,t,\tau) = \int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{w^l (i\xi)^\alpha e^{wt}}{p^-(i\xi,w)(w-z)}\, dw dz\, e^{i(x-y)\cdot\xi}\, d\xi.$$

Hence,

$$L_{x,t}(k(x-y,t,\tau)) = \int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{p(i\xi,w)e^{wt}}{p^-(i\xi,w)(w-z)}\, dwdz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$= \int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{p^+(i\xi,w)e^{wt}}{(w-z)}\, dwdz\, e^{i(x-y)\cdot\xi}\, d\xi = 0,$$

since $\int_{\gamma^-(\xi)} \frac{p^+(i\xi,w)e^{wt}}{(w-z)}\, dw = 0$, because for each $z \in \gamma^+(\xi)$, the function $f(w) = \frac{p^+(i\xi,w)e^{wt}}{w-z}$ is analytic inside $\gamma^-(\xi)$.

Next, we show that $L^\star_{y,\tau}(g(x-y,t,\tau)) = 0$. Since $D^\alpha_y D^l_\tau(h(x-y,t-\tau)) = (-1)^{|\alpha|+l} D^\alpha_x D^l_t(h(x-y,t-\tau))$, it follows that $L^\star_{y,\tau}(h(x-y,t-\tau)) = 0$. Also,

$$D^\alpha_y D^l_\tau(k(x-y,t,\tau))$$
$$= (-1)^{|\alpha|+l} \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{z^l(i\xi)^\alpha e^{-\tau z}}{p^+(i\xi,z)(w-z)}\, dzdw\, e^{i(x-y)\cdot\xi} d\xi.$$

Hence

$$L^\star_{y,\tau}(k(x-y,t,\tau)) = \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{p^-(i\xi,z)e^{-\tau z}}{(w-z)}\, dzdw\, e^{i(x-y)\cdot\xi} d\xi = 0,$$

since $\int_{\gamma^+(\xi)} \frac{p(^-(i\xi,z)e^{-\tau z}}{(w-z)}\, dz = 0$.

We now check the boundary conditions.

$$g(x-y,0,\tau) = -\int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p(i\xi,z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$-\int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{dw}{p(i\xi,w)(w-z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi = 0,$$

since $\int_{\gamma^-(\xi)} \frac{dw}{p^-(i\xi,w)(w-z)} = -\frac{1}{p^-(i\xi,z)}$. For $t < \tau$, we have

$$\frac{\partial g}{\partial t}(x-y,t,\tau) = -\int\int_{\gamma^+(\xi)} \frac{ze^{(t-\tau)z}}{p(i\xi,z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$-\int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{w\, e^{tw}\, dw}{p^-(i\xi,w)(w-z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi,$$

and hence

$$\frac{\partial g}{\partial t}(x-y,0,\tau) = -\int\int_{\gamma^+(\xi)} \frac{ze^{(-\tau)z}}{p(i\xi,z)} dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$-\int\int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{w\, dw}{p^-(i\xi,w)(w-z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi = 0,$$

since $\int_{\gamma^-(\xi)} \frac{w\, dw}{p^-(i\xi,w)(w-z)} = -\frac{z}{p^-(i\xi,z)}$.

Continue up to the derivative of order $m-1$.

We now proceed to prove the $m$ boundary conditions at $\tau = 0$.

$$g(x-y,t,0) = \int\int_{\gamma^-(\xi)} \frac{e^{tz}}{p(i\xi,z)} dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$- \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi = 0,$$

since $\int_{\gamma^+(\xi)} \frac{dz}{p^+(i\xi,z)(w-z)} = \frac{1}{p^+(i\xi,w)}$. For $\tau < t$ we have

$$\frac{\partial g}{\partial \tau}(x-y,t,\tau) = -\int\int_{\gamma^-(\xi)} \frac{ze^{(t-\tau)z}}{p(i\xi,z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$+ \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{z\, e^{-\tau z}\, dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi.$$

Hence

$$\frac{\partial g}{\partial \tau}(x-y,t,0) = -\int\int_{\gamma^-(\xi)} \frac{ze^{(t)z}}{p(i\xi,z)}\, dz\, e^{i(x-y)\cdot\xi}\, d\xi$$
$$+ \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{z\, dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi = 0,$$

since $\int_{\gamma^+(\xi)} \frac{z\,dz}{p^+(i\xi,z)(w-z)} = \frac{w}{p(i\xi,w)}$.

Continue up to the derivative of order $m-1$. □

We use the function $g$ to prove a representation formula, which is the basis for the a priori estimates.

**Theorem 3.7.** *Let $u \in C^{2m+\delta}(\mathbb{R}^{n+1}_+)$ satisfy the $m$ boundary conditions at $t = 0$:*

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0,\ \ldots,\ \frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) = 0.$$

*Then*

$$u(x,t) = \int_0^\infty \int Lu(y,\tau) g(x-y,t,\tau)\, dyd\tau$$

*for all $x \in \mathbb{R}^n$, $t > 0$.*

*Proof.* Fix $(x,t) \in \mathbb{R}^{n+1}_+$ and $\epsilon > 0$. Write the $m$ conditions on $u$ in $(y,\tau)$ variables,

$$u(y,0) = 0,\ \frac{\partial u}{\partial \tau}(y,0) = 0,\ \ldots,\ \frac{\partial^{m-1} u}{\partial \tau^{m-1}}(y,0) = 0,$$

and let $v(y,\tau) = g(x-y,t,\tau)$. Recall $L^\star v(y,\tau) = 0$ and $v(y,0) = 0$, $\frac{\partial v}{\partial \tau} v(y,0) = 0, \ldots, \frac{\partial^{m-1} v}{\partial \tau^{m-1}} v(y,0) = 0$. First we need to prove the next three claims.

*Claim 1.* For $|\alpha| + l = 2m$ and $l \geq 1$ we have

$$\int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} D^\alpha_y D^l_\tau u(y,\tau) v(y,\tau)\, dyd\tau = \int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} u(y,\tau) D^\alpha_y D^l_\tau v(y,\tau)\, dyd\tau$$
$$- \int_{\partial B_\epsilon(x,t)} u(y,\tau) D^\alpha_y D^{l-1}_\tau v(y,\tau) \eta_\tau(y,\tau)\, dS(y,\tau) + O(\epsilon),$$

where $\eta_\tau(y,\tau) = \frac{t-\tau}{\epsilon}$.

*Claim 2.* For $|\alpha| = 2m$ we have

$$\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^\alpha_y u(y,\tau)v(y,\tau)\,dyd\tau = \int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} u(y,\tau)D^\alpha_y v(y,\tau)\,dyd\tau \\ - \int_{\partial B_\epsilon(x,t)} u(y,\tau)D^{\alpha'}_y v(y,\tau)\eta_{\alpha'}(y,\tau)\,dS(y,\tau) + O(\epsilon),$$

where $\alpha = \alpha' + e_i$, $\eta_{\alpha'}(y,\tau) = \frac{x_i-y_i}{\epsilon}$ and $i$ is any integer $1 \le i \le n$.

*Claim 3.* For $|\alpha| + l < 2m$ we have

$$\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^\alpha_y D^l_\tau u(y,\tau)v(y,\tau)\,dyd\tau \\ = (-1)^{|\alpha|+l}\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} u(y,\tau)D^\alpha_y D^l_\tau v(y,\tau)\,dyd\tau + O(\epsilon).$$

The proof of the claims follows by succesive integration by parts.

Use the boundary conditions on $u$ and $v$ at $\tau = 0$ that complement each other so that no boundary terms appear at $\tau = 0$, and the decay of $v$ and its derivatives at $\infty$.

The terms $O(\epsilon)$ come from the derivatives of $v$ of orders less than $2m-1$ integrated over $\partial B_\epsilon(x,t)$.

We illustrate with two cases to see how it goes.

$$\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^{2m}_\tau u(y,\tau)v(y,\tau)\,dyd\tau \\ = \int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^{2m-1}_\tau u(y,\tau)D_\tau v(y,\tau)\,dyd\tau \\ + \int_{\partial B_\epsilon(x,t)} D^{2m-1}_\tau u(y,\tau)v(y,\tau)\,\eta_\tau\,dS \\ + \int_{\mathbb{R}^n} \lim_{R\to\infty}(D^{2m-1}_\tau u(y,R)\,v(y,R)) - D^{2m-1}_\tau u(y,0)\,v(y,0)\,dy.$$

The second term is $O(\epsilon)$ and the last term is 0.

After $m$ steps, using the $m$ conditions on $v$ at $\tau = 0$, we get

$$\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^{2m}_\tau u(y,\tau)v(y,\tau)\,dyd\tau \\ = (-1)^m\int_{\mathbb{R}^{n+1}_+\setminus B_\epsilon(x,t)} D^m_\tau u(y,\tau)D^m_\tau v(y,\tau)\,dyd\tau + O(\epsilon).$$

Continue integrating by parts, now using the $m$ conditions on $u$ at $\tau = 0$, to get, after $m$ steps,

$$\int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} D_\tau^{2m} u(y,\tau) v(y,\tau)\, dy d\tau = \int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} u(y,\tau) D_\tau^{2m} v(y,\tau)\, dy d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_\tau^{2m-1} v(y,\tau)\, \eta_\tau \, dS + O(\epsilon).$$

Again as an illustration, we have

$$\int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} D_\tau^{2m-1} D_{y_j} u(y,\tau) v(y,\tau)\, dy d\tau = (-1)^m \int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} D_\tau^{m-1} D_{y_j} u(y,\tau) D_\tau^m v(y,\tau)\, dy d\tau + O(\epsilon),$$

where we have used the $m$ conditions on $v$ at $\tau = 0$. Next, we continue integrating by parts, now using $D_{y_j} D_\tau^l u = 0$ and $u = 0$ at $\tau = 0$ for $l = 0, \ldots, m-2$ to get

$$\int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} D_\tau^{2m-1} D_{y_j} u(y,\tau) v(y,\tau)\, dy d\tau = \int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} u(y,\tau) D_\tau^{2m-1} D_{y_j} v(y,\tau)\, dy d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_\tau^{2m-1} v(y,\tau)\, \eta_{y_j}\, dS + O(\epsilon).$$

From the three claims we get

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} & Lu(y,\tau) v(y,\tau) dy d\tau \\ &= \int_{\mathbb{R}^{n+1}_+ \setminus B_\epsilon(x,t)} u(y,\tau) L^\star v(y,\tau)\, dy d\tau \\ &\quad - \sum_{|\alpha|+l=m, l\geq 1} a_{\alpha,l} \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_y^\alpha D_\tau^{l-1} v(y,\tau)\, \eta_\tau\, dS \\ &\quad - \sum_{|\alpha|=m} a_{\alpha,0} \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_y^{\alpha'} v(y,\tau)\, \eta_{\alpha'}\, dS + O(\epsilon).\end{aligned}$$

Since $L^\star v(y,\tau) = 0$ for all $(y,\tau) \neq (x,t)$ the theorem will follow once we show that

$$\lim_{\epsilon \to 0} \Big\{ \sum_{|\alpha|+l=m, l\geq 1} a_{\alpha,l} \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_y^\alpha D_\tau^{l-1} v(y,\tau)\, \eta_\tau\, dS + \sum_{|\alpha|=m} a_{\alpha,0} \int_{\partial B_\epsilon(x,t)} u(y,\tau) D_y^{\alpha'} v(y,\tau)\, \eta_{\alpha'}\, dS \Big\} = c u(x,t),$$

for some $c$ depending on $n$.

Since $v(y,\tau)=h(x-y,t-\tau)-k(x-y,t,\tau)$ and $k$ is regular for all $(y,\tau)$, we have to prove the limit above with $h$ replacing $v$.

We now proceed to prove this limit. Notice that there is no longer the need to emphasize the last variable, so we just write $h=h(y)$ and $u=u(y)$, and $y\in\mathbb{R}^n$.

Let $|\alpha|=2m-1$ and write

$$\begin{aligned}\int_{\partial B_\epsilon(x)} u(y)D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y)\\ &= \int_{\partial B_\epsilon(x)} (u(y)-u(x))D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y)\\ &\qquad + u(x)\int_{\partial B_\epsilon(x)} D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y).\end{aligned}$$

The first term goes to 0 as $\epsilon\to 0$, since $|u(y)-u(x)|\le C|x-y|$ and $|D^\alpha h(x-y)|\le \frac{C}{|x-y|^{n-1}}$. So, it is enough to show that

$$\lim_{\epsilon\to 0}\sum_{|\alpha|=2m} a_\alpha\int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y)=c_n.$$

Recall that $h_R\to h$ uniformly on compacts of $\mathbb{R}^n\setminus\{0\}$ and $D^\alpha h_R\to D^\alpha h$ uniformly on compacts of $\mathbb{R}^n\setminus\{0\}$ as $R\to\infty$, where $h_R(y)=\int\varphi(\frac{\xi}{R})\frac{e^{iy\cdot\xi}}{p(i\xi)}d\xi$.

Now,

$$\int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y)=\int_{\partial B_1(0)}\epsilon^{n-1}D^{\alpha-e_j}h(\epsilon y)y_j\,dS(y).$$

Let $\beta=\alpha-e_j$, where $|\alpha|=2m$, so $|\beta|=2m-1$. We have

$$\epsilon^{n-1}D^\beta h_R(\epsilon y)=\int\varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta e^{iy\cdot\xi}}{\epsilon^m p(\frac{i\xi}{\epsilon})}\,d\xi$$

and

$$\begin{aligned}\int_{\partial B_1(0)}\epsilon^{n-1}D^\beta h_R(\epsilon y)y_j\,dS(y) &= \int\varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}\int_{\partial B_1(0)}e^{iy\cdot\xi}y_j\,dS(y)d\xi\\ =\int\varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}\xi_j\int_{B_1(0)}e^{iy\cdot\xi}\,dyd\xi &= \int_{B_1(0)}\int\varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\alpha}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}\,d\xi dy.\end{aligned}$$

Therefore, for fixed $\epsilon$,

$$\begin{aligned}\sum_{|\alpha|=2m} a_\alpha\int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}\,dS(y)\\ &= \lim_{R\to\infty}\int_{B_1(0)}\int\varphi(\frac{\xi}{\epsilon R})\frac{p_{2m}(i\xi)}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}\,d\xi dy,\end{aligned}$$

where $p_{2m}$ is the principal part of $p$. We show that this last limit is independent of $\epsilon$. Note that $\frac{p_{2m}(i\xi)}{\epsilon^{2m}} = p_{2m}(\frac{i\xi}{\epsilon})$ and write

$$\int_{B_1(0)} \int \varphi(\frac{\xi}{\epsilon R}) \frac{p_{2m}(i\xi)}{\epsilon^{2m} p(\frac{i\xi}{\epsilon})} e^{iy\cdot\xi} d\xi dy$$
$$= \int_{B_1(0)} \int \varphi(\frac{\xi}{\epsilon R}) \frac{p_{2m}(\frac{i\xi}{\epsilon}) - p(\frac{i\xi}{\epsilon})}{p(\frac{i\xi}{\epsilon})} e^{iy\cdot\xi} d\xi dy$$
$$+ \int_{B_1(0)} \int \varphi(\frac{\xi}{\epsilon R}) e^{iy\cdot\xi} d\xi dy = A + B.$$

The first term goes to 0 and the second has a limit independent of $\epsilon$. To see this, let $p_{2m}(\frac{i\xi}{\epsilon}) - p(\frac{i\xi}{\epsilon}) = \tilde{p}(\frac{i\xi}{\epsilon})$, $\tilde{p}$ has degree $\leq 2m-1$. Write $A = \int_{|y|\leq \frac{1}{\epsilon R}} \dots dy + \int_{\frac{1}{\epsilon R}\leq |y| \leq 1} \dots dy = A_1 + A_2$. Note that

$$A_1 = \int_{|y|\leq 1} \int \varphi(\xi) \frac{\tilde{p}(iR\xi)}{p(iR\xi)} e^{iy\cdot\xi}\, d\xi dy \to 0 \quad \text{as } R \to \infty,$$
$$A_2 = \int_{1\leq |y| \leq \epsilon R} \int \varphi(\xi) \frac{\tilde{p}(iR\xi)}{p(iR\xi)} e^{iy\cdot\xi}\, d\xi dy$$
$$= \int_{1\leq |y| \leq \epsilon R} \frac{1}{|y|^{2l}} \int (-\Delta_\xi)^l (\varphi(\xi) \frac{\tilde{p}(iR\xi)}{p(iR\xi)}) e^{iy\cdot\xi}\, d\xi dy \to 0 \quad \text{as } R \to \infty,$$

where in the last limit we note that setting $q(\xi) = \frac{\tilde{p}(i\xi)}{p(i\xi)}$ we have, for any $\alpha$, $|D^\alpha q(\xi)| \leq \frac{C}{1+|\xi|^{|\alpha|+1}}$, and hence $|D^\alpha(q(R\xi))| \leq C \frac{R^{|\alpha|}}{1+(R|\xi|)^{|\alpha|+1}}$.

The same argument shows that the limit of $B$ is

$$\int_{|y|\leq 1} \int \varphi(\xi) e^{iy\cdot\xi}\, d\xi dy + \int_{1\leq |y|} \frac{1}{|y|^{2l}} \int (-\Delta)^l (\varphi(\xi)) e^{iy\cdot\xi}\, d\xi dy,$$

where $l$ is any integer such that $2l \geq n+1$. This finishes the proof of the theorem. □

In the next theorem we solve the Dirichlet problem for the upper half space $\mathbb{R}^{n+1}_+$ with $m$ conditions at the boundary $t = 0$. The theorem is also used in combination with Theorem 3.7 to prove a priori estimates in the space $C^{2m+\delta}$.

**Theorem 3.8.** *Let $f \in C^\delta(\mathbb{R}^{n+1}_+)$ and define $u(x,t) = \int_0^\infty \int f(y,\tau) g(x-y,t,\tau)\, dy d\tau$. Then,*

$$u \in C^{2m+\delta}(\mathbb{R}^{n+1}_+) \quad \textit{and} \quad Lu(x,t) = f(x,t),$$
$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0,\ \dots,\ \frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) = 0.$$

*Moreover,*

$$|u|_{2m+\delta;\mathbb{R}^{n+1}_+} \leq C|f|_{\delta;\mathbb{R}^{n+1}_+}.$$

*Proof.* The boundary conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0, \dots, \frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) = 0$ follow immediately from the analogous properties of $g$.

To prove $Lu = f$, write $u = v - w$, where $v(x,t) = \int_0^\infty \int f(y,\tau)h(x-y,t-\tau)\,dyd\tau$ and $w(x,t) = \int_0^\infty \int f(y,\tau)k(x-y,t,\tau)\,dyd\tau$. Clearly, $Lw(x,t) = 0$. We show that $Lv(x,t) = f(x,t)$. Fix $(x_0,t_0)$ and $\epsilon > 0$ such that $B_{3\epsilon}(x_0,t_0) \subseteq \mathbb{R}_+^{n+1}$ and $\eta \in C_0^\infty(B_{3\epsilon}(x_0,t_0))$, with $\eta = 1$ on $B_{2\epsilon}(x_0,t_0)$.

Write $v(x,t) = \int_0^\infty \int \eta(y,\tau)f(y,\tau)h(x-y,t-\tau)\,dyd\tau + \int_0^\infty \int (1-\eta(y,\tau))f(y,\tau)\cdot h(x-y,t-\tau)\,dyd\tau = v_1(y,\tau) + v_2(y,\tau)$. Clearly, $Lv_2 = 0$ on $B_\epsilon(x_0,t_0)$.

Note that $\bar{f} = \eta f \in C^\delta(\mathbb{R}^{n+1})$ and it follows (see the argument below) that for $1 \leq |\alpha| + l \leq 2m$,

$$D_x^\alpha D_t^l v_1(x,t) = \int_{\mathbb{R}^{n+1}} (\bar{f}(y,\tau) - \bar{f}(x,t)) D_x^\alpha D_t^l h(x-y,t-\tau)\,dyd\tau.$$

Therefore,

$$\begin{aligned} Lv_1(x,t) &= a_{0,0}v_1(x,t) \\ &\quad + \int_{\mathbb{R}^{n+1}} (\bar{f}(y,\tau) - \bar{f}(x,t)) \sum_{1\leq|\alpha|+l} a_{\alpha,l} D_x^\alpha D_t^l h(x-y,t-\tau)\,dyd\tau \\ &= \int_{\mathbb{R}^{n+1}} (\bar{f}(y,\tau) - \bar{f}(x,t)) L_{x,t}(h(x-y,t-\tau))\,dyd\tau \\ &\quad + a_{0,0}\bar{f}(x,t) \int_{\mathbb{R}^{n+1}} h(x-y,t-\tau)\,dyd\tau = \bar{f}(x,t). \end{aligned}$$

We proceed to show that $|u|_{2m+\delta;\mathbb{R}_+^{n+1}} \leq C|f|_{\delta;\mathbb{R}_+^{n+1}}$. First we prove a claim.

For $|\alpha| + l = 2m$ such that $|\alpha| \geq 1$ we have

$$D_x^\alpha D_t^l u(x,t) = \int_{\mathbb{R}^{n+1}} (f(y,\tau) - f(x,t)) D_x^\alpha D_t^l (g(x-y,t,\tau)\,dyd\tau.$$

Write $\alpha = \beta + e_j$. Since $|\beta| + l = 2m - 1$, it follows that

$$D_x^\alpha D_t^l u(x,t) = \int_{\mathbb{R}^{n+1}} f(y,\tau) D_x^\beta D_t^l (g(x-y,t,\tau))\,dyd\tau.$$

Let $w_\epsilon(x,t) = \int_{\mathbb{R}^{n+1}} f(y,\tau)\eta_\epsilon(x-y,t-\tau) D_x^\beta D_t^l (g(x-y,t,\tau))\,dyd\tau$, where now $\eta_\epsilon(x-y,t-\tau) = 0$ for $|x-y|^2 + (t-\tau)^2 \leq \epsilon^2$ and $\eta_\epsilon(x-y,t-\tau) = 1$ for $|x-y|^2 + (t-\tau)^2 \geq 4\epsilon^2$. Then

$$w_\epsilon \to D_x^\beta D_t^l u \quad \text{uniformly in } \mathbb{R}_+^{n+1} \text{ as } \epsilon \to 0,$$

and

$$\begin{aligned} \frac{\partial w_\epsilon(x,t)}{\partial x_j} &= \int_{\mathbb{R}^{n+1}} f(y,\tau) \frac{\partial}{\partial x_j} (\eta_\epsilon(x-y,t-\tau) D_x^\beta D_t^l (g(x-y,t,\tau)))\,dyd\tau \\ &= \int_{\mathbb{R}^{n+1}} (f(y,\tau) - f(x,t)) \frac{\partial}{\partial x_j} (\eta_\epsilon(x-y,t-\tau) D_x^\beta D_t^l (g(x-y,t,\tau)))\,dyd\tau \\ &\quad - f(x,t) \int_{\mathbb{R}^{n+1}} \frac{\partial}{\partial y_j} (\eta_\epsilon(x-y,t-\tau) D_x^\beta D_t^l (g(x-y,t,\tau)))\,dyd\tau \\ &\qquad \to \int_{\mathbb{R}^{n+1}} (f(y,\tau) - f(x,t)) D_x^\alpha D_t^l (g(x-y,t,\tau)\,dyd\tau \end{aligned}$$

uniformly in $\mathbb{R}^{n+1}_+$ as $\epsilon \to 0$, which proves the claim.

Now, let $|\alpha|+l = 2m$ and $|\alpha| \geq 1$ and write $\alpha = \beta + e_j$. Let $(x,t)$, $(\bar{x},\bar{t}) \in \mathbb{R}^{n+1}_+$. Let $(\hat{x},\hat{t}) = \frac{1}{2}(x+\bar{x}, t+\bar{t})$. Let $r = 2((x-\bar{x})^2+(t-\bar{t})^2)^{1/2}$ and $B^+ = B_r(\hat{x},\hat{t}) \cap \mathbb{R}^{n+1}_+$. We have

$$\begin{aligned}
&D^\alpha_x D^l_t u(x,t) - D^\alpha_x D^l_t u(\bar{x},\bar{t})\\
&\quad = \int_{B^+} (f(y,\tau) - f(x,t)) D^{|\alpha|+l}_{x,t} g(x-y,t,\tau)\, dyd\tau\\
&\quad - \int_{B^+} (f(y,\tau) - f(\bar{x},\bar{t})) D^{|\alpha|+l}_{x,t} g(\bar{x}-y,\bar{t},\tau)\, dyd\tau\\
&\quad + \int_{\mathbb{R}^{n+1}_+ \setminus B^+} (f(y,\tau) - f(\bar{x},\bar{t})) \big(D^{|\alpha|+l}_{x,t} g(x-y,t,\tau) - D^{|\alpha|+l}_{x,t} g(\bar{x}-y,\bar{t},\tau)\big)\, dyd\tau\\
&\quad + (f(\bar{x},\bar{t}) - f(x,t)) \int_{\mathbb{R}^{n+1}_+ \setminus B^+} D^{|\alpha|+l}_{x,t} g(x-y,t,\tau)\, dyd\tau\\
&\quad = A - B + C + (f(\bar{x},\bar{t}) - f(x,t))D.
\end{aligned}$$

We estimate these integrals as follows:

$$\begin{aligned}
&|A| \leq C[f]_\delta \int_{B^+} \frac{(|x-y|^2+(t-\tau)^2)^{\delta/2}}{(|x-y|^2+(t-\tau)^2)^{(n+1)/2}}\, dyd\tau \leq C[f]_\delta r^\delta,\\
&|B| \leq C[f]_\delta r^\delta,\\
&C \leq C[f]_\delta ((|\bar{x}-x|^2+(\bar{t}-t)^2)^{1/2}) \int_{\mathbb{R}^{n+1}_+ \setminus B^+} \frac{(|\bar{x}-y|^2+(\bar{t}-\tau)^2)^{\delta/2}}{(|\tilde{x}-y|^2+(\tilde{t}-\tau)^2)^{(n+2)/2}}\, dyd\tau\\
&\quad \leq C[f]_\delta r^\delta.
\end{aligned}$$

To estimate $D$, we use that $|\alpha| \geq 1$ to write

$$\begin{aligned}
D = \int_{\mathbb{R}^{n+1}_+ \setminus B^+} &\frac{\partial}{\partial x_j} (D^{\beta+l}_{x,t} g(x-y,t,\tau))\, dyd\tau\\
&= \int_{\mathbb{R}^{n+1}_+ \setminus B^+} \frac{\partial}{\partial x_j} (D^{\beta+l}_{x,t} h(x-y,t-\tau))\, dyd\tau\\
&\quad + \int_{\mathbb{R}^{n+1}_+ \setminus B^+} \frac{\partial}{\partial x_j} (D^{\beta+l}_{x,t} k(x-y,t,\tau))\, dyd\tau = D_1 + D_2.
\end{aligned}$$

For the first term we have:

$$\begin{aligned}
|D_1| &= \left| \int_{\mathbb{R}^{n+1}_+ \setminus B^+} \frac{\partial}{\partial y_j} (D^{\beta+l}_{x,t} h(x-y,t-\tau))\, dyd\tau \right|\\
&= \left| \int_{(\partial B^+) \setminus \{\tau=0\}} D^{\beta+l}_{x,t} h(x-y,t-\tau)\, dS(y,\tau) \right| \leq C.
\end{aligned}$$

Similarly, $|D_2| \leq C$. This proves that $[D^\alpha_x D^l_t u]_{\delta;\mathbb{R}^{n+1}_+} \leq C[f]_{\delta;\mathbb{R}^{n+1}_+}$ for $|\alpha|+l = 2m$ and $|\alpha| \geq 1$.

To estimate $D_t^{2m}u$, note that

$$D_t^{2m}u(x,t) = f(x,t) - \sum_{|\alpha|+l=2m;|\alpha|\geq 1} a_{\alpha,l}D_x^\alpha D_t^l u(x,t) - \sum_{|\alpha|+l\leq 2m-1} a_{\alpha,l}D_x^\alpha D_t^l u(x,t),$$

which implies that $[D_t^{2m}u]_{\delta;\mathbb{R}_+^{n+1}} \leq C[f]_\delta + |u|_{2m-1+\delta}$. Because $g \in L^1(\mathbb{R}_+^{n+1})$, we also have $|u|_{0;\mathbb{R}_+^{n+1}} \leq C|f|_{0;\mathbb{R}_+^{n+1}}$. By interpolation, it follows that $|u|_{2m+\delta;\mathbb{R}_+^{n+1}} \leq C|f|_{\delta;\mathbb{R}_+^{n+1}}$. This finishes the proof of the theorem. Note that we have used in an essential way the estimates of lemmas 3.3 and 3.4. □

In order to consider operators with variable coefficients and apply the freezing coefficient method we need the following observation, which we state as a theorem.

**Theorem 3.9.** *Let $u \in C^{2m+\delta}(\mathbb{R}_+^{n+1})$ with*

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0,\ \ldots,\ \frac{\partial^{m-1}u}{\partial t^{m-1}}(x,0) = 0.$$

*Let $(x_0,t_0) \in (\mathbb{R}_+^{n+1})$ be any point, $R > 0$, and $\eta \in C_0^\infty(B_{2R}(x_0,t_0))$, with $\eta = 1$ in $B_R(x_0,t_0)$. Then*

$$|u|_{2m+\delta;B_R^+(x_0,t_0)} \leq C|L(u\eta)|_{\delta;B_{2R}^+(x_0,t_0)}.$$

*Proof.* The proof follows from Theorems 3.7 and 3.8. Since $u\eta$ satisfies the same hypothesis as $u$, we have

$$|u|_{2m+\delta;B_R^+(x_0,t_0)} \leq |u\eta|_{2m+\delta;\mathbb{R}_+^{n+1}} \leq C|L(u\eta)|_{\delta;\mathbb{R}_+^{n+1}} = C|L(u\eta)|_{\delta;B_{2R}^+(x_0,t_0)},$$

thus finishing the proof. □

## 4. Variable coefficients

In this section, we apply the constant coefficient estimates to prove Schauder estimates for variable Hölder coefficients. There will be a restriction on the size of the Hölder constant of the coefficient to get full estimates. We make a slight change in notation: Take $x \in \mathbb{R}_+^n$ , write $x = (x', x_n)$ and consider $u \in C^{2m+\delta}(\mathbb{R}_+^n)$, such that $u(x',0) = 0$, $\frac{\partial u(x',0)}{\partial x_n} = 0, \ldots, \frac{\partial^{m-1}u(x',0)}{\partial x_n^{m-1}} = 0$.

We consider an operator $Lu(x) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u(x)$, where we assume that $a_\alpha(x)$ are complex valued functions which are Hölder continuous. Let $K_0 = \sum_{|\alpha|\leq 2m} |a_\alpha|_{0;\mathbb{R}_+^n}$ and $K_\delta = \sum_{\alpha|\leq 2m}[a_\alpha]_{\delta;\mathbb{R}_+^n}$. Let $p(x,i\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(i\xi)^\alpha$ and assume the ellipticity condition $|p(x,i\xi)| \geq \lambda(1+|\xi|^{2m})$, for all $x \in \mathbb{R}_+^n$ and $\xi \in \mathbb{R}^n$.

As before, let $\{\lambda_1^-(x,\xi'),\ldots,\lambda_m^-(x,\xi')\}$ denote roots of the equation $p(x,i\xi',z) = 0$ with negative real part and $\{\lambda_1^+(x,\xi'),\ldots,\lambda_m^+(x,\xi')\}$ denote roots of the equation $p(x,i\xi',z) = 0$ with positive real part.

In order to apply the estimates for the constant coefficient case, we assume that if $\lambda(x_0,\xi')$ denotes any root of $p(x_0,i\xi',z)=0$, then

$$|D^\alpha_\xi(\lambda(x_0,\xi')|\le C(1+|\xi|)^{1-|\alpha|}$$

with a constant that depends only on $n,\alpha,\lambda$ and $K_0$.

Finally, for $x_0\in\mathbb{R}^n_+$, let $L_0(x)=\sum_{|\alpha|\le 2m}a_\alpha(x_0)D^\alpha u(x)$.

**Theorem 4.1.** *There exists a constant $\gamma$ depending on $K_0$ such that if $K_\delta\le\gamma$, then $|u|_{2m+\delta;\mathbb{R}^n_+}\le C|Lu|_{\delta;\mathbb{R}^n_+}$, where $C$ is a universal constant.*

*Proof.* Let $x_0$ and $\bar{x}$ be points in $\mathbb{R}^n_+$ and $R\ge 1$ to be chosen. Fix $|\alpha|=2m$. Let $\eta=1$ in $B_R(x_0)$ and $\eta\in C_0^\infty(B_{2R}(x_0))$.

If $\bar{x}\notin B_R(x_0)$, then

$$\frac{|D^\alpha u(\bar{x})-D^\alpha u(x_0|}{|\bar{x}-x_0|^\delta}\le\frac{2}{R}[u]_{2m;\mathbb{R}^n_+}.$$

If $\bar{x}\in B_R(x_0)$, then

$$\frac{|D^\alpha u(\bar{x})-D^\alpha u(x_0|}{|\bar{x}-x_0|^\delta}\le[u\eta]_{2m+\delta;B^+_R(x_0)}\le C|L_0(u\eta)|_{\delta;B^+_{2R}(x_0)},$$

by Theorem 3.9 applied to the constant coefficient operator $L_0$.

We have

$$L_0(u\eta)(x)=\eta(x)L_0u(x)+\sum_{|\alpha|\le 2m}\sum_{|\beta|+|\gamma|=|\alpha|;|\gamma|\ge 1}a_{\gamma;\beta}(x_0)D^\beta u(x)D^\gamma\eta(x).$$

We have the estimates

$$|L_0(u\eta)|_{0;B^+_{2R}(x_0)}\le|L_0(u)|_{0;B^+_{2R}(x_0)}+\frac{CK_0}{R}|u|_{2m;\mathbb{R}^n_+}$$

and

$$[L_0(u\eta)]_{\delta;B^+_{2R}(x_0)}\le[L_0(u)]_{\delta;B^+_{2R}(x_0)}+\frac{CK_0}{R^\delta}|u|_{2m;\mathbb{R}^n_+}.$$

These estimates follow easily from the following four facts:

- $[uv]_\delta\le|u|_0[v]_\delta+|v|_0[u]_\delta$;
- $|D^\gamma\eta|_0\le\frac{C}{R^{|\gamma|}}$ and $[D^\gamma\eta]_\delta\le\frac{C}{R^{|\gamma|+\delta}}$;
- interpolation to absorbe $[D^\beta u]_\delta\le C|u|_{2m}$, for $|\beta|\le 2m-1$;
- $R\ge 1$.

To continue, write $L_0u(x)=Lu(x)+(L_0-L)u(x)$, so $[L_0u]_{\delta;B^+_{2R}(x_0)}\le[Lu]_{\delta;\mathbb{R}^n_+}+[(L_0-L)u]_{\delta;B^+_{2R}(x_0)}$ and

$$\begin{aligned}[(L_0-L)u]_{\delta;B^+_{2R}(x_0)}&\le\sum_{|\alpha|\le 2m}[a_\alpha(.)-a_\alpha(x_0))D^\alpha u]_{\delta;B^+_{2R}(x_0)}\\&\le\sum_{|\alpha|\le 2m}[a_\alpha(.)-a_\alpha(x_0))]_{\delta;B^+_{2R}(x_0)}|D^\alpha u|_{0;B^+_{2R}(x_0)}\\&\quad+\sum_{|\alpha|\le 2m}|a_\alpha(.)-a_\alpha(x_0))|_{0;B^+_{2R}(x_0)}[D^\alpha u]_{\delta;B^+_{2R}(x_0)}\\&\le C(K_\delta|u|_{2m;\mathbb{R}^n_+}+K_\delta R^\delta|u|_{2m+\delta;\mathbb{R}^n_+}).\end{aligned}$$

Similarly,

$$|L_0u|_{0;B^+_{2R}(x_0)} \leq |Lu|_{0;\mathbb{R}^n_+} + |(L_0 - L)u|_{0;B^+_{2R}(x_0)}$$

and

$$|(L_0 - L)u|_{0;B^+_{2R}(x_0)} \leq \sum_{|\alpha|\leq 2m} |a_\alpha(.) - a_\alpha(x_0)|_{0;B^+_{2R}(x_0)}|D^\alpha u|_{0;\mathbb{R}^n_+} \leq CK_\delta R^\delta |u|_{2m;\mathbb{R}^n_+}.$$

Combining estimates, we obtain

$$|L_0(u\eta)|_{\delta;B^+_{2R}(x_0)} \leq |Lu|_{\delta;\mathbb{R}^n_+} + C(K_\delta + \frac{K_0}{R^\delta})|u|_{2m;\mathbb{R}^n_+} + CK_\delta R^\delta |u|_{2m+\delta;\mathbb{R}^n_+}.$$

Now, take sup over $\bar{x}$, $x_0 \in \mathbb{R}^n_+$ and maximum over $|\alpha| = 2m$, to obtain

$$[u]_{2m+\delta;\mathbb{R}^n_+} \leq C|Lu|_{\delta;\mathbb{R}^n_+} + C(\frac{K_0}{R^\delta} + K_\delta(1 + R^\delta))|u|_{2m+\delta;\mathbb{R}^n_+} \tag{4}$$

with a constant $C$ that depends only on $n$, $\lambda$, $\delta$ and $K_0$.

Also, for $x_0 \in \mathbb{R}^n_+$ and $R \geq 1$, we have

$$\begin{aligned}|u|_{0;B^+_R(x_0)} &\leq |u\eta|_{0;\mathbb{R}^n_+} \leq C|L_0(u\eta)|_{0;B^+_{2R}(x_0)} \leq C|L_0(u)|_{0;B^+_{2R}(x_0)} + \frac{C}{R}|u|_{2m;\mathbb{R}^n_+}\\ &\leq C|Lu|_{0;\mathbb{R}^n_+} + C(K_\delta R^\delta + \frac{1}{R})|u|_{2m+\delta;\mathbb{R}^n_+}.\end{aligned}$$

Taking sup over $x_0 \in \mathbb{R}^n_+$, with $R \geq 1$ fixed, we get

$$|u|_{0;\mathbb{R}^n_+} \leq C|Lu|_{0;\mathbb{R}^n_+} + C(K_\delta R^\delta + \frac{1}{R})|u|_{2m+\delta;\mathbb{R}^n_+}. \tag{5}$$

By (4) and (5) and interpolation we have the estimate

$$|u|_{2m+\delta;\mathbb{R}^n_+} \leq C|Lu|_{\delta;\mathbb{R}^n_+} + C(\frac{1}{R^\delta} + K_\delta(1 + R^\delta))|u|_{2m+\delta;\mathbb{R}^n_+}, \tag{6}$$

with a constant $C$ that depends only on $n$, $\lambda$, $\delta$ and $K_0$.

We now choose $R \geq 1$ large enough so that $C\frac{1}{R^\delta} \leq \frac{1}{4}$ and then $\gamma$ small enough so that $C\gamma(1 + R^\delta) \leq \frac{1}{4}$. Then, if $K_\delta \leq \gamma$, from (6) we get

$$|u|_{2m+\delta;\mathbb{R}^n_+} \leq C|Lu|_{\delta;\mathbb{R}^n_+}.$$

□

Notice that in the theorem above we can take $\gamma$ of the form

$$\gamma = \frac{1}{4C(1 + 4C)} \tag{7}$$

for some constant $C$ that depends only on $n$, $\lambda$, $\delta$ and $K_0$.

In order to apply the continuity method to solve the Dirichlet problem we need estimates with no restriction on the coefficients. We can overcome this difficulty by scaling each term as follows.

For $s > 0$ let

$$L_su(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)s^{2m-|\alpha|}D^\alpha u(x)$$

and

$$p_s(x, i\xi) = \sum_{|\alpha|\leq 2m} a_\alpha(x)s^{2m-|\alpha|}(i\xi)^\alpha = s^{2m}p(x, \frac{i\xi}{s});$$

hence $|p_s(x,i\xi)| \geq \lambda(s^{2m}+|\xi|^{2m})$.

It follows that $p_s(x,i\xi',z)=0$ has $m$ roots with negative real part for all $x \in \mathbb{R}^n_+$, for all $\xi' \in \mathbb{R}^{n-1}$ and for all $s>0$.

**Theorem 4.2.** *There exists $s_0$ depending on $K_\delta$ and $K_0$ so that if $s \geq s_0$, then $|u|_{2m+\delta;\mathbb{R}^n_+} \leq C|L_s u|_{\delta;\mathbb{R}^n_+}$. The constant $C$ depends on $s$.*

*Proof.* Let $v(x)=u(\frac{x}{\lambda})$ and $\tilde{a}_\alpha(x)=a_\alpha(\frac{x}{s})$. Note that

$$\tilde{L}v(x)=\sum_{|\alpha|\leq 2m}\tilde{a}_\alpha(x)D^\alpha v(x)=s^{-2m}L_s u(\frac{x}{s}):=\tilde{f}_s(x).$$

It follows that $\tilde{K_0}=K_0$ and $\tilde{K_\delta}=s^{-\delta}K_\delta$. Therefore, the constant $C$ in Theorem 4.1 is the same.

In order to apply Theorem 4.1 to $v$, we need, according to (7), that $\tilde{K_\delta} \leq \frac{1}{4C(1+4C)}$ which amounts to $\frac{K_\delta}{s^\delta} \leq \frac{1}{4C(1+4C)}$ and so we take $s_0=(4CK_\delta(1+4C))^{\frac{1}{\delta}}$, where we emphasize that $C$ depends only on $n$, $\lambda$, $\delta$, and $K_0$.

An application of Theorem 4.1 to $v$ gives, for $s \geq s_0$, $|v|_{2m+\delta;\mathbb{R}^n_+} \leq C|\tilde{L}v|_{\delta;\mathbb{R}^n_+}$ with $C$ universal. Therefore, $|u|_{2m+\delta;\mathbb{R}^n_+} \leq C|L_s u|_{\delta;\mathbb{R}^n_+}$. □

A consequence of the a priori estimate of Theorem 4.2 following a standard application of the continuity method is this theorem on solvability of the Dirichlet problem:

**Theorem 4.3.** *Let $s_0$ be defined as in Theorem 4.2 and $s \geq s_0$. Then, given $f \in C^\delta(\mathbb{R}^n_+)$, there exists a (unique) solution $u \in C^{2m+\delta}(\mathbb{R}^n_+)$ of $L_s u=f$ in $\mathbb{R}^n_+$ such that $u(x',0)=0$, $\frac{\partial u(x',0)}{\partial x_n}=0, \ldots, \frac{\partial^{m-1}u(x',0)}{\partial x_n^{m-1}}=0$.*

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