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THE INITIAL VALUE PROBLEM FOR THE SCHRÖDINGER EQUATION INVOLVING THE HENSTOCK–KURZWEIL INTEGRAL

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Abstract. Let L be the one-dimensional Schrödinger operator defined by $Ly = -y'' + qy$. We investigate the existence of a solution to the initial value problem for the differential equation $(L-\lambda)y = g$, when q and g are Henstock–Kurzweil integrable functions on $[a, b]$. Results presented in this article are generalizations of classical results for the Lebesgue integral.

1. Introduction

Let q be a real valued function defined on $[a, b]$ and let L be the one-dimensional Schrödinger operator defined by $Ly = -y'' + qy$. It is well known that if q, g are Lebesgue integrable functions on $[a, b]$, then there exists a unique solution $f, f' \in AC([a, b])$ of the differential equation $(L - \lambda)y = g$ satisfying the initial condition $f(c) = \alpha$ and $f'(c) = \beta$, where $c \in [a, b]$, $\lambda, \alpha, \beta \in \mathbb{C}$ and $AC([a, b])$ denotes the space of all absolutely continuous functions on $[a, b]$. See, for example, [5]. In this paper, we generalize this result when q, g are Henstock–Kurzweil integrable functions on $[a, b]$.

2. Preliminaries

In this section, the definition of the Henstock–Kurzweil integral and their main properties needed in this paper are presented.

Definition 2.1. Let $f : [a, b] \to \mathbb{C}$ be a function. We say that f is Henstock–Kurzweil (shortly, HK-) integrable on $[a, b]$, if there is an $A \in \mathbb{C}$ such that, for each $\epsilon > 0$, there exists a function $\gamma_\epsilon : [a, b] \to (0, \infty)$ (named a gauge) for which

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \epsilon,$$

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for any partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ such that $t_i \in [x_{i-1}, x_i]$ and $[x_{i-1}, x_i] \subseteq [t_i - \gamma_\epsilon(t_i), t_i + \gamma_\epsilon(t_i)]$ for all $i = 1, 2, \ldots, n$. The number A is called the integral of f over $[a, b]$ and it is denoted by $\int_a^b f$.

The set of all Henstock–Kurzweil integrable functions on $[a, b]$ is denoted by $HK([a, b])$. This set is a vector space and contains the union of $L([a, b])$, the space of Lebesgue integrable functions on $[a, b]$, and the Cauchy-Lebesgue integrable functions (i.e., improper Lebesgue integrals). It is well known that if $f \in HK([a, b])$ then not necessarily $|f| \in HK([a, b])$. If both f and $|f|$ are HK-integrable on $[a, b]$, we say that f is Henstock–Kurzweil absolutely integrable on $[a, b]$. The space of Henstock–Kurzweil absolutely integrable functions on $[a, b]$ coincides with the space $L([a, b])$.

For each $f \in HK([a, b])$ and $I \subseteq [a, b]$ the Alexiewicz seminorm of f on I is defined as

$$\|f\|_I = \sup_{J \subseteq I} \left| \int_J f \right|,$$

where the supremum is taken over all intervals J contained in I.

Definition 2.2. Let $\varphi : [c, d] \to \mathbb{C}$ be a function. The variation of φ on the interval $[c, d]$ is defined as

$$V_{[c,d]}\varphi = \sup \left\{ \sum_{i=1}^n |\varphi(x_i) - \varphi(x_{i-1})| \;\middle|\; \{x_i\}_{i=0}^n \text{ is a partition of } [c, d] \right\}.$$

We say that the function φ is of bounded variation on $[c, d]$ if $V_{[c,d]}\varphi < \infty$. The space of all bounded variation functions on $[c, d]$ is denoted by $BV([c, d])$.

The next theorem shows that absolutely integrable functions are precisely those integrable functions whose indefinite integrals have bounded variation.

Theorem 2.3 ([1, Theorem 7.5]). *Let $f \in HK([a, b])$. Then $|f|$ is HK-integrable if and only if the indefinite integral $F(x) = \int_a^x f$ has bounded variation on $[a, b]$. In this case*

$$V_{[a,b]}F = \int_a^b |f|.$$

The space of Henstock–Kurzweil integrable functions is not multiplicative in general. However, the multipliers for HK-integrable functions are the functions of bounded variation.

Theorem 2.4 (Multiplier Theorem, [1, Theorem 10.12]). *If g is a real valued function such that $g \in HK([a, b])$ and $f \in BV([a, b])$ then the product gf belongs to $HK([a, b])$.*

The following theorem gives an estimate of the integral of a product. This theorem will be useful to prove the existence and uniqueness theorem.

Theorem 2.5 ([4, Lemma 24]). *If g is a real valued function such that $g \in HK([a,b])$ and $f \in BV([a,b])$, then*

$$\left| \int_a^b fg \right| \leq \inf_{t\in[a,b]} |f(t)| \left| \int_a^b g(t)\,dt \right| + \|g\|_{[a,b]} V_{[a,b]} f.$$

Theorem 2.6 ([3, Corollary 3.2]). *If g is a real valued function such that $g \in HK([a,b])$ and (f_n) is a sequence in $BV([a,b])$ such that $V_{[a,b]} f_n \leq M$ for all $n \in \mathbb{N}$, and $g_n \to g$ pointwise on $[a,b]$, then*

$$\lim_{n\to\infty} \int_a^b f g_n = \int_a^b fg.$$

It is well known that the Lebesgue integral may be characterized by the fact that the indefinite integral is absolutely continuous. A similar characterization is possible with the Henstock–Kurzweil integral.

Definition 2.7. Let $F : [a,b] \to \mathbb{C}$. We say that F is absolutely continuous in the restricted sense on a set $E \subseteq [a,b]$ ($F \in AC_*(E)$), if for every $\epsilon > 0$ there exists $\eta_\epsilon > 0$ such that if $\{[u_i, v_i]\}_{i=1}^s$ is a collection of nonoverlapping intervals with endpoints in E and such that $\sum_{i=1}^s (v_i - u_i) < \eta_\epsilon$, then

$$\sum_{i=1}^s \sup\{|F(x) - F(y)| : x, y \in [u_i, v_i]\} < \epsilon.$$

Moreover, F is said to be generalized absolutely continuous in the restricted sense on $[a,b]$ ($F \in ACG_*([a,b])$), if F is continuous on $[a,b]$ and there is a countable collection $(E_n)_{n=1}^\infty$ of sets in $[a,b]$ with $[a,b] = \cup_{i=1}^\infty E_n$ and $F \in AC_*(E_n)$ for all $n \in \mathbb{N}$.

Theorem 2.8 (Fundamental Theorem of Calculus, [2]). *Let $f, F : [a,b] \to \mathbb{C}$ be functions and $c \in [a,b]$.*

(1) $f \in HK([a,b])$ and $F(x) = \int_c^x f$ for all $x \in [a,b]$ if and only if $F \in ACG_*([a,b])$, $F(c) = 0$, and $F' = f$ almost everywhere on $[a,b]$.
If $f \in HK([a,b])$ and f is continuous at $x \in [a,b]$ then $\frac{d}{dx}\int_c^x f = f(x)$.

(2) $F \in ACG_*([a,b])$ if and only if F' exists almost everywhere on $[a,b]$, and $\int_c^x F' = F(x) - F(c)$ for all $x \in [a,b]$.

3. The existence and uniqueness theorem

In this section q is a real valued function such that $q \in HK([a,b])$ and L is the Schrödinger operator defined as

$$Ly = -y'' + qy.$$

Lemma 3.1. *Let $c \in [a,b]$, $\lambda, \alpha, \beta \in \mathbb{C}$ and let $A = \begin{pmatrix} 0 & 1 \\ q-\lambda & 0 \end{pmatrix}$. If $g \in HK([a,b])$, then there exists a unique solution f, $f' \in ACG_*([a,b])$ of the initial value problem*

$$\begin{aligned} (L-\lambda)y &= g \quad a.e. \\ y(c) &= \alpha \\ y'(c) &= \beta \end{aligned} \tag{3.1}$$

if and only if there exists a unique solution $u \in C([a,b],\mathbb{C}^2)$ of the equation

$$u = \int_c^{(\cdot)} A(s)u(s)\,ds + w, \tag{3.2}$$

where $w : [a,b] \to \mathbb{C}^2$ is defined as $w(x) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \int_c^x \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds$.

Proof. Let f, with $f, f' \in ACG_*([a,b])$, be a solution of the initial value problem (3.1). Since $f' \in C([a,b])$, it follows that $f(x) = \int_a^x f'(s)\,ds + f(a)$ and hence from Theorem 2.3 f is of bounded variation on $[a,b]$. This implies, by Theorem 2.4, that $qf \in HK([a,b])$. We set $u = \begin{pmatrix} f \\ f' \end{pmatrix}$, then $u \in C([a,b],\mathbb{C}^2)$ and $Au = \begin{pmatrix} f' \\ qf - \lambda f \end{pmatrix} \in HK([a,b])$. Therefore for all $x \in [a,b]$,

$$\begin{aligned} \int_c^x A(s)u(s)\,ds + w(x) &= \begin{pmatrix} \int_c^x f'(s)\,ds + \alpha \\ \int_c^x [q(s)f(s) - \lambda f(s)]\,ds + \beta - \int_c^x g(s)\,ds \end{pmatrix} \\ &= \begin{pmatrix} \int_c^x f'(s)\,ds + f(c) \\ \int_c^x [f''(s) + g(s)]\,ds + f'(c) - \int_c^x g(s)\,ds \end{pmatrix} \\ &= \begin{pmatrix} f(c) + \int_c^x f'(s)\,ds \\ \int_c^x f''(s)\,ds + f'(c) \end{pmatrix} \\ &= \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix}, \end{aligned}$$

where the last equality is due to Theorem 2.8 (2) because $f, f' \in ACG_*([a,b])$. Therefore u satisfies the equation (3.2).

Conversely, let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in C([a,b],\mathbb{C}^2)$ be a solution of equation (3.2); then for every $x \in [a,b]$,

$$\begin{aligned} u_1(x) &= \int_c^x u_2(s)\,ds + \alpha, \\ u_2(x) &= \int_c^x [(q(s)-\lambda)u_1(s) - g(s)]\,ds + \beta. \end{aligned}$$

Therefore $u_1' = u_2$ on $[a,b]$ and, from Theorem 2.8 (1), u_2 is ACG_* on $[a,b]$ and $u_2' = (q-\lambda)u_1 - g$ almost everywhere on $[a,b]$. This implies that $u_1, u_1' \in ACG_*([a,b])$ and $(L-\lambda)u_1 = g$ almost everywhere on $[a,b]$.

The uniqueness in any of the situations follows from the above. □

Theorem 3.2. *Let $c \in [a,b]$, $\lambda, \alpha, \beta \in \mathbb{C}$. If $g \in HK([a,b])$, then there exists a unique solution f, $f' \in ACG_*([a,b])$ of the initial-value problem*

$$\begin{aligned} (L-\lambda)y &= g \quad a.e. \\ y(c) &= \alpha \\ y'(c) &= \beta. \end{aligned}$$

Proof. Let (y_n), (z_n) be two sequences of functions defined on $[a,b]$ as $y_0(x) = \alpha$, $z_0(x) = \beta$, and for all $n \in \mathbb{N}$,

$$y_n(x) = \alpha + \int_c^x z_{n-1}(s)\,ds \quad \text{and} \quad z_n(x) = \beta - \int_c^x g(s)\,ds + \int_c^x (q(s)-\lambda)y_{n-1}(s)\,ds.$$

Clearly y_0, z_0, y_1 and z_1 are well defined. Suppose that $y_2, z_2, \ldots, y_n, z_n$ exist. Since $z_{n-1} \in L([a,b])$, from Theorem 2.3, y_n is of bounded variation on $[a,b]$ and hence, by Theorem 2.4, $(q-\lambda)y_n \in HK([a,b])$. Then z_{n+1} exists. Also observe that clearly y_{n+1} exists. Therefore (y_n) and (z_n) are well defined.

We claim that for every $x \in [a,b]$ with $x > c$,

$$\int_c^x |z_n(s) - z_{n-1}(s)|\,ds \leq \begin{cases} \|(q-\lambda)\alpha - g\|_{[a,b]} \dfrac{(x-c)^{k+1}}{(k+1)!} \|q-\lambda\|_{[a,b]}^k, & \text{if } n = 2k+1; \\ |\beta| \dfrac{(x-c)^{k+1}}{(k+1)!} \|q-\lambda\|_{[a,b]}^k, & \text{if } n = 2k. \end{cases} \tag{3.3}$$

We prove this only for n odd, by induction on k. For $k=0$ we have that

$$\begin{aligned} \int_c^x |z_1(s) - z_0(s)|\,ds &= \int_c^x \left| \int_c^s [(q(t)-\lambda)\alpha - g(t)]\,dt \right| ds \\ &\leq \int_c^x \|(q-\lambda)\alpha - g\|_{[a,b]}\,ds \\ &= \|(q-\lambda)\alpha - g\|_{[a,b]}(x-c) \end{aligned}$$

for all $x > c$. Now, suppose that for every $x > c$,

$$\int_c^x |z_{2k+1}(s) - z_{2k}(s)|\,ds \leq \|(q-\lambda)\alpha - g\|_{[a,b]} \frac{(x-c)^{k+1}}{(k+1)!} \|q-\lambda\|_{[a,b]}^k.$$

Let $x > c$, observe that

$$\int_c^x \left| z_{2(k+1)+1}(s) - z_{2(k+1)}(s) \right| ds = \int_c^x \left| \int_c^s (q(t)-\lambda)[y_{2(k+1)}(t) - y_{2k+1}(t)]\,dt \right| ds.$$

Now, since $y_{2(k+1)} - y_{2k+1}$ is of bounded variation on $[c,s]$ and

$$V_{[c,s]}(y_{2(k+1)} - y_{2k+1}) \leq \int_c^s |z_{2k+1}(t) - z_{2k}(t)|\,dt,$$

it follows by Theorem 2.5 that

$$\begin{aligned}
&\int_c^x \left|\int_c^s (q(t)-\lambda)[y_{2(k+1)}(t) - y_{2k+1}(t)]\,dt\right| ds \\
&\quad\leq \int_c^x \left[\inf_{t\in[c,s]} |y_{2(k+1)}(t) - y_{2k+1}(t)| + V_{[c,s]}(y_{2(k+1)} - y_{2k+1})\right] \|q-\lambda\|_{[a,b]}\,ds \\
&\quad\leq \int_c^x \left[|y_{2(k+1)}(c) - y_{2k+1}(c)| + V_{[c,s]}(y_{2(k+1)} - y_{2k+1})\right] \|q-\lambda\|_{[a,b]}\,ds \\
&\quad= \int_c^x V_{[c,s]}(y_{2(k+1)} - y_{2k+1})\,ds\,\|q-\lambda\|_{[a,b]} \\
&\quad\leq \int_c^x \left[\int_c^s |z_{2k+1}(t) - z_{2k}(t)|\,dt\right] ds\,\|q-\lambda\|_{[a,b]} \\
&\quad\leq \int_c^x \left[\|(q-\lambda)\alpha - g\|_{[a,b]} \frac{(s-c)^{k+1}}{(k+1)!}\|q-\lambda\|_{[a,b]}^k\right] ds\,\|q-\lambda\|_{[a,b]} \\
&\quad\leq \|(q-\lambda)\alpha - g\|_{[a,b]} \frac{(x-c)^{k+2}}{(k+2)!}\|q-\lambda\|_{[a,b]}^{k+1}.
\end{aligned}$$

Thus (3.3) follows by induction. Similarly we can prove that for every $x \in [a,b]$ with $x < c$,

$$\begin{aligned}
&\int_x^c |z_n(s) - z_{n-1}(s)|\,ds \\
&\qquad\leq \begin{cases} \|(q-\lambda)\alpha - g\|_{[a,b]} \dfrac{(c-x)^{k+1}}{(k+1)!}\|q-\lambda\|_{[a,b]}^k, & \text{if } n = 2k+1; \\ |\beta| \dfrac{(c-x)^{k+1}}{(k+1)!}\|q-\lambda\|_{[a,b]}^k, & \text{if } n = 2k. \end{cases}
\end{aligned} \tag{3.4}$$

For each $k \in \mathbb{N}$ define $l_k = y_{2k+1} - y_{2k}$. Take $k \in \mathbb{N}$ and $x \in [a,b]$. If $x < c$, then

$$\begin{aligned}
|l_k(x)| &= |y_{2k+1}(x) - y_{2k}(x)| \\
&= \left|\int_c^x [z_{2k}(s) - z_{2k-1}(s)]\,ds\right| \\
&\leq \int_x^c |z_{2k}(s) - z_{2k-1}(s)|\,ds \\
&\leq |\beta| \frac{(c-x)^{k+1}}{(k+1)!}\|q-\lambda\|_{[a,b]}^k.
\end{aligned}$$

Therefore

$$|l_k(x)| \leq \frac{|\beta|}{\|q-\lambda\|_{[a,b]}} \frac{(b-a)^{k+1}\|q-\lambda\|_{[a,b]}^{k+1}}{(k+1)!}.$$

This inequality also holds when $x \geq c$.

Now, since

$$\sum_{k=1}^{\infty} \frac{|\beta|}{\|q-\lambda\|_{[a,b]}} \frac{(b-a)^{k+1}\|q-\lambda\|_{[a,b]}^{k+1}}{(k+1)!} \leq \frac{|\beta|}{\|q-\lambda\|_{[a,b]}} e^{(b-a)\|q-\lambda\|_{[a,b]}} < \infty,$$

it follows that $\sum_{k=1}^{\infty} l_k$ converges uniformly on $[a,b]$. Again using the equations (3.3) and (3.4), we obtain that if $h_k = y_{2k} - y_{2k-1}$ then $\sum_{k=1}^{\infty} h_k$ converges uniformly on $[a,b]$. Therefore $y_0 + \sum_{n=1}^{\infty}[y_n - y_{n-1}] = y_0 + [y_1 - y_0] + \sum_{k=1}^{\infty} l_k + \sum_{k=1}^{\infty} h_k$ converges uniformly on $[a,b]$. Thus its sequences of partial sums s_n converges uniformly to a limit function y on $[a,b]$. But

$$s_n(x) = y_0 + \sum_{k=1}^{n}[y_k(x) - y_{k-1}(x)] = y_n(x).$$

In other words, the sequence (y_n) converges uniformly to y on $[a,b]$.

On the other hand, from the inequalities

$$|z_{2k+1}(x) - z_{2k}(x)| \leq \|(q-\lambda)\alpha - g\|_{[a,b]} \frac{(b-a)^k\|q-\lambda\|_{[a,b]}^k}{k!}$$

and

$$|z_{2k}(x) - z_{2k-1}(x)| \leq |\beta| \frac{(b-a)^k\|q-\lambda\|_{[a,b]}^k}{k!},$$

it follows that $z_0 + \sum_{k=0}^{\infty}[z_n - z_{n-1}]$ converges uniformly on $[a,b]$. If z denotes its sum then z_n converges uniformly to z on $[a,b]$.

Consequently, for each $x \in [a,b]$,

$$\begin{aligned} y(x) = \lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty}\left[\alpha + \int_c^x z_{n-1}(s)\,ds\right] \\ &= \alpha + \lim_{n\to\infty}\int_c^x z_{n-1}(s)\,ds \\ &= \alpha + \int_c^x \lim_{n\to\infty} z_{n-1}(s)\,ds \\ &= \alpha + \int_c^x z(s)\,ds. \end{aligned}$$

It is not possible to apply the above idea to the sequence (y_n) because $(q-\lambda)y_{n-1}$ might not converge uniformly to $(q-\lambda)y$. However, we can use Theorem 2.6 in order to have a similar result to the above. Let $x \in [a,b]$ with $x > c$. We first show that (y_n) is of uniformly bounded variation on $[c,x]$.

For every $n \in \mathbb{N}$,

$$\begin{aligned}
V_{[c,x]}y_n &= V_{[c,x]}\left[y_1 + \sum_{k=1}^{n-1}[y_{k+1} - y_k]\right] \\
&\le V_{[c,x]}y_1 + \sum_{k=1}^{n-1} V_{[c,x]}[y_{k+1} - y_k] \\
&\le V_{[c,x]}y_1 + \sum_{k=1}^{n-1} \int_c^x |z_k(s) - z_{k-1}(s)|\, ds \\
&\le V_{[c,x]}y_1 + \sum_{k=1}^{\infty} \int_c^x |z_k(s) - z_{k-1}(s)|\, ds \\
&= V_{[c,x]}y_1 + \int_c^x |z_1(s) - z_0(s)|\, ds \\
&\quad + \sum_{k=1}^{\infty} \int_c^x |z_{2k+1}(s) - z_{2k}(s)|\, ds + \sum_{k=1}^{\infty} \int_c^x |z_{2k}(s) - z_{2k-1}(s)|\, ds \\
&\le \left[\frac{\|(q-\lambda)\alpha - q\|_{[a,b]}}{\|q-\lambda\|_{[a,b]}} + \frac{|\beta|}{\|q-\lambda\|_{[a,b]}}\right] \sum_{k=0}^{\infty} \frac{(b-a)^{k+1}\|q-\lambda\|_{[a,b]}^{k+1}}{(k+1)!} \\
&= \frac{\|(q-\lambda)\alpha - q\|_{[a,b]} + |\beta|}{\|q-\lambda\|_{[a,b]}} e^{(b-a)\|q-\lambda\|}.
\end{aligned}$$

Therefore, by Theorem 2.6, it follows that

$$\lim_{n\to\infty} \int_c^x (q(s) - \lambda)y_{n-1}(s) = \int_c^x (q(s) - \lambda)y(s)\, ds.$$

Thus

$$\begin{aligned}
z(x) = \lim_{n\to\infty} z_n(x) &= \lim_{n\to\infty}\left[\beta - \int_c^x g(s)\, ds + \int_c^x (q(s)-\lambda)y_{n-1}(s)\, ds\right] \\
&= \beta - \int_c^x g(s)\, ds + \lim_{n\to\infty}\int_c^x (q(s)-\lambda)y_{n-1}(s)\, ds \qquad (3.5) \\
&= \beta - \int_c^x g(s)\, ds + \int_c^x (q(s)-\lambda)y(s)\, ds.
\end{aligned}$$

A similar reasoning shows that the equality (3.5) holds when $x < c$.

Consequently, $u = \begin{pmatrix} y \\ z \end{pmatrix}$ is a solution of equation (3.2). We shall now prove that this solution is unique. Assume that $\begin{pmatrix} \overline{y} \\ \overline{z} \end{pmatrix}$ is another solution of equation (3.2). Therefore

$$\overline{y}(x) = \alpha + \int_c^x \overline{z}(s)\, ds$$

and

$$\overline{z}(x) = \beta - \int_c^x g(s)\,ds + \int_c^x [q(s) - \lambda]\overline{y}(s)\,ds$$

for all $x \in [a,b]$. Observe that for every $k \in \mathbb{N}$,

$$\int_c^x |z_{2k}(s) - \overline{z}(s)|\,ds \;\le\; \|(q-\lambda)\overline{y} - g\|_{[a,b]} \frac{(x-c)^{k+1}}{(k+1)!}\|q-\lambda\|^k, \text{ if } c < x,$$

and

$$\int_x^c |z_{2k}(s) - \overline{z}(s)|\,ds \;\le\; \|(q-\lambda)\overline{y} - g\|_{[a,b]} \frac{(c-x)^{k+1}}{(k+1)!}\|q-\lambda\|^k, \text{ if } x < c.$$

From this it follows that

$$|y_{2k+1}(x) - \overline{y}(x)| \;\le\; \frac{\|(q-\lambda)\overline{y} - g\|_{[a,b]}}{\|q-\lambda\|} \frac{(b-a)^{k+1}}{(k+1)!}\|q-\lambda\|^{k+1}$$

and

$$|z_{2(k+1)}(x) - \overline{z}(x)| \;\le\; \frac{\|(q-\lambda)\overline{y} - g\|_{[a,b]}}{\|q-\lambda\|} \frac{(b-a)^{k+1}}{(k+1)!}\|q-\lambda\|^{k+1}$$

for all $x \in [a,b]$. Thus $y_{2k+1}(x) \to \overline{y}(x)$ and $z_{2(k+1)}(x) \to \overline{z}(x)$, but $y_{2k+1}(x) \to y(x)$ and $z_{2(k+1)}(x) \to z(x)$. Therefore $\overline{y}(x) = y(x)$ and $\overline{z}(x) = z(x)$. □

4. Example

In this section, we give an example for the application of Theorem 3.2.

Example 4.1. Let q be a function defined on $[0,1]$ as

$$q(x) = \begin{cases} \dfrac{2\pi}{x}\sin\left(\dfrac{\pi}{x^2}\right), & \text{if } x \in (0,1]; \\ 0, & \text{if } x = 0, \end{cases}$$

and let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \dfrac{(-1)^{k+1}2^k}{k}, & \text{for } x \in [c_{k-1}, c_k),\ k \in \mathbb{N}; \\ 0, & \text{for } x = 1, \end{cases}$$

where $c_k = 1 - \frac{1}{2^k}$, $k = 0, 1, 2, \dots$ Then q and g are unbounded HK-integrable functions on $[0,1]$. Therefore, by Theorem 3.2, the initial value problem

$$-y'' + q(x)y - 2y = g(x) \quad \text{a.e.}$$
$$y\left(\frac{1}{2}\right) = 0$$
$$y'\left(\frac{1}{2}\right) = 1$$

has a solution.

The functions q and g are not Lebesgue integrable on $[0,1]$. Hence, this example is not covered by any result using the Lebesgue integral. Thus, Theorem 3.2 is more general than the classical result of existence and uniqueness given at the introduction.

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