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FRACTIONAL TYPE INTEGRAL OPERATORS OF VARIABLE ORDER

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Abstract. We study the boundedness on the variable Lebesgue spaces of the operators of variable order

$$T_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(y)} \ldots |y - A_m x|^{-\alpha_m(y)} f(y)\, dy$$

and

$$U_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(x)} \ldots |y - A_m x|^{-\alpha_m(x)} f(y)\, dy,$$

where $\alpha_i : \Omega \subset \mathbb{R}^n \to (0, n)$ are positive measurable functions, for $i = 1, \ldots, m$ ($m \geq 2$), such that $\alpha_1(x) + \cdots + \alpha_m(x) = n - \alpha(x)$ with $0 \leq \alpha(x) < n$ for all $x \in \Omega$, and the A_i's are $n \times n$ invertible real matrices such that $A_i - A_j$ is invertible if $i \neq j$.

1. Introduction

The Lebesgue spaces with variable exponents are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. In the last years many authors have extended the machinery of classical harmonic analysis to these spaces. The basic properties were developed in [23] and [11], and subsequently compiled —with many others results— in the books [6] and [4].

Once the basic properties are established, the second step was to study the boundedness of the Hardy–Littlewood maximal operator and the fractional Hardy–Littlewood maximal operator on the variable Lebesgue spaces; see Theorem 1 and Theorem 2 below and references therein. With these tools one can then study other operators such as convolution operators, singular integral operators, fractional type operators, and Riesz potentials. Several results about these operators can be found in the papers [1, 2, 7, 10, 17, 18, 20, 21].

Besides the intrinsic mathematical interest, the variable Lebesgue spaces are also important for their many applications in fluid dynamics, elasticity theory and differential equations with non-standard growth conditions (see [5, 19] and [6, Chapter III]).

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The Riesz potential can be generalized by replacing the constant exponent with a variable exponent $\alpha(\cdot)$.

Given $\Omega \subset \mathbb{R}^n$, let $\alpha : \Omega \to (0, n)$ be a measurable function. The variable Riesz potential is defined by

$$I_{\Omega}^{\alpha(x)} f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha(x)}}\, dy. \tag{1}$$

These operators were introduced by S. Samko (see [20, 21]); they have applications to the theory of fractional differentiation and integration. If Ω is bounded, in [20] S. Samko proved a Sobolev type theorem (see Theorem 3 below). A similar result on $\Omega = (0, 1)$ was proved by Edmunds and Meskhi in [7]. Samko in [22] asked whether these results could be extended to unbounded domains. P. Hästö in [10] gives an example whereby there is no point in investigating the global behavior of the potential $I_{\mathbb{R}^n}^{\alpha(\cdot)}$.

For the above reasons, we find it interesting to study fractional type operators analogous to variable Riesz potentials.

Given an open set $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}$, $m \geq 2$, let $A_1, \ldots, A_m$ be $n \times n$ invertible real matrices such that $A_i - A_j$ are invertible for each $1 \leq i \neq j \leq m$, and $A_i(\Omega) = \Omega$ for each $1 \leq i \leq m$. We consider measurable functions $\alpha_i : \Omega \to (0, n)$, $1 \leq i \leq m$, and $\alpha : \Omega \to [0, n)$ such that

$$\alpha_1(x) + \cdots + \alpha_m(x) = n - \alpha(x), \quad x \in \Omega.$$

We now define the positive operators T_Ω and U_Ω by

$$T_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(y)} \ldots |y - A_m x|^{-\alpha_m(y)} f(y)\, dy \tag{2}$$

and

$$U_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(x)} \ldots |y - A_m x|^{-\alpha_m(x)} f(y)\, dy, \quad x \in \Omega. \tag{3}$$

Also for $s \in \mathbb{N}$, $s < m$, we define

$$T_{\Omega,s} f(x) = \int_\Omega K_s(x, y) f(y)\, dy, \tag{4}$$

where K_s is the kernel given by

$$K_s(x, y) = \left(\prod_{i=1}^{s} |y - A_i x|^{-\alpha_i(x)} \right) \left(\prod_{i=s+1}^{m} |y - A_i x|^{-\alpha_i(y)} \right). \tag{5}$$

If the functions α_i are constant for all $1 \leq i \leq m$ and $0 \leq \alpha < n$, then $T_\Omega = U_\Omega = T_{\Omega,s}$. In [16], [17] and [18] we obtain the $H^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$ boundedness, the $L^{p(\cdot)}(\mathbb{R}^n)$-$L^{q(\cdot)}(\mathbb{R}^n)$ boundedness and the $H^{p(\cdot)}(\mathbb{R}^n)$-$L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of such operator, for certain variable exponents $p(\cdot)$ and $q(\cdot)$, where $H^p(\mathbb{R}^n)$ are the classical Hardy spaces and $H^{p(\cdot)}(\mathbb{R}^n)$ are the variable Hardy spaces defined by E. Nakai and Y. Sawano in [12].

We recall that a weight w is a locally integrable and non negative function. The Muckenhoupt class A_p, $1<p<\infty$, is defined as the class of weights w such that

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} \right] < \infty.$$

For $p=1$, A_1 is the class of weights w satisfying that there exists $c>0$ such that $Mf(x) \leq cw(x)$ a.e. $x \in \mathbb{R}^n$, where M is the Hardy–Littlewood maximal function. In [13] and [14] M. S. Riveros and M. Urciuolo obtain the $L^p(\mathbb{R}^n, w)$-$L^q(\mathbb{R}^n, w)$ boundedness of $T_{\mathbb{R}^n}$ for certain Muckenhoupt weights w.

In this paper we extend many of these results to the fractional type integral operators of variable order defined in (2), (3) and (4). We start in Section 2 with some definitions and well known results necessary for Sections 4 and 5. In Section 3 we show that $T_{\mathbb{R}^n}$ and $U_{\mathbb{R}^n}$ are of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$, in the case $\alpha(x) \equiv 0$ for all $x \in \mathbb{R}^n$. Next we give an example that shows that $T_{\mathbb{R}^n,s}$ is not bounded on $L^p(\mathbb{R}^n)$, but if Ω is a bounded open set we obtain the boundedness of $T_{\Omega,s}$ on $L^p(\Omega)$, with certain additional hypotheses. We also obtain the boundedness of $T_{\mathbb{R}^n}$ on $L^p(\mathbb{R}^n, w)$ for certain Muckenhoupt weights w. These results are used in Section 4 to prove the boundedness on $L^{p(\cdot)}(\mathbb{R}^n)$ of the operators $T_{\mathbb{R}^n}$ and $U_{\mathbb{R}^n}$ and the $H^{p(\cdot)}(\mathbb{R}^n)$-$L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of $U_{\mathbb{R}^n}$. Finally, in Section 5 we obtain the $L^{p(\cdot)}(\Omega)$-$L^{q(\cdot)}(\Omega)$ boundedness of the operators T_Ω and U_Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded set and $0<\alpha(x)<n$, for $x \in \Omega$.

2. Preliminaries

Given a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : \Omega \to (0,\infty)$, let $L^{p(\cdot)}(\Omega)$ denote the set of all measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

As a special case of the theory of Nakano and Luxemburg, we see that $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a quasi-normed space. These spaces are known as *variable exponent spaces* and are a generalization of classical Lebesgue spaces $L^p(\Omega)$. The Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. The first step was to determine sufficient conditions on $p(\cdot)$ for the boundedness on $L^{p(\cdot)}$ of the Hardy–Littlewood maximal operator.

Given a function $p(\cdot) : \Omega \to (0, \infty)$ we say that $p(\cdot)$ is locally log-Hölder continuous, and denote this by $p(\cdot) \in LH_0(\Omega)$, if there exists a positive constant C_0 such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x-y|}, \quad \forall\, x, y \in \Omega,\ |x-y| < \frac{1}{2}.$$

We say that $p(\cdot)$ is log-Hölder continuous at infinity, and denote this by $p(\cdot) \in LH_\infty(\Omega)$, if there exists a positive constant C_∞ such that

$$|p(x) - p(y)| \leq \frac{C_\infty}{\log(e + |x|)}, \quad \forall\, x, y \in \Omega,\ |y| \geq |x|\,.$$

In [3], D. Cruz Uribe, A. Fiorenza and C.J. Neugebauer proved the following result. To state this, let $p_- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

Theorem 1. *If $\Omega \subset \mathbb{R}^n$ is an open set and $p(\cdot) \in LH_0 \cap LH_\infty(\Omega)$ is such that $1 < p_- \leq p_+ < \infty$, then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.*

In [2], C. Capone, D. Cruz Uribe and A. Fiorenza generalize Theorem 1 to the fractional maximal operator $M_\alpha f$, where

$$M_\alpha f(x) = \sup_B \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_{B\cap\Omega} |f(y)|\, dy,$$

$0 < \alpha < n$, and the supremun is taken over all balls B containing x. They obtain the following result.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $0 < \alpha < n$, and let $p(\cdot) \in LH_0 \cap LH_\infty(\Omega)$ be such that $1 < p_- \leq p_+ < \frac{n}{\alpha}$. If $q(\cdot) : \Omega \to [1, \infty)$ is defined by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \Omega,$$

then the fractional maximal operator M_α is bounded from $L^{p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$.

As a consequence of this last result the authors also obtain the same norm inequality for the fractional integral operator of order α, $0 < \alpha < n$, given by

$$I_\alpha f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

In [20] S. Samko introduces and studies potential type operators of variable order given by (1) and obtains the following Sobolev type theorem.

Theorem 3 ([1, 20])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p(\cdot) \in LH_0(\Omega)$. If $\sup_{x\in\Omega} p(x)\alpha(x) < n$ and $\inf_{x\in\Omega} \alpha(x) > 0$, then $I_\Omega^{\alpha(\cdot)}$ is a bounded operator from $L^{p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$ for $x \in \Omega$.*

Recall that P. Hästö in [10] showed that the above result is false if $\Omega = \mathbb{R}^n$.

In the paper [12], E. Nakai and Y. Sawano give a variety of distinct approaches, based on differing definitions, all leading to the same notion of the variable Hardy space $H^{p(\cdot)}$.

We recall the definition and the atomic decomposition of the Hardy spaces with variable exponents.

Topologize $\mathcal{S}(\mathbb{R}^n)$ by the collection of semi-norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) = \sum_{|\beta|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \left|\partial^\beta \varphi(x)\right|,$$

for each $N \in \mathbb{N}$. We set $\mathcal{F}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \leq 1\}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We denote by $\mathcal{M}_{\mathcal{F}_N}$ the grand maximal operator given by

$$\mathcal{M}_{\mathcal{F}_N} f(x) = \sup_{t>0} \sup_{\varphi\in\mathcal{F}_N} \left|\left(t^{-n}\varphi(t^{-1}\cdot) * f\right)(x)\right|,$$

where N is a large and fixed integer.

The Hardy space with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ is the set of all $f \in S'(\mathbb{R}^n)$ for which $\mathcal{M}_{\mathcal{F}_N} f \in L^{p(\cdot)}(\mathbb{R}^n)$. In this case we define $\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}_{\mathcal{F}_N} f\|_{p(\cdot)}$.

Definition 4 (($p(\cdot), p_0, d$)-atom)**.** Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, $0 < p_- \leq p_+ < p_0 \leq \infty$ and $p_0 \geq 1$. Fix an integer $d \geq d_{p(\cdot)} = \min\{l \in \mathbb{N}\cup\{0\} : p_-(n+l+1) > n\}$. A function a on $\mathbb{R}^n$ is called a $(p(\cdot), p_0, d)$-atom if there exists a cube Q such that

(1) $\operatorname{supp}(a) \subset Q$,
(2) $\|a\|_{p_0} \leq \frac{|Q|^{\frac{1}{p_0}}}{\|\chi_Q\|_{p(\cdot)}}$,
(3) $\int a(x)x^\alpha\, dx = 0$ for all $|\alpha| \leq d$.

Definition 5. For sequences of nonnegative numbers $\{k_j\}_{j=1}^\infty$ and cubes $\{Q_j\}_{j=1}^\infty$ and for a function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$, we define

$$\mathcal{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right) = \left\| \left\{ \sum_{j=1}^\infty \left(\frac{k_j \chi_{Q_j}}{\left\|\chi_{Q_j}\right\|_{p(\cdot)}} \right)^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{p(\cdot)}.$$

The space $H^{p(\cdot),p_0,d}_{\text{atom}}(\mathbb{R}^n)$ is the set of all distributions $f \in S'(\mathbb{R}^n)$ that can be written as

$$f = \sum_{j=1}^\infty k_j a_j \tag{6}$$

in $S'(\mathbb{R}^n)$, where $\{k_j\}_{j=1}^\infty$ is a sequence of non negative numbers, the a_j's are $(p(\cdot), p_0, d)$-atoms and $\mathcal{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right) < \infty$. One defines

$$\|f\|_{H^{p(\cdot),p_0,d}_{\text{atom}}} = \inf \mathcal{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right),$$

where the infimum is taken over all admissible expressions as in (6).

Theorem 4.6 in [12] asserts that the quantities $\|f\|_{H^{p(\cdot),p_0,d}_{\text{atom}}}$ and $\|f\|_{H^{p(\cdot)}}$ are comparable. Moreover, f admits an atomic decomposition $f = \sum_{j=1}^\infty k_j a_j$ such that

$$\mathcal{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right) \leq c\,\|f\|_{H^{p(\cdot)}}.$$

3. Boundedness on classical Lebesgue spaces: Case $\alpha(x) \equiv 0$

For $m \in \mathbb{N}$, $m \geq 2$, let $A_1, \dots, A_m$ be $n \times n$ invertible real matrices such that $A_i - A_j$ are invertible for each $1 \leq i \neq j \leq m$. For $1 \leq i \leq m$ we consider measurable functions $\alpha_i : \mathbb{R}^n \to (0, n)$ such that

$$\alpha_1(x) + \cdots + \alpha_m(x) = n, \quad x \in \mathbb{R}^n.$$

We define

$$\alpha_i^- = \inf_{x \in \mathbb{R}^n} \alpha_i(x) \quad \text{and} \quad \alpha_i^+ = \sup_{x \in \mathbb{R}^n} \alpha_i(x).$$

Our first result is the following.

Theorem 6. *If the operator $T_{\mathbb{R}^n}$ is defined by* (2) *with $\alpha_i^- > 0$, $\alpha_i^+ < n$, then $T_{\mathbb{R}^n}$ is weak type* $(1,1)$ *and bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Proof. Without loss of generality, we suppose $f \geq 0$. For each $1 \leq i < j \leq m$, we put $d_{ij} = \min\{|A_i x - A_j x| : |x| \leq 1\}$, $d = \min_{1 \leq i < j \leq m} d_{ij}$, and $D = \max_{1 \leq i \leq m} |A_i|$. We observe that the constants d, $d_{i,j}$ and D are all positive since the matrices $A_1, A_2, \dots, A_m$ are invertible and $A_i - A_j$ is also invertible for $i \neq j$.
For $x \neq 0$ and $1 \leq i \leq m$, we define

$$R_i = \{y \in \mathbb{R}^n : |y - A_i x| \leq \frac{d}{2}|x|\}.$$

We also define

$$R_{m+1} = \left(\bigcap_{i=1}^{m} R_i^c \right) \cap \{y \in \mathbb{R}^n : |y| \leq (1+D)|x|\},$$
$$R_{m+2} = \left(\bigcap_{i=1}^{m} R_i^c \right) \cap \{y \in \mathbb{R}^n : |y| > (1+D)|x|\}.$$

Now

$$T_{\mathbb{R}^n} f(x) = \sum_{i=1}^{m+2} \int_{R_i} |y - A_1 x|^{-\alpha_1(y)} \dots |y - A_m x|^{-\alpha_m(y)} f(y)\, dy,$$

so we will estimate each term of the above sum. Fix $x \neq 0$ and $1 \leq i \leq m$; for $y \in R_i$ and $j \neq i$ we have $|y - A_i x| \leq \frac{d}{2}|x|$ and $|A_i x - A_j x| \geq d|x|$. Then we get

$$|y - A_j x| \geq |A_i x - A_j x| - |y - A_i x| \geq d|x| - \frac{d}{2}|x| = \frac{d}{2}|x|,$$

so

$$\begin{aligned}
&\int_{R_i} |y-A_1x|^{-\alpha_1(y)} \dots |y-A_mx|^{-\alpha_m(y)} f(y)\,dy \\
&\quad \le \int_{R_i} \left(\frac{d}{2}|x|\right)^{\alpha_i(y)-n} |y-A_ix|^{-\alpha_i(y)} f(y)\,dy \\
&\quad = \sum_{k=0}^{\infty} \int_{\frac{d}{2}|x|2^{-(k+1)}<|y-A_ix|\le \frac{d}{2}|x|2^{-k}} \left(\frac{d}{2}|x|\right)^{\alpha_i(y)-n} |y-A_ix|^{-\alpha_i(y)} f(y)\,dy \\
&\quad \le \sum_{k=0}^{\infty} \int_{\frac{d}{2}|x|2^{-(k+1)}<|y-A_ix|\le \frac{d}{2}|x|2^{-k}} \left(\frac{d}{2}|x|\right)^{\alpha_i(y)-n} \left(\frac{d}{2}|x|2^{-(k+1)}\right)^{-\alpha_i(y)} f(y)\,dy \\
&\quad \le 2^{\alpha_i^+} \sum_{k=0}^{\infty} 2^{-k(n-\alpha_i^+)} |2^{-k}(d/2)x|^{-n} \int_{|y-A_ix|\le \frac{d}{2}|x|2^{-k}} f(y)\,dy \\
&\quad \le 2^{\alpha_i^+} Mf(A_ix) \left(\sum_{k=0}^{\infty} 2^{-k(n-\alpha_i^+)}\right) = \frac{2^{\alpha_i^+}}{1-(1/2)^{n-\alpha_i^+}} Mf(A_ix).
\end{aligned}$$

On the region R_{m+1} we obtain

$$\begin{aligned}
&\int_{R_{m+1}} |y-A_1x|^{-\alpha_1(y)} \dots |y-A_mx|^{-\alpha_m(y)} f(y)\,dy \\
&\quad \le (2/d)^n |x|^{-n} \int_{|y|\le(1+D)|x|} f(y)\,dy \\
&\quad \le (2/d)^n |x|^{-n} \int_{|y-x|\le 4(1+D)|x|} f(y)\,dy \\
&\quad \le (2/d)^n (4+4D)^n Mf(x).
\end{aligned}$$

Finally if $y \in R_{m+2}$, we have that for each $1 \le i \le m$, $|y-A_ix| \ge (1/(1+D))|y|$ so

$$\begin{aligned}
&\int_{R_{m+2}} |y-A_1x|^{-\alpha_1(y)} \dots |y-A_mx|^{-\alpha_m(y)} f(y)\,dy \\
&\qquad \le (1+D)^n \int_{|y|>(1+D)|x|} |y|^{-n} f(y)\,dy \le (1+D)^{\frac{n}{p'}} \|f\|_p |x|^{-\frac{n}{p}},
\end{aligned}$$

where the last inequality follows from Hölder's inequality. Then for $\lambda > 0$ and $1 \le p < \infty$

$$\begin{aligned}
&\left|\left\{x : \int_{R_{m+2}} |y-A_1x|^{-\alpha_1(y)} \dots |y-A_mx|^{-\alpha_m(y)} f(y)\,dy \ge \lambda\right\}\right| \\
&\qquad\qquad \le (1+D)^{\frac{np}{p'}} \left(\frac{\|f\|_p}{\lambda}\right)^p.
\end{aligned}$$

Thus $T_{\mathbb{R}^n}$ is weak type (p,p) for all $1 \le p < \infty$. The Marcinkiewicz interpolation theorem gives the boundedness of $T_{\mathbb{R}^n}$ on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. $\square$

The argument used to prove Theorem 6 works for the operator $U_{\mathbb{R}^n}$ given by (3), and thus we obtain the following result.

Theorem 7. *If the operator $U_{\mathbb{R}^n}$ is defined by (3) with $\alpha_i^- > 0$, $\alpha_i^+ < n$, then $U_{\mathbb{R}^n}$ is weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Remark 8. Given a subset $\Omega \subset \mathbb{R}^n$, from Theorem 6 it follows that if T_Ω is defined by (2) with $\alpha_i^- > 0$ and $\alpha_i^+ < n$, then T_Ω is bounded on $L^p(\Omega)$ for $1 < p < \infty$.

The following example shows that if $\Omega = \mathbb{R}^n$, then the operator $T_{\mathbb{R}^n,s}$ cannot be bounded on $L^p(\mathbb{R}^n)$. The proof is an adaptation of the example given by P. Hästö in [10] for the variable Riesz potential.

Example 9. Let $\gamma : \mathbb{R}^n \to (0,n)$ be a smooth function such that $\gamma(x) = \gamma_0$ for $x \in B(0,1)$ and $\gamma(x) = \gamma_\infty$ for $x \notin B(0,3)$. Let $f(y) = 2^{\gamma_\infty} |y|^{\gamma_\infty - \beta} \chi_{\mathbb{R}^n \setminus B(0,3)}(y)$. For $m = 2$ and $s = 1$, let $T_{\mathbb{R}^n,s}$ be the integral operator defined by (4), with $\alpha_1(x) = n - \gamma(x)$, $\alpha_2(y) = \gamma(y)$, $A_1 = I$ and $A_2 = -I$. For $x \in B(0,1)$ we have:

$$\begin{aligned} T_{\mathbb{R}^n,s} f(x) &\geq \int_{\mathbb{R}^n \setminus B(0,3)} |x-y|^{\gamma_0 - n} \, |x+y|^{-\gamma_\infty} 2^{\gamma_\infty} |y|^{\gamma_\infty - \beta} \, dy \\ &\geq \int_{\mathbb{R}^n \setminus B(0,3)} |x-y|^{\gamma_0 - n} \, |y|^{-\beta} \, dy \approx \int_3^\infty r^{-\beta - n + \gamma_0 + n - 1} dr = \infty \end{aligned}$$

if $\gamma_0 \geq \beta$. On the other hand,

$$\|f\|_p^p \approx \int_3^\infty r^{(\gamma_\infty - \beta)p + n - 1} dr < \infty$$

if $n < (\beta - \gamma_\infty)p$. Thus there exists a function $f \in L^p(\mathbb{R}^n)$ with $T_{\mathbb{R}^n,s} f(x) = \infty$ in $B(0,1)$ provided that $n < (\beta - \gamma_\infty)p$.

However, we have the following result in the case that Ω is a bounded open subset of $\mathbb{R}^n$.

Proposition 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and let $\alpha_i : \Omega \to (0,n)$ be such that $\alpha_i \in LH_0(\Omega)$, $A_i(\Omega) = \Omega$, and $\alpha_i(A_i x) = \alpha_i(x)$ for each $i = 1, \dots, m$, and $x \in \Omega$. If $s \in \mathbb{N}$, $s < m$, then the operators $T_{\Omega,s}$ and U_Ω defined by (4) and (3) are bounded on $L^p(\Omega)$ for $1 < p < \infty$.*

Proof. If $\alpha_i \in LH_0(\Omega)$, then

$$-|\alpha_i(A_i x) - \alpha_i(y)| \ln(|y - A_i x|) \leq C, \quad \forall x, y \in \Omega : |y - A_i x| \leq \frac{1}{2}.$$

Since Ω is a bounded open subset of $\mathbb{R}^n$ and $A_i(\Omega) = \Omega$ for each $1 \leq i \leq m$, then there exists a positive constant C_0 such that

$$|(\alpha_i(A_i x) - \alpha_i(y)) \ln(|y - A_i x|)| \leq C_0, \quad \forall x, y \in \Omega. \tag{7}$$

Now, the hypothesis $\alpha_i(A_i x) = \alpha_i(x)$ gives

$$|y - A_i x|^{-\alpha_i(x) + \alpha_i(y)} = e^{-(\alpha_i(A_i x) - \alpha_i(y)) \ln(|y - A_i x|)}. \tag{8}$$

We set $c_1 = e^{-C_0}$ and $c_2 = e^{C_0}$, then the equality in (8) and the inequality in (7) allow us to obtain

$$c_1|y - A_i x|^{-\alpha_i(y)} \leq |y - A_i x|^{-\alpha_i(x)} \leq c_2|y - A_i x|^{-\alpha_i(y)}, \quad \forall\, x, y \in \Omega.$$

Finally, if $f \geq 0$ we have that $T_{\Omega,s}f(x) \leq cT_\Omega f(x)$ and $U_\Omega f(x) \leq cT_\Omega f(x)$ for all $x \in \Omega$. The proposition follows from Remark 8. □

Next we give a nontrivial example of functions $\alpha_i : \Omega \to (0, n)$, $1 \leq i \leq m$, with $\Omega = B(0, 1)$ (where $B(0, 1)$ is the n-dimensional open ball centered at the origin with radius 1) and $A_1, A_2, \ldots, A_m$ are $n \times n$ matrices that satisfy the hypothesis of Proposition 10.

Example 11. Let $\gamma_i : \mathbb{R} \to (0, \infty)$ be functions belonging to $LH_0(\mathbb{R})$ for each $1 \leq i \leq m$, and $\gamma(t) = \sum_{i=1}^m \gamma_i(t)$. Without loss of generality, we can assume that $0 < \gamma(t) < n$ for all $t \in [-1, 1]$. Now, for $x \in B(0, 1)$ and $1 \leq i \leq m$, we set $\alpha_i(x) = \gamma_i(|x|)$ and $\alpha(x) = n - \gamma(|x|)$. It is easy to check that $\alpha_i \in LH_0(B(0, 1))$, $1 \leq i \leq m$, and $\alpha_1(x)+\cdots+\alpha_m(x) = n-\alpha(x)$ for all $x \in B(0, 1)$. If $A_1, A_2, \ldots, A_m$ are $n \times n$ orthogonal matrices, then $A_i(B(0, 1)) = B(0, 1)$ and $\alpha_i(A_i x) = \alpha_i(x)$ for all $1 \leq i \leq m$ and all $x \in B(0, 1)$. Finally, the orthogonal matrices $A_1, A_2, \ldots, A_m$ can be chosen such that $A_i - A_j$ is invertible for $i \neq j$.

We recall that for a locally integrable function f, the *sharp* operator is defined by

$$M^\sharp f(x) = \sup_{B \ni x} \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y) - a|\, dy.$$

We obtain the following pointwise estimate.

Proposition 12. *If $T_{\mathbb{R}^n}$ is the integral operator given by* (2) *then for each* $0 < \delta < 1$ *there exists $C = C_\delta > 0$ such that*

$$M^\sharp_\delta(T_{\mathbb{R}^n} f)(x) \leq C \sum_{i=1}^m Mf(A_i x), \quad f \in C_0^\infty(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Proof. By Theorem 6, we have that $T_{\mathbb{R}^n} f \in L^s(\mathbb{R}^n)$, for all $1 < s < \infty$ and all $f \in C_0^\infty(\mathbb{R}^n)$, and $M^\sharp_\delta(T_{\mathbb{R}^n} f)(x)$ is well defined for all $x \in \mathbb{R}^n$ and all $0 < \delta < 1$. Let $x \in \mathbb{R}^n$ and let $B = B(x_0, r)$ be a ball that contains x, centered at x_0 with radius r, and $T_{\mathbb{R}^n} f(x_0) < \infty$. We write $\widetilde{B} = B(x_0, 4r)$, and for $i = 1, \ldots, m$ we also set $\widetilde{B}_i = A_i \widetilde{B}$. Let $f_1 = f\chi_{\cup_{1 \leq i \leq m} \widetilde{B}_i}$ and let $f_2 = f - f_1$. We choose $a = T_{\mathbb{R}^n} f_2(x_0)$. By Jensen's inequality and the inequality

$$||t|^\delta - |s|^\delta| \leq |t - s|^\delta,$$

valid for $0 < \delta < 1$, it follows that

$$\begin{aligned}\left(\frac{1}{|B|} \int_B |T_{\mathbb{R}^n} f(y)|^\delta - |a|^\delta|\, dy\right)^{1/\delta} &\leq \frac{1}{|B|} \int_B |T_{\mathbb{R}^n} f(y) - a|\, dy \\ &\leq \frac{1}{|B|} \int_B |T_{\mathbb{R}^n} f_1(y)|\, dy + \frac{1}{|B|} \int_B |T_{\mathbb{R}^n} f_2(y) - a|\, dy = I + II.\end{aligned}$$

We use now Kolmogorov's inequality (see [8, exercise 2.1.5, p. 91]) and that $T_{\mathbb{R}^n}$ is weak type $(1,1)$ to get

$$I \leq \frac{C}{|B|}\int_{\mathbb{R}^n} |f_1(y)|\,dy \leq \sum_{i=1}^{m} \frac{C}{|B|}\int_{\widetilde{B}_i} |f(y)|\,dy \leq C\sum_{i=1}^{m} Mf(A_i x).$$

On the other hand, since $f_2 = \chi_{\left(\cup_{1\leq i\leq m}\widetilde{B}_i\right)^c} f$ we have

$$\begin{aligned} II &= \frac{1}{|B|}\int_B |T_{\mathbb{R}^n} f_2(y) - T_{\mathbb{R}^n} f_2(x_0)|\,dy \\ &\leq \frac{1}{|B|}\int_B \int_{\left(\bigcup_{1\leq k\leq m}\widetilde{B}_k\right)^c} |K(y,z) - K(x_0,z)||f(z)|\,dzdy, \end{aligned}$$

taking into account that

$$\mathbb{R}^n = \bigcup_{1\leq j\leq m} \{z : |z - A_j x_0| \leq |z - A_i x_0|, \text{ for all } 1 \leq i \leq m\},$$

for each $1 \leq j \leq m$ we set

$$Z_j = \left(\bigcup_{1\leq k\leq m} \widetilde{B}_k\right)^c \bigcap \{z : |z - A_j x_0| \leq |z - A_i x_0|, \text{ for all } 1 \leq i \leq m\},$$

so

$$II \leq \sum_{j=1}^{m} \frac{1}{|B|}\int_B \int_{Z_j} |K(y,z) - K(x_0,z)||f(z)|\,dzdy.$$

We estimate now $|K(y,z) - K(x_0,z)|$ for $y \in B$ and $z \in Z_j$. By the mean value theorem we obtain

$$\begin{aligned} |K(y,z) - K(x_0,z)| &\leq |y - x_0| \sum_{i=1}^{m} \frac{\alpha_i(z)}{|z - A_i\xi|^{\alpha_i(z)+1}\prod_{l\neq i}|z - A_l\xi|^{\alpha_l(z)}} \\ &\leq |y - x_0| \sum_{i=1}^{m} \frac{\alpha_i^+}{|z - A_i\xi|^{\alpha_i(z)+1}\prod_{l\neq i}|z - A_l\xi|^{\alpha_l(z)}} \end{aligned}$$

for some ξ between x_0 and y, and where $\alpha_i^+ = \sup_{z\in\mathbb{R}^n} \alpha_i(z)$. Since $|z - A_i\xi| \geq \frac{|z - A_i x_0|}{2}$ and $\alpha_1(z) + \cdots + \alpha_m(z) = n$ we obtain

$$|K(y,z) - K(x_0,z)| \leq c\frac{|y - x_0|}{|z - A_i x_0|^{n+1}}.$$

This inequality together with the argument used in the proof of Theorem 2.1 in [14] allow us to obtain

$$II \leq C\sum_{i=1}^{m} Mf(A_i x). \qquad \square$$

This proposition suggests that weight estimates could be obtained for some Muckenhoupt weights w. In fact, we obtain the following result.

Theorem 13. *Let $T_{\mathbb{R}^n}$ be the integral operator given by* (2). *Let $1 < p < \infty$ and let w be a weight in the Muckenhoupt class A_p, such that $w(A_i^{-1}x) \leq cw(x)$ for some positive constant c. Then $T_{\mathbb{R}^n}$ is bounded on $L^p(w)$.*

Proof. Let $0 < \delta < 1$. From Proposition 12 we get that if $f \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\int |T_{\mathbb{R}^n} f(x)|^p w(x)\,dx &\leq \int \left| M\left(T_{\mathbb{R}^n} f\right)^\delta (x)\right|^{\frac{p}{\delta}} w(x)\,dx \\
&\leq C \int \left| M_\delta^\sharp \left(T_{\mathbb{R}^n} f\right)(x)\right|^p w(x)\,dx \leq C \int \left(\sum_{i=1}^m Mf(A_i x)\right)^p w(x)\,dx \\
&\leq C \sum_{i=1}^m \int \left(Mf\right)^p (A_i x) w(x)\,dx \leq C \sum_{i=1}^m \int \left(Mf\right)^p (x) w(A_i^{-1}x)\,dx \\
&\leq C \sum_{i=1}^m \int \left(Mf\right)^p (x) w(x)\,dx \leq C \int |f|^p (x) w(x)\,dx,
\end{aligned}$$

where the last inequality follows from the boundedness on $L^p(w)$ of the Hardy–Littlewood maximal operator M. Now the theorem follows since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(w)$. □

Example 14. The nonnegative function $w(x) = |x|^a$, $x \in \mathbb{R}^n \setminus \{0\}$, is a weight in the Muckenhoupt class A_p, $1 < p < \infty$, if and only if $-n < a < n(p-1)$ (see [9, p. 286]). If $A_1, A_2, \dots, A_m$ are $n \times n$ orthogonal matrices, then $w(A_i^{-1}x) = w(x)$ for all $1 \leq i \leq m$ and all $x \in \mathbb{R}^n \setminus \{0\}$. Finally, the orthogonal matrices $A_1, A_2, \dots, A_m$ can be chosen such that $A_i - A_j$ is invertible for $i \neq j$.

4. Boundedness on variable Lebesgue spaces and on variable Hardy spaces: Case $\alpha(x) \equiv 0$

Theorem 15. *Let $T_{\mathbb{R}^n}$ be the integral operator given by* (2). *Let $p(\cdot) \in LH_0 \cap LH_\infty(\mathbb{R}^n)$ be such that $1 < p_- \leq p_+ < \infty$ and $p(A_i x) = p(x)$ for each $i = 1, \dots, m$, and all $x \in \mathbb{R}^n$. Then $T_{\mathbb{R}^n}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. From Theorem 6, we have that $T_{\mathbb{R}^n} f \in L^s(\mathbb{R}^n)$, for all $1 < s < \infty$ and all $f \in C_0^\infty(\mathbb{R}^n)$. Since

$$|T_{\mathbb{R}^n} f(x)|^{p(x)} \leq |T_{\mathbb{R}^n} f(x)|^{p_+} \chi_{\{|T_{\mathbb{R}^n} f(x)| \geq 1\}} + |T_{\mathbb{R}^n} f(x)|^{p_-} \chi_{\{|T_{\mathbb{R}^n} f(x)| < 1\}},$$

it follows that $T_{\mathbb{R}^n} f \in L^{p(\cdot)}(\mathbb{R}^n)$. From Theorem 3.6 in [5] we get

$$\begin{aligned}
\|T_{\mathbb{R}^n} f\|_{p(\cdot)} &= \left\| |T_{\mathbb{R}^n} f|^\delta \right\|_{\frac{p(\cdot)}{\delta}}^{\frac{1}{\delta}} \leq c \left\| M^\sharp \left(|T_{\mathbb{R}^n} f|^\delta\right) \right\|_{\frac{p(\cdot)}{\delta}}^{\frac{1}{\delta}} \\
&\leq c \left\| \left(M^\sharp \left(|T_{\mathbb{R}^n} f|^\delta\right)\right)^{1/\delta} \right\|_{p(\cdot)} = c \left\| M_\delta^\sharp \left(T_{\mathbb{R}^n} f\right) \right\|_{p(\cdot)}.
\end{aligned}$$

Now, from Proposition 12 and Theorem 1 we obtain

$$\begin{aligned}\left\|M_\delta^\sharp\left(T_{\mathbb{R}^n} f\right)\right\|_{p(\cdot)} \leq c\sum_{i=1}^{m}\|Mf(A_i\cdot)\|_{p(\cdot)} &= c\sum_{i=1}^{m}\|Mf\|_{p(A_i^{-1}\cdot)}\\ &= c\sum_{i=1}^{m}\|Mf\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}.\end{aligned}$$

Finally the theorem follows since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$. □

Corollary 16. *Let $U_{\mathbb{R}^n}$ be the integral operator given by* (3)*. Let $p(\cdot) \in LH_0 \cap LH_\infty(\mathbb{R}^n)$ be such that $1 < p_- \leq p_+ < \infty$ and $p(A_i x) = p(x)$ for each $i = 1, \dots, m$, and all $x \in \mathbb{R}^n$. Then $U_{\mathbb{R}^n}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $\widetilde{T_{\mathbb{R}^n}}$ be the integral operator defined by

$$\widetilde{T_{\mathbb{R}^n}} f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1(y)} \dots |x - A_m y|^{-\alpha_m(y)} f(y)\, dy$$

and let $p'(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be given by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Since the operator $\widetilde{T_{\mathbb{R}^n}}$ is similar to $T_{\mathbb{R}^n}$, we can apply Theorem 15 to $\widetilde{T_{\mathbb{R}^n}}$ and $p'(\cdot)$ to obtain $\|\widetilde{T_{\mathbb{R}^n}} f\|_{p'(\cdot)} \leq c\|f\|_{p'(\cdot)}$. Since $U_{\mathbb{R}^n} = (\widetilde{T_{\mathbb{R}^n}})^*$ by the Hölder inequality we have

$$\begin{aligned}\left|\int (U_{\mathbb{R}^n} f)(x) g(x)\, dx\right| &= \left|\int f(x)(\widetilde{T_{\mathbb{R}^n}} g)(x)\, dx\right|\\ &\leq c\|f\|_{p(\cdot)}\|\widetilde{T_{\mathbb{R}^n}} g\|_{p'(\cdot)} \leq c\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.\end{aligned}$$

Taking the supremum on all $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ such that $\|g\|_{p'(\cdot)} \leq 1$, we obtain

$$\|U_{\mathbb{R}^n} f\|_{p(\cdot)} \leq c\|f\|_{p(\cdot)}.$$ □

Example 17. Suppose $h : \mathbb{R} \to (1, \infty)$ belongs to $LH_0 \cap LH_\infty(\mathbb{R})$ and $1 < h_- \leq h_+ < \infty$. Let $p(x) = h(|x|)$ for all $x \in \mathbb{R}^n$ and for $m \geq 2$ let $A_1, \dots, A_m$ be $n \times n$ orthogonal matrices such that $A_i - A_j$ is invertible for $i \neq j$. It is easy to check that $p \in LH_0 \cap LH_\infty(\mathbb{R}^n)$, and also that $1 < p_- \leq p_+ < \infty$ and $p(A_i x) = p(x)$, for each $1 \leq i \leq m$.

The next result deals with the $H^{p(\cdot)}(\mathbb{R}^n)$-$L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of the operator $U_{\mathbb{R}^n}$.

Theorem 18. *Let $U_{\mathbb{R}^n}$ be the operator defined by* (3)*. Suppose $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ belongs to $LH_0 \cap LH_\infty(\mathbb{R}^n)$, $0 < p_- \leq p_+ < \infty$ and $p(A_i x) = p(x)$, $1 \leq i \leq m$. Then $U_{\mathbb{R}^n}$ can be extended to a $H^{p(\cdot)}(\mathbb{R}^n)$-$L^{p(\cdot)}(\mathbb{R}^n)$ bounded operator.*

Proof. Let $\max\{1, p_+\} < p_0 < \infty$. Given $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$, from [12, Theorem 4.6] we have that there exist a sequence of nonnegative numbers $\{k_j\}_{j=1}^\infty$, a sequence of cubes $Q_j = Q(z_j, r_j)$ centered at z_j with side length r_j, and $(p(\cdot), p_0, d)$ atoms a_j supported on Q_j, satisfying

$$\mathcal{A}\left(\{k_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty, p(\cdot)\right) \leq c\|f\|_{H^{p(\cdot)}},$$

such that f can be decomposed as $f = \sum_{j\in\mathbb{N}} k_j a_j$, where the convergence is in $H^{p(\cdot)}$ and in L^{p_0} (for the convergence in L^{p_0} see [15, Theorem 5]; the same argument works for $f \in H^{p(\cdot)} \cap L^{p_0}(\mathbb{R}^n)$).

We will study the behavior of $U_{\mathbb{R}^n}$ on atoms. Define $D = \max\{|A_i^{-1}x| : 1 \le i \le m, |x| \le 1\}$. Fix $j \in \mathbb{N}$, let a_j be a $(p(\cdot), p_0, d)$-atom supported on a cube $Q_j = Q(z_j, r_j)$, for each $1 \le i \le m$ let $Q_{ji}^* = Q(A_i^{-1}z_j, 4Dr_j)$. Theorem 7 gives that $U_{\mathbb{R}^n}$ is bounded on $L^{p_0}(\mathbb{R}^n)$, thus $\|U_{\mathbb{R}^n} a_j\|_{L^{p_0}(Q_{ji}^*)} \le c\|a_j\|_{p_0} \le c|Q_{ji}^*|^{1/p_0}\|\chi_{Q_{ji}^*}\|_{p(\cdot)}^{-1}$. So if $\|U_{\mathbb{R}^n} a_j\|_{L^{p_0}(Q_{ji}^*)} \ne 0$ we get

$$1 \le c\frac{|Q_{ji}^*|^{1/p_0}}{\|\chi_{Q_{ji}^*}\|_{p(\cdot)}\|U_{\mathbb{R}^n} a_j\|_{L^{p_0}(Q_{ji}^*)}}, \tag{9}$$

with c independent of the cube Q_j.

Let $\widetilde{K}$ be the kernel defined by

$$\widetilde{K}(x,y) = |y - A_1x|^{-\alpha_1(x)} \dots |y - A_mx|^{-\alpha_m(x)}.$$

In view of the moment condition of a_j we have

$$U_{\mathbb{R}^n} a_j(x) = \int_{Q_j} \widetilde{K}(x,y)a_j(y)\,dy = \int_{Q_j} \left(\widetilde{K}(x,y) - q_{d,j}(x,y)\right) a_j(y)\,dy,$$

where $q_{d,j}$ is the degree d Taylor polynomial of the function $y \to \widetilde{K}(x,y)$ about z_j. By the standard estimate of the remainder term of the Taylor expansion, there exists ξ between y and z_j such that

$$|\widetilde{K}(x,y) - q_{d,j}(x,y)| \le C|y - z_j|^{d+1}\left(\prod_{i=1}^m |\xi - A_ix|^{-\alpha_i(x)}\right)\left(\sum_{l=1}^m |\xi - A_lx|^{-1}\right)^{d+1}.$$

Now we decompose $\mathbb{R}^n = \bigcup_{i=1}^m Q_{ji}^* \cup R_j$, where $R_j = \left(\bigcup_{i=1}^m Q_{ji}^*\right)^c$; at the same time we decompose $R_j = \bigcup_{k=1}^m R_{jk}$ with

$$R_{jk} = \{x \in R_j : |x - A_k^{-1}z_j| \le |x - A_i^{-1}z_j| \text{ for all } i \ne k\}.$$

If $x \in R_j$ then $|x - A_i^{-1}z_j| \ge 2Dr_j$. Since $\xi \in Q_j$ it follows that $|A_i^{-1}z_j - A_i^{-1}\xi| \le Dr_j \le \frac{1}{2}|x - A_i^{-1}z_j|$, so

$$\begin{aligned}|x - A_i^{-1}\xi| &= |x - A_i^{-1}z_j + A_i^{-1}z_j - A_i^{-1}\xi| \\ &\ge |x - A_i^{-1}z_j| - |A_i^{-1}z_j - A_i^{-1}\xi| \ge \frac{1}{2}|x - A_i^{-1}z_j|.\end{aligned}$$

Thus if $x \in R_j$ and $\xi \in Q_j$, then there exists a positive constant c such that $|\xi - A_ix| \ge c|z_j - A_ix|$. If $x \in R_j$, then $x \in R_{jk}$ for some k and since $\alpha_1(x) + \cdots + \alpha_m(x) = n$ we obtain

$$\begin{aligned}\left|\widetilde{K}(x,y) - q_{d,j}(x,y)\right| &\le c\,|y - z_j|^{d+1}\left(\prod_{i=1}^m |z_j - A_ix|^{-\alpha_i(x)}\right)\left(\sum_{l=1}^m |z_j - A_lx|^{-1}\right)^{d+1} \\ &\le c\,r_j^{d+1}\,|z_j - A_kx|^{-(n+d+1)}.\end{aligned}$$

So for $x \in R_{jk}$ we have

$$\begin{aligned}|U_{\mathbb{R}^n} a_j(x)| &\leq c\, \|a_j\|_1\, r_j^{d+1}\, |z_j - A_k x|^{-(n+d+1)} \\ &\leq c\, |Q_j|^{1-\frac{1}{p_0}}\, \|a\|_{p_0}\, r_j^{d+1}\, |z_j - A_k x|^{-(n+d+1)} \\ &\leq c\, \frac{r_j^{n+d+1}}{\left\|\chi_{Q_j}\right\|_{p(\cdot)}} |z_j - A_k x|^{-(n+d+1)}.\end{aligned}$$

This inequality implies

$$|U_{\mathbb{R}^n} a_j(x)| \leq c\, \frac{\left(M\left(\chi_{Q_j}\right)(A_k x)\right)^{\frac{n+d+1}{n}}}{\left\|\chi_{Q_j}\right\|_{p(\cdot)}}, \quad \text{if } x \in R_{j,k}. \tag{10}$$

Since $f = \sum_{j=1}^{\infty} k_j a_j$ in L^{p_0} we have that $|U_{\mathbb{R}^n} f(x)| \leq \sum_{j=1}^{\infty} k_j |U_{\mathbb{R}^n} a_j(x)|$. Finally, the estimates (9), (10) and a similar argument to that used in the proof of Theorem 7 in [18] allow us to obtain the result. □

5. Boundedness on variable Lebesgue spaces: Case $0 < \alpha(x) < n$

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}$, $m \geq 2$, let $A_1, \dots, A_m$ be $n \times n$ invertible real matrices such that $A_i - A_j$ are invertible for each $1 \leq i \neq j \leq m$, and $A_i(\Omega) = \Omega$ for each $1 \leq i \leq m$. We consider measurable functions $\alpha_i : \Omega \to (0, n)$, $(1 \leq i \leq m)$ and $\alpha : \Omega \to (0, n)$ such that

$$\alpha_1(x) + \cdots + \alpha_m(x) = n - \alpha(x), \quad \forall\, x \in \Omega.$$

Now we consider the operators T_Ω, U_Ω and for $s \in \mathbb{N}$, $1 \leq s < m$, the operator $T_{\Omega,s}$ defined by

$$T_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(y)} \dots |y - A_m x|^{-\alpha_m(y)} f(y)\, dy, \tag{11}$$

$$U_\Omega f(x) = \int_\Omega |y - A_1 x|^{-\alpha_1(x)} \dots |y - A_m x|^{-\alpha_m(x)} f(y)\, dy, \tag{12}$$

and

$$T_{\Omega,s} f(x) = \int_\Omega \left(\prod_{i=1}^{s} |y - A_i x|^{-\alpha_i(x)}\right) \left(\prod_{i=s+1}^{m} |y - A_i x|^{-\alpha_i(y)}\right) f(y)\, dy, \quad x \in \Omega. \tag{13}$$

We obtain the following results.

Theorem 19. *Let U_Ω be the operator defined by* (12) *with* $\inf_{x\in\Omega} \alpha(x) > 0$. *If $p(\cdot) \in LH_0(\Omega)$, $p(A_i x) = p(x)$ for $1 \leq i \leq m$, and $\sup_{x\in\Omega} p(x)\alpha(x) < n$, then U_Ω is bounded from $L^{p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$, for $x \in \Omega$.*

Proof. For each $x \in \Omega$ we decompose $\Omega = \cup_{i=1}^m \Omega_i$, where $\Omega_i = \{y \in \Omega : |y - A_i x| \leq |y - A_j x|, 1 \leq j \leq m\}$; then for $f \geq 0$ and since $A_i(\Omega) = \Omega$, for each $1 \leq i \leq m$, a computation gives

$$U_\Omega f(x) \leq \sum_{i=1}^m U_{\Omega_i} f(x) \leq c \sum_{i=1}^m I^{\alpha(x)}_{A_i^{-1}(\Omega)} (f \circ A_i)(x) = c \sum_{i=1}^m I^{\alpha(x)}_{\Omega} (f \circ A_i)(x), \quad \forall\, x \in \Omega.$$

By Theorem 3 and since $p(A_i x) = p(x)$ for $1 \le i \le m$, we obtain

$$\begin{aligned} \|U_\Omega f\|_{L^{q(\cdot)}(\Omega)} &\le c \sum_{i=1}^{m} \|I_\Omega^{\alpha(\cdot)}(f \circ A_i)\|_{L^{q(\cdot)}(\Omega)} \le c \sum_{i=1}^{m} \|(f \circ A_i)\|_{L^{p(\cdot)}(\Omega)} \\ &= mc\,\|f\|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$ □

Corollary 20. *Let $\alpha_i(\cdot) \in LH_0(\Omega)$ such that $\alpha_i(A_i x) = \alpha_i(x)$ for each $1 \le i \le m$ and $\inf_{x\in\Omega} \alpha(x) > 0$. If $p(\cdot) \in LH_0(\Omega)$, $\sup_{x\in\Omega} p(x)\alpha(x) < n$ and $p(A_i x) = p(x)$ for each $1 \le i \le m$, then the operators T_Ω and $T_{\Omega,s}$ defined by* (11) *and* (13) *are bounded from $L^{p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$, for $x \in \Omega$.*

Proof. Since $\Omega \subset \mathbb{R}^n$ is a bounded subset, $\alpha_i(\cdot) \in LH_0(\Omega)$ and $\alpha_i(A_i x) = \alpha_i(x)$ for each $i = 1, \ldots, m$, the computation done in the proof of Proposition 10 gives us two positive constants c_1 and c_2 such that for each $i = 1, \ldots, m$

$$c_1|y - A_i x|^{-\alpha_i(y)} \le |y - A_i x|^{-\alpha_i(x)} \le c_2|y - A_i x|^{-\alpha_i(y)}, \quad \forall\, x, y \in \Omega.$$

Thus if $f \ge 0$, we have that $T_\Omega f(x) \le cU_\Omega f(x)$ and $T_{\Omega,s} f(x) \le cU_\Omega f(x)$ for all $x \in \Omega$, then the corollary follows from Theorem 19. □

Remark 21. It is easy to check that for $f \ge 0$, $\sum_{i=1}^{m} I_{\mathbb{R}^n}^{\alpha(x)}(f \circ A_i)(x) \le cU_{\mathbb{R}^n} f(x)$, for $x \in \mathbb{R}^n$. In view of the example of Hästö (see [10, p. 271]) we cannot expect the $L^{p(\cdot)}(\mathbb{R}^n)$-$L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of this kind of operators. From this and by an argument of duality, we have also that the operator $T_{\mathbb{R}^n}$ cannot be bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$. This is a notable difference with respect to the case $\alpha(x) \equiv 0$.

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