REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 60, No. 1, 2019, Pages 9–20
Published online: February 11, 2019
https://doi.org/10.33044/revuma.v60n1a02

# LIE $n$-MULTIPLICATIVE MAPPINGS ON TRIANGULAR $n$-MATRIX RINGS

BRUNO L. M. FERREIRA AND HENRIQUE GUZZO JR.

Abstract. We extend to triangular $n$-matrix rings and Lie $n$-multiplicative maps a result about Lie multiplicative maps on triangular algebras due to Xiaofei Qi and Jinchuan Hou.

## 1. Introduction

Let $\mathfrak{R}$ be an associative ring and $[x_1, x_2] = x_1x_2 - x_2x_1$ denote the usual Lie product of $x_1$ and $x_2$. Let us define the following sequence of polynomials: $p_1(x) = x$ and $p_n(x_1, x_2, \ldots, x_n) = [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]$ for all integers $n \geq 2$. Thus, $p_2(x_1, x_2) = [x_1, x_2]$, $p_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$, etc. Let $n \geq 2$ be an integer. Assume that $\mathfrak{S}$ is any ring. A map $\varphi : \mathfrak{R} \to \mathfrak{S}$ is called a *Lie $n$-multiplicative mapping* if

$$\varphi(p_n(x_1, x_2, \ldots, x_n)) = p_n(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)). \tag{1}$$

In particular, if $n = 2$, $\varphi$ will be called a *Lie multiplicative mapping.* And, if $n = 3$, $\varphi$ will be called a *Lie triple multiplicative mapping.*

The study on the question of when a particular application between two rings is additive has become an area of great interest in the theory of rings. One of the first results ever recorded was given by Martindale III, which in his condition requires that the ring possess idempotents, see [4]. Xiaofei Qi and Jinchuan Hou [6] also considered this question in the context of triangular algebras. They proved the following theorem.

**Theorem 1.1** (Xiaofei Qi and Jinchuan Hou [6]). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and let $M$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra and $\mathcal{V}$ any algebra over $\mathcal{R}$. Assume that $\Phi : \mathcal{U} \to \mathcal{V}$ is a Lie multiplicative isomorphism, that is, $\Phi$ satisfies*

$$\Phi(ST - TS) = \Phi(S)\Phi(T) - \Phi(T)\Phi(S) \quad \forall S, T \in \mathcal{U}.$$

*Then $\Phi(S + T) = \Phi(S) + \Phi(T) + Z_{S,T}$ for all $S, T \in \mathcal{U}$, where $Z_{S,T}$ is an element in the centre $\mathcal{Z}(\mathcal{V})$ of $\mathcal{V}$ depending on $S$ and $T$.*

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2010 *Mathematics Subject Classification.* 47L35; 16W25.
*Key words and phrases.* Triangular $n$-matrix rings; Additivity; Lie $n$-multiplicative maps.

This motivated us to discuss the additivity of Lie $n$-multiplicative mappings on another kind of rings: triangular $n$-matrix rings. In this paper, we give a full answer to this question, where Xiaofei Qi and Jinchuan Hou's result is a consequence of our case.

## 2. Motivation and definitions

For any unital ring $\mathfrak{R}$, let $\mathrm{Mod}(\mathfrak{R})$ denote the category of unitary $\mathfrak{R}$-modules, i.e., satisfying $1m = m$ for all elements $m$. This category is important in many areas of mathematics such as ring theory, representation theory, and homological algebra. The purpose of this paper is to work with a more general category, that is, $\mathrm{Mod}(\mathfrak{R})$ where $\mathfrak{R}$ does not have identity element. In this paper $\widetilde{\mathfrak{R}}$ denotes the unital ring $\mathfrak{R} \times \mathbb{Z}$. Define operations on $\widetilde{\mathfrak{R}}$ by

$$(r, \lambda) + (t, \mu) := (r + t, \lambda + \mu),$$
$$(r, \lambda) \cdot (t, \mu) := (rt + \lambda t + \mu r, \lambda\mu).$$

Then $\widetilde{\mathfrak{R}}$ is a ring with $(0_{\mathfrak{R}}, 1) := 1_{\widetilde{\mathfrak{R}}}$ as multiplicative identity. If $M$ is a non unitary $\mathfrak{R}$-module, define a right $\widetilde{\mathfrak{R}}$-module operation by

$$(r, \lambda)m := rm + \lambda m$$

and a left $\widetilde{\mathfrak{R}}$-module operation by

$$m(r, \lambda) := mr + \lambda m,$$

where the action $\mathbb{Z}$ on $M$ is the usual of $M$ as a $\mathbb{Z}$-module. A module over $\mathfrak{R}$ is the same thing as a unitary $\widetilde{\mathfrak{R}}$-module.

The following definition is a generalization of the definition that arises in the work of W. S. Cheung [1]. This definition appears in Ferreira's paper [3].

**Definition 2.1.** Let $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_n$ be rings and $\mathfrak{M}_{ij}$ $(\mathfrak{R}_i, \mathfrak{R}_j)$-bimodules with $\mathfrak{M}_{ii} = \mathfrak{R}_i$ for all $1 \le i \le j \le n$. Let $\varphi_{ijk} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{M}_{jk} \longrightarrow \mathfrak{M}_{ik}$ be $(\mathfrak{R}_i, \mathfrak{R}_k)$-bimodules homomorphisms with $\varphi_{iij} : \mathfrak{R}_i \otimes_{\mathfrak{R}_i} \mathfrak{M}_{ij} \longrightarrow \mathfrak{M}_{ij}$ and $\varphi_{ijj} : \mathfrak{M}_{ij} \otimes_{\mathfrak{R}_j} \mathfrak{R}_j \longrightarrow \mathfrak{M}_{ij}$ the canonical multiplication maps for all $1 \le i \le j \le k \le n$. Write $ab = \varphi_{ijk}(a \otimes b)$ for $a \in \mathfrak{M}_{ij}$, $b \in \mathfrak{M}_{jk}$. We consider

(i) $\mathfrak{M}_{ij}$ is faithful as a left $\mathfrak{R}_i$-module and faithful as a right $\mathfrak{R}_j$-module $i < j$.
(ii) if $m_{ij} \in \mathfrak{M}_{ij}$ is such that $\mathfrak{R}_i m_{ij} \mathfrak{R}_j = 0$ then $m_{ij} = 0$ $i < j$.

Let

$$\mathfrak{T} = \left\{ \begin{pmatrix} r_{11} & m_{12} & \ldots & m_{1n} \\ & r_{22} & \ldots & m_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix}_{n \times n} : \underbrace{r_{ii} \in \mathfrak{R}_i\ (= \mathfrak{M}_{ii}),\ m_{ij} \in \mathfrak{M}_{ij}}_{(1 \le i < j \le n)} \right\}$$

be the set of all $n \times n$ matrices $[m_{ij}]$ with the $(i, j)$-entry $m_{ij} \in \mathfrak{M}_{ij}$ for all $1 \le i \le j \le n$. Observe that, with the obvious matrix operations of addition and multiplication, $\mathfrak{T}$ is a ring iff $a(bc) = (ab)c$ for all $a \in \mathfrak{M}_{ik}$, $b \in \mathfrak{M}_{kl}$ and $c \in \mathfrak{M}_{lj}$

for all $1 \leq i \leq k \leq l \leq j \leq n$. When $\mathfrak{T}$ is a ring, it is called a *triangular $n$-matrix ring.*

Note that if $n = 2$ we have the triangular matrix ring. As in [3] we denote by $\bigoplus_{i=1}^{n} r_{ii}$ the element $\begin{pmatrix} r_{11} & 0 & \dots & 0 \\ & r_{22} & \dots & 0 \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix}$ in $\mathfrak{T}$.

Set $\mathfrak{T}_{ij} = \left\{ (m_{kt}) : m_{kt} = \begin{cases} m_{ij}, & \text{if } (k,t) = (i,j) \\ 0, & \text{if } (k,t) \neq (i,j) \end{cases}, \ i \leq j \right\}$. Then we can write $\mathfrak{T} = \bigoplus_{1 \leq i \leq j \leq n} \mathfrak{T}_{ij}$. Henceforth the element $a_{ij}$ belongs to $\mathfrak{T}_{ij}$ and the corresponding elements are in $\mathfrak{R}_1, \dots, \mathfrak{R}_n$ or $\mathfrak{M}_{ij}$. By a direct calculation $a_{ij}a_{kl} = 0$ if $j \neq k$. Also as in [3] we define natural projections $\pi_{\mathfrak{R}_i} : \mathfrak{T} \longrightarrow \mathfrak{R}_i$ $(1 \leq i \leq n)$ by

$$\begin{pmatrix} r_{11} & m_{12} & \dots & m_{1n} \\ & r_{22} & \dots & m_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{pmatrix} \longmapsto r_{ii}.$$

**Definition 2.2.** Let $\mathfrak{R}$, $\mathfrak{S}$ be rings; we shall say that the Lie $n$-multiplicative mapping $\varphi : \mathfrak{R} \to \mathfrak{S}$ is almost additive if there exist $S_{A,B}$ in the centre $\mathcal{Z}(\mathfrak{S})$ of $\mathfrak{S}$ depending on $A$ and $B$ such that

$$\varphi(A + B) = \varphi(A) + \varphi(B) + S_{A,B}$$

for all $A, B \in \mathfrak{R}$.

The proposition below appears in [3]; it is a generalization of Proposition 3 of [1] and will be very useful.

**Proposition 2.1.** *Let $\mathfrak{T}$ be a triangular $n$-matrix ring. The center of $\mathfrak{T}$ is*

$$\mathfrak{Z}(\mathfrak{T}) = \left\{ \bigoplus_{i=1}^{n} r_{ii} \mid r_{ii}m_{ij} = m_{ij}r_{jj} \text{ for all } m_{ij} \in \mathfrak{M}_{ij},\ i < j \right\}.$$

*Furthermore, $\mathfrak{Z}(\mathfrak{T})_{ii} \cong \pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{T})) \subseteq \mathfrak{Z}(\mathfrak{R}_i)$, and there exists a unique ring isomorphism $\tau_i^j$ from $\pi_{\mathfrak{R}_i}(\mathfrak{Z}(\mathfrak{T}))$ to $\pi_{\mathfrak{R}_j}(\mathfrak{Z}(\mathfrak{T}))$ $i \neq j$ such that $r_{ii}m_{ij} = m_{ij}\tau_i^j(r_{ii})$ for all $m_{ij} \in \mathfrak{M}_{ij}$.*

**Remark 2.1.** Throughout this paper we shall make some identifications. For example: let $r_{kk} \in \mathfrak{R}_k$ and $m_{ij} \in \mathfrak{M}_{ij}$; then

$$r_{kk} \equiv \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & \dots & 0 \\ & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ & & 0 & \dots & 0 & \cdots & 0 \\ & & & r_{kk} & 0 & \cdots & 0 \\ & & & & 0 & \cdots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 0 \end{pmatrix}$$

and

$$m_{ij} \equiv \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & \dots & 0 \\ & \ddots & \dots & \dots & \vdots & \dots & \vdots \\ & & 0 & \cdots & m_{ij} & \cdots & 0 \\ & & & \ddots & \vdots & & \vdots \\ & & & & 0 & \cdots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 0 \end{pmatrix}$$

where $i, j, k \in \{1, 2, \dots, n\}$.

In addition we have the following identifications:

Let $\widetilde{\mathfrak{R}} = \mathfrak{R} \times \mathbb{Z}$ be a unital ring. If $r \in \mathfrak{R}$ then $r \equiv (r, 0)$. And $\varphi \times Id : \mathfrak{R} \times \mathbb{Z} \to \mathfrak{S} \times \mathbb{Z}$ with $(\varphi \times Id)(r, \lambda) = (\varphi(r), Id(\lambda)) = (\varphi(r), \lambda)$, where $Id$ is the identity map on $\mathbb{Z}$. By a straightforward calculus it is shown that $\varphi \times Id$ is a Lie $n$-multiplicative mapping. Sometimes we shall do $\varphi \times Id \equiv \varphi$.

## 3. A key lemma

The following results are generalizations of those that appear in [6].

**Lemma 3.1** (Standard lemma)**.** *Let $A, B, C \in \mathfrak{R}$ and $\varphi(C) = \varphi(A) + \varphi(B)$. Then for any $T_1, T_2, \dots, T_{n-1} \in \mathfrak{R}$, we have*

$$\begin{aligned} \varphi(p_n(C, T_1, T_2, \dots, T_{n-1})) = \varphi(p_n(A, T_1, T_2, \dots, T_{n-1})) \\ + \varphi(p_n(B, T_1, T_2, \dots, T_{n-1})). \end{aligned}$$

*Proof.* Using (1) we have

$$\begin{aligned} \varphi(p_n(C, T_1, T_2, \dots, T_{n-1})) &= p_n(\varphi(C), \varphi(T_1), \varphi(T_2), \dots, \varphi(T_{n-1})) \\ &= p_n(\varphi(A) + \varphi(B), \varphi(T_1), \varphi(T_2), \dots, \varphi(T_{n-1})) \\ &= p_n(\varphi(A), \varphi(T_1), \varphi(T_2), \dots, \varphi(T_{n-1})) \\ &\quad + p_n(\varphi(B), \varphi(T_1), \varphi(T_2), \dots, \varphi(T_{n-1})) \\ &= \varphi(p_n(A, T_1, T_2, \dots, T_{n-1})) \\ &\quad + \varphi(p_n(B, T_1, T_2, \dots, T_{n-1})). \end{aligned}$$ □

Note that if

$$\begin{aligned} \varphi(p_n(A, T_1, T_2, \dots, T_{n-1})) + \varphi(p_n(B, T_1, T_2, \dots, T_{n-1})) \\ = \varphi(p_n(A, T_1, T_2, \dots, T_{n-1}) + p_n(B, T_1, T_2, \dots, T_{n-1})), \end{aligned}$$

then by the injectivity of $\varphi$ we get

$$p_n(C, T_1, T_2, \dots, T_{n-1}) = p_n(A, T_1, T_2, \dots, T_{n-1}) + p_n(B, T_1, T_2, \dots, T_{n-1}).$$

**Lemma 3.2.** *Let $\mathfrak{R}_1, \mathfrak{R}_2, \dots, \mathfrak{R}_n$ be rings and $\mathfrak{M}_{ij}$ $(\mathfrak{R}_i, \mathfrak{R}_j)$-bimodules as in Definition 2.1. Let $\mathfrak{T}$ be the triangular $n$-matrix ring. If $p_2(A, \mathfrak{T}) \subseteq \mathcal{Z}(\mathfrak{T})$ then $A \in \mathcal{Z}(\mathfrak{T})$ for each $A \in \mathfrak{T}$. Moreover, if $p_n(A, \mathfrak{T}, \dots, \mathfrak{T}) \subseteq \mathcal{Z}(\mathfrak{T})$ then $A \in \mathcal{Z}(\mathfrak{T})$ for each $A \in \mathfrak{T}$.*

*Proof.* Let $A = \bigoplus_{1\leq i\leq j\leq n} a_{ij}$ such that $p_2(A,\mathfrak{T}) \subseteq \mathcal{Z}(\mathfrak{T})$. Thus $p_2(A,\mathfrak{R}_k) \subseteq \mathcal{Z}(\mathfrak{T})$ for $k = 1,\ldots,n-1$, and it follows that

$$p_2(A,\mathfrak{R}_k) = \bigoplus_i p_2(a_{ik},\mathfrak{R}_k) + \bigoplus_j p_2(a_{kj},\mathfrak{R}_k) + p_2(a_{kk},\mathfrak{R}_k),$$

because

- If $i < j$, $k \neq i$ and $k \neq j$ then $p_2(a_{ij},\mathfrak{R}_k) = 0$;
- If $i = j$ and $k \neq i$ then $p_2(a_{ii},\mathfrak{R}_k) = 0$.

As $p_2(A,\mathfrak{R}_k) \subseteq \mathcal{Z}(\mathfrak{T})$ we have $p_2(A,\mathfrak{R}_k) = p_2(a_{kk},\mathfrak{R}_k)$ for $k = 1,\ldots,n-1$ by Proposition 2.1. Consequently, $p_2(A,\mathfrak{R}_k)m_{kj} = 0$ for $k < j$. Since $\mathfrak{M}_{kj}$ is faithful as a left $\mathfrak{R}_k$-module, we see that $p_2(A,\mathfrak{R}_k) = 0$ for $k = 1,\ldots,n-1$. In the case $k = n$ we use an analogous argument and we obtain $m_{in}p_2(A,\mathfrak{R}_n) = 0$. And since $\mathfrak{M}_{in}$ is faithful as a right $\mathfrak{R}_n$-module, it follows that $p_2(A,\mathfrak{R}_n) = 0$.

Now note that $p_2(A,\mathfrak{M}_{ij}) = p_2(A_{ii},\mathfrak{M}_{ij}) + p_2(A_{jj},\mathfrak{M}_{ij})$ and as $p_2(A,\mathfrak{M}_{ij}) \subseteq \mathcal{Z}(\mathfrak{T})$ we get $p_2(A,\mathfrak{M}_{ij}) = 0$ for $i < j$. Therefore $p_2(A,\mathfrak{T}) = 0$, that is, $A \in \mathcal{Z}(\mathfrak{T})$. □

## 4. Main results

Let's state our main result in this section, which is a generalization of Theorem 2.1 in [6].

**Theorem 4.1.** *Let $\mathfrak{T}$ be the triangular $n$-matrix ring and $\mathfrak{S}$ any ring. Consider $\varphi : \mathfrak{T} \to \mathfrak{S}$ a bijection Lie $n$-multiplicative mapping satisfying*

*(i) $\varphi(\mathcal{Z}(\mathfrak{T})) \subset \mathcal{Z}(\mathfrak{S})$.*

*Then*

$$\varphi(A + B) = \varphi(A) + \varphi(B) + S_{A,B}$$

*for all $A, B \in \mathfrak{T}$, where $S_{A,B}$ is an element in the centre $\mathcal{Z}(\mathfrak{S})$ of $\mathfrak{S}$ depending on $A$ and $B$.*

To prove Theorem 4.1 we introduce a set of lemmas, almost all of which are generalizations of claims in [6]. We begin with the following lemma.

**Lemma 4.1.** $\varphi(0) = 0$.

*Proof.* Indeed, $\varphi(0) = \varphi(p_n(0,0,\ldots,0)) = p_n(\varphi(0),\varphi(0),\ldots,\varphi(0)) = [\cdots[[\varphi(0), \varphi(0)],\varphi(0)],\ldots] = 0$. □

**Lemma 4.2.** *For any $A \in \mathfrak{T}$ and any $Z \in \mathcal{Z}(\mathfrak{T})$, there exists $S \in \mathcal{Z}(\mathfrak{S})$ such that $\varphi(A + Z) = \varphi(A) + S$.*

*Proof.* Note that $\varphi^{-1}$ is also a bijection Lie $n$-multiplicative map. Let $A \in \mathfrak{T}$, $Z' \in \mathcal{Z}(\mathfrak{T})$ and $T_2,\ldots,T_n \in \mathfrak{T}$. As $\varphi^{-1}$ is surjective we have $\varphi^{-1}(S') = A$ and

$\varphi^{-1}(S) = Z'$. Now as $\varphi(\mathcal{Z}(\mathfrak{T})) \subset \mathcal{Z}(\mathfrak{S})$ by condition (i) of Theorem 4.1 we have

$$\begin{aligned}\varphi(p_n(\varphi^{-1}(S'+S), T_2, \ldots, T_n)) &= p_n(S'+S, \varphi(T_2), \ldots, \varphi(T_n)) \\ &= p_n(S', \varphi(T_2), \ldots, \varphi(T_n)) \\ &= \varphi(p_n(\varphi^{-1}(S'), T_2, \ldots, T_n)).\end{aligned}$$

Thus, $p_n(\varphi^{-1}(S'+S) - \varphi^{-1}(S'), T_2, \ldots, T_n) = 0$, and it follows that $\varphi^{-1}(S'+S) - \varphi^{-1}(S') \in \mathcal{Z}(\mathfrak{T})$ by Lemma 3.2. Therefore $\varphi(A+Z) = \varphi(A) + S$. □

**Lemma 4.3.** *For any $a_{kk} \in \mathfrak{R}_k$, $a_{ii} \in \mathfrak{R}_i$, $a_{jj} \in \mathfrak{R}_j$ and $m_{ij} \in \mathfrak{M}_{ij}$, $i<j$, there exist $S, S_1, S_2 \in \mathcal{Z}(\mathfrak{S})$ such that*

- $\varphi(a_{kk} + \bigoplus_{i<j} m_{ij}) = \varphi(a_{kk}) + \varphi(\bigoplus_{i<j} m_{ij}) + S$;
- $\varphi(a_{ii} + m_{ij}) = \varphi(a_{ii}) + \varphi(m_{ij}) + S_1$;
- $\varphi(a_{jj} + m_{ij}) = \varphi(a_{jj}) + \varphi(m_{ij}) + S_2$.

*Proof.* We shall only prove the first item because the demonstrations of the others are similar. It is important to note that if $\varphi$ is Lie 2-multiplicative mapping (i.e., Lie multiplicative mapping), then $\varphi$ is also Lie $n$-multiplicative mapping for any $n \geq 3$. Therefore, the following proof is valid for all $n \geq 2$. As $\varphi$ is surjective, there is an element $H = \bigoplus_{1\leq i\leq j\leq n} h_{ij} \in \mathfrak{T}$ such that

$$\varphi(H) = \varphi(a_{kk}) + \varphi(\bigoplus_{i<j} m_{ij}).$$

Let $b_{jj} \in \mathfrak{R}_j$, $j \neq k$ and $T_3, \ldots, T_n \in \mathfrak{T}$; by Lemma 3.1 we have

$$\begin{aligned}\varphi(p_n(H, b_{jj}, T_3, \ldots, T_n)) &= \varphi(p_n(a_{kk}, b_{jj}, T_3, \ldots, T_n)) \\ &\quad + \varphi(p_n(\bigoplus_{i<j} m_{ij}, b_{jj}, T_3, \ldots, T_n)) \\ &= \varphi(p_n(m_{ij}, b_{jj}, T_3, \ldots, T_n)).\end{aligned}$$

It follows that $p_n(H - m_{ij}, b_{jj}, T_3, \ldots, T_n) \in \mathcal{Z}(\mathfrak{T})$ and by Lemma 3.2 we get $p_2(H - m_{ij}, b_{jj}) \in \mathcal{Z}(\mathfrak{T})$, thus $p_2(h_{jj}, b_{jj}) \in \mathcal{Z}(\mathfrak{T})$. Therefore $\bigoplus_{i<j} p_2(h_{ij} - m_{ij}, b_{jj}) = 0$, that is $(h_{ij} - m_{ij})b_{jj} = 0$ for all $b_{jj} \in \mathfrak{R}_j$ and by condition (ii) of Definition 2.1 we get $h_{ij} = m_{ij}$. Now consider $b_{kj} \in \mathfrak{M}_{kj}$; by standard Lemma 3.1 we have

$$\begin{aligned}\varphi(p_n(H, b_{kj}, T_3, \ldots, T_n)) &= \varphi(p_n(a_{kk}, b_{kj}, T_3, \ldots, T_n)) \\ &\quad + \varphi(p_n(\bigoplus_{i<j} m_{ij}, b_{kj}, T_3, \ldots, T_n)) \\ &= \varphi(p_n(a_{kk}, b_{kj}, T_3, \ldots, T_n)) + \varphi(0) \\ &= \varphi(p_n(a_{kk}, b_{kj}, T_3, \ldots, T_n)).\end{aligned}$$

It follows that $p_n(H - a_{kk}, b_{kj}, T_3, \ldots, T_n) = 0$ and by Lemma 3.2 we have $p_2(H - a_{kk}, b_{kj}) \in \mathcal{Z}(\mathfrak{T})$. Thus by Proposition 2.1 we get $p_2(H - a_{kk}, b_{kj}) = 0$, which implies that $(h_{kk} - a_{kk})b_{kj} = b_{kj}h_{jj}$ for all $b_{kj} \in \mathfrak{M}_{kj}$. Therefore by Proposition 2.1

we obtain $\bigoplus_{l\neq\{k,j\}} h_{ll} + (h_{kk} - a_{kk} + h_{jj}) \in \mathcal{Z}(\mathfrak{T})$. And finally by Lemma 4.2 we verify that the lemma is valid. □

**Lemma 4.4.** *For any $m_{ij}, s_{ij} \in \mathfrak{M}_{ij}$ with $i < j$, we have $\varphi(m_{ij} + s_{ij}) = \varphi(m_{ij}) + \varphi(s_{ij})$.*

*Proof.* Firstly we note that for any $m_{ij}, s_{ij} \in \mathfrak{M}_{ij}$, $i < j$, the following identity is valid:

$$m_{ij} + s_{ij} = p_n(1_{\mathfrak{R}_i} + s_{ij}, 1_{\mathfrak{R}_j} + m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j}).$$

Indeed, due to Remark 2.1 we get $m_{ij} + s_{ij} = p_2(1_{\mathfrak{R}_i} + s_{ij}, 1_{\mathfrak{R}_j} + m_{ij})$. It follows that $m_{ij} + s_{ij} = p_n(1_{\mathfrak{R}_i} + s_{ij}, 1_{\mathfrak{R}_j} + m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j})$. Finally by Lemma 4.3 we have

$$\begin{aligned}
\varphi(m_{ij} + s_{ij}) &= \varphi(p_n(1_{\mathfrak{R}_i} + s_{ij}, 1_{\mathfrak{R}_j} + m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j}))\\
&= p_n(\varphi(1_{\mathfrak{R}_i} + s_{ij}), \varphi(1_{\mathfrak{R}_j} + m_{ij}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&= p_n(\varphi(1_{\mathfrak{R}_i}) + \varphi(s_{ij}) + S_1, \varphi(1_{\mathfrak{R}_j}) + \varphi(m_{ij}) + S_2, \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&= p_n(\varphi(1_{\mathfrak{R}_i}) + \varphi(s_{ij}), \varphi(1_{\mathfrak{R}_j}) + \varphi(m_{ij}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&= p_n(\varphi(1_{\mathfrak{R}_i}), \varphi(1_{\mathfrak{R}_j}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&\quad + p_n(\varphi(1_{\mathfrak{R}_i}), \varphi(m_{ij}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&\quad + p_n(\varphi(s_{ij}), \varphi(1_{\mathfrak{R}_j}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&\quad + p_n(\varphi(s_{ij}), \varphi(m_{ij}), \varphi(1_{\mathfrak{R}_j}), \ldots, \varphi(1_{\mathfrak{R}_j}))\\
&= \varphi(p_n(1_{\mathfrak{R}_i}, 1_{\mathfrak{R}_j}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j})) + \varphi(p_n(1_{\mathfrak{R}_i}, m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j}))\\
&\quad + \varphi(p_n(s_{ij}, 1_{\mathfrak{R}_j}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j})) + \varphi(p_n(s_{ij}, m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j}))\\
&= \varphi(p_n(1_{\mathfrak{R}_i}, m_{ij}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j})) + \varphi(p_n(s_{ij}, 1_{\mathfrak{R}_j}, 1_{\mathfrak{R}_j}, \ldots, 1_{\mathfrak{R}_j}))\\
&= \varphi(0) + \varphi(m_{ij}) + \varphi(s_{ij}) + \varphi(0)\\
&= \varphi(m_{ij}) + \varphi(s_{ij}).
\end{aligned}$$

□

**Lemma 4.5.** *For any $a_{ii}, b_{ii} \in \mathfrak{R}_i$, $i = 1, 2, \ldots, n$, there exists $S_i \in \mathcal{Z}(\mathfrak{S})$ such that $\varphi(a_{ii} + b_{ii}) = \varphi(a_{ii}) + \varphi(b_{ii}) + S_i$.*

*Proof.* As $\varphi$ is surjective, there is an element $H = \bigoplus_{1\leq i\leq j\leq n} h_{ij} \in \mathfrak{T}$ such that

$$\varphi(H) = \varphi(a_{ii}) + \varphi(b_{ii}).$$

Let $c_{kk} \in \mathfrak{R}_k$, $k \neq i$ and $T_3, \ldots, T_n \in \mathfrak{T}$; by Lemma 3.1 we have

$$\begin{aligned}
\varphi(p_n(H, c_{kk}, T_3, \ldots, T_n)) &= \varphi(p_n(a_{ii}, c_{kk}, T_3, \ldots, T_n))\\
&\quad + \varphi(p_n(b_{ii}, c_{kk}, T_3, \ldots, T_n))\\
&= \varphi(0) + \varphi(0) = 0
\end{aligned}$$

It follows that $p_n(H, c_{kk}, T_3, \ldots, T_n) \in \mathcal{Z}(\mathfrak{T})$ and by Lemma 3.2 we get $p_2(H, c_{kk}) \in \mathcal{Z}(\mathfrak{T})$, thus $p_2(h_{kk}, c_{kk}) \in \mathcal{Z}(\mathfrak{T})$. Therefore by condition (i) of Definition 2.1 we have $h_{kk} \in \mathcal{Z}(\mathfrak{R}_k)$. Moreover, $\bigoplus_{i<j} p_2(h_{ij}, c_{kk}) = 0$, that is, $h_{ik}c_{kk} = 0$ for all $c_{kk} \in \mathfrak{R}_k$ and by condition (ii) of Definition 2.1 we get $h_{ik} = 0$. Now consider $c_{ij} \in \mathfrak{M}_{ij}$ and $r_{jj} \in \mathfrak{R}_j$; by standard Lemma 3.1 and Lemma 4.4 we have

$$\begin{aligned}\varphi(p_n(H, c_{ij}, r_{jj}, \dots, r_{jj})) &= \varphi(p_n(a_{ii}, c_{ij}, r_{jj}, \dots, r_{jj})) \\ &\quad + \varphi(p_n(b_{ii}, c_{ij}, r_{jj}, \dots, r_{jj})) \\ &= \varphi(p_n(a_{ii} + b_{ii}, c_{ij}, r_{jj}, \dots, r_{jj})).\end{aligned}$$

It follows that $p_n(H - (a_{ii} + b_{ii}), c_{ij}, r_{jj}, \dots, r_{jj}) = 0$ and by (ii) of Definition 2.1 we get $(h_{ii} - (a_{ii} + b_{ii}))c_{ij} = c_{ij}h_{jj}$ for all $c_{ij} \in \mathfrak{M}_{ij}$. Therefore by Proposition 2.1 we obtain $\bigoplus_{l \neq \{i,j\}} h_{ll} + h_{ii} - (a_{ii} + b_{ii}) + h_{jj} \in \mathcal{Z}(\mathfrak{T})$. And finally by Lemma 4.2 we see that the Lemma is valid. □

**Lemma 4.6.** *For any $T \in \mathfrak{T}$ with $T = \bigoplus_{1 \leq i \leq j \leq n} T_{ij}$, there exists $S \in \mathcal{Z}(\mathfrak{S})$ such that $\varphi(T) = \bigoplus_{1 \leq i \leq j \leq n} \varphi(T_{ij}) + S$.*

*Proof.* As $\varphi$ is surjective, there is an element $H = \bigoplus_{1 \leq i \leq j \leq n} H_{ij} \in \mathfrak{T}$ such that

$$\varphi(H) = \bigoplus_{t=1}^{n} \varphi(T_{tt}) + \bigoplus_{1 \leq i \leq j \leq n} \varphi(T_{ij}).$$

Let $c_{kk} \in \mathfrak{R}_k$, $k = 1, 2, \dots, n$; by Lemma 3.1 we have

$$\begin{aligned}\varphi(p_n(H, c_{kk}, c_{kk}, \dots, c_{kk})) &= \varphi(p_n(T_{kk}, c_{kk}, c_{kk}, \dots, c_{kk})) \\ &\quad + \sum_{i=1}^{k-1} \varphi(p_n(T_{ik}, c_{kk}, c_{kk}, \dots, c_{kk})) \\ &\quad + \sum_{j=k+1}^{n} \varphi(p_n(T_{kj}, c_{kk}, c_{kk}, \dots, c_{kk})).\end{aligned}$$

Now let $c_{ll} \in \mathfrak{R}_l$ with $l \in \{1, \dots, k-1\}$; again by Lemma 3.1 we get

$$\begin{aligned}&\varphi(p_n(p_n(H, c_{kk}, c_{kk}, \dots, c_{kk}), c_{ll}, c_{ll}, \dots, c_{ll})) \\ &\qquad = \varphi(p_n(p_n(T_{lk}, c_{kk}, c_{kk}, \dots, c_{kk}), c_{ll}, c_{ll}, \dots, c_{ll})).\end{aligned}$$

Since $\varphi$ is injective we have

$$\begin{aligned}&p_n(p_n(H, c_{kk}, c_{kk}, \dots, c_{kk}), c_{ll}, c_{ll}, \dots, c_{ll}) \\ &\qquad = p_n(p_n(T_{lk}, c_{kk}, c_{kk}, \dots, c_{kk}), c_{ll}, c_{ll}, \dots, c_{ll}).\end{aligned}$$

It follows that $p_n(p_n(H - T_{lk}, c_{kk}, c_{kk}, \dots, c_{kk}), c_{ll}, c_{ll}, \dots, c_{ll}) = 0$ and by (ii) of Definition 2.1 we obtain $H_{lk} = T_{lk}$. Again let $c_{qq} \in \mathfrak{R}_q$ with $q \in \{k+1, \dots, n\}$; by

Lemma 3.1 we get

$$\varphi(p_n(p_n(H, c_{kk}, c_{kk}, \ldots, c_{kk}), c_{qq}, c_{qq}, \ldots, c_{qq})) = \varphi(p_n(p_n(T_{kq}, c_{kk}, c_{kk}, \ldots, c_{kk}), c_{qq}, c_{qq}, \ldots, c_{qq})).$$

Since $\varphi$ is injective we have

$$p_n(p_n(H, c_{kk}, c_{kk}, \ldots, c_{kk}), c_{qq}, c_{qq}, \ldots, c_{qq}) = p_n(p_n(T_{kq}, c_{kk}, c_{kk}, \ldots, c_{kk}), c_{qq}, c_{qq}, \ldots, c_{qq}).$$

It follows that $p_n(p_n(H - T_{kq}, c_{kk}, c_{kk}, \ldots, c_{kk}), c_{qq}, c_{qq}, \ldots, c_{qq}) = 0$ and by (ii) of Definition 2.1 we obtain $H_{kq} = T_{kq}$. Finally let $c_{tt} \in \mathfrak{R}_t$ and $c_{kt} \in \mathfrak{M}_{kt}$, $k < t$; by Lemma 3.1 we have

$$\begin{aligned}
&\varphi(p_n(H, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\quad = \varphi(p_n(T_{kk}, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\qquad + \varphi(p_n(T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt})) + \sum_{i=1}^{k-1} \varphi(p_n(T_{ik}, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\qquad + \sum_{j=t+1}^{n} \varphi(p_n(T_{tj}, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\quad = \varphi(p_n(T_{kk}, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\qquad + \varphi(p_n(T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt})) + \sum_{i=1}^{k-1} \varphi(p_n(T_{ik}, c_{kt}, c_{tt}, \ldots, c_{tt})) + \varphi(0) \\
&\quad = \varphi(p_n(T_{kk}, c_{kt}, c_{tt}, \ldots, c_{tt})) + \varphi(p_n(T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt})) \\
&\qquad + \sum_{i=1}^{k-1} \varphi(p_n(T_{ik}, c_{kt}, c_{tt}, \ldots, c_{tt})).
\end{aligned}$$

Now let $c_{kk} \in \mathfrak{R}_k$; by Lemma 3.1 and Lemma 4.4 we obtain

$$\begin{aligned}
&\varphi(p_n(p_n(H, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})) \\
&\quad = \varphi(p_n(p_n(T_{kk}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})) \\
&\qquad + \varphi(p_n(p_n(T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})) \\
&\qquad + \sum_{i=1}^{k-1} \varphi(p_n(p_n(T_{ik}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})) \\
&\quad = \varphi(p_n(p_n(T_{kk}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk}) \\
&\qquad + p_n(p_n(T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})) + \sum_{i=1}^{k-1} \varphi(0) \\
&\quad = \varphi(p_n(p_n(T_{kk} + T_{tt}, c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk})).
\end{aligned}$$

Since $\varphi$ is injective we have

$$p_n(p_n(H - (T_{kk} + T_{tt}), c_{kt}, c_{tt}, \ldots, c_{tt}), c_{kk}, c_{kk}, \ldots, c_{kk}) = 0.$$

By (ii) of Definition 2.1 it follows that $(H_{kk} - T_{kk})c_{kt} = c_{kt}(H_{tt} - T_{tt})$ for all $c_{kt} \in \mathfrak{M}_{kt}$. Therefore $\bigoplus_{i=1}^{n} H_{ii} = \bigoplus_{i=1}^{n} T_{ii} + Z$, where $Z \in \mathcal{Z}(\mathfrak{T})$. Now by Lemma 4.2 the result is true. □

We are ready to prove our Theorem 4.1.

*Proof of Theorem 4.1.* Let $A, B \in \mathfrak{T}$. By the previous lemmas we have

$$\begin{aligned}
\varphi(A+B) &= \varphi\Big(\bigoplus_{1\le i\le j\le n} A_{ij} + \bigoplus_{1\le i\le j\le n} B_{ij}\Big)\\
&= \varphi\Big(\bigoplus_{k=1}^{n}(A_{kk}+B_{kk}) + \bigoplus_{1\le i\le j\le n}(A_{ij}+B_{ij})\Big)\\
&= \bigoplus_{k=1}^{n}\varphi((A_{kk}+B_{kk})) + \bigoplus_{1\le i\le j\le n}\varphi((A_{ij}+B_{ij})) + S'_{A,B}\\
&= \bigoplus_{k=1}^{n}\varphi(A_{kk}) + \bigoplus_{k=1}^{n}\varphi(B_{kk}) + \bigoplus_{k=1}^{n} S_{\mathfrak{R}_k} + \bigoplus_{1\le i\le j\le n}\varphi(A_{ij})\\
&\quad + \bigoplus_{1\le i\le j\le n}\varphi(B_{ij}) + S'_{A,B}\\
&= \varphi\Big(\bigoplus_{k=1}^{n} A_{kk} + \bigoplus_{1\le i\le j\le n}\varphi(A_{ij})\Big) - S_A\\
&\quad + \varphi\Big(\bigoplus_{k=1}^{n} B_{kk} + \bigoplus_{1\le i\le j\le n}\varphi(B_{ij})\Big) - S_B + \bigoplus_{k=1}^{n} S_{\mathfrak{R}_k} + S'_{A,B}\\
&= \varphi(A) + \varphi(B) + S_{A,B},
\end{aligned}$$

where $S_{A,B} = \bigoplus_{k=1}^{n} S_{\mathfrak{R}_k} - S_A - S_B + S'_{A,B}$, so the theorem is proved. □

## 5. Final remarks

**Corollary 5.1.** *Let $\mathfrak{T}$ be the triangular $n$-matrix unital ring and $\mathfrak{S}$ be a ring satisfying the condition*

$$\textit{If } p_2(s, \mathfrak{S}) \subseteq \mathcal{Z}(\mathfrak{S}) \textit{ then } s \in \mathcal{Z}(\mathfrak{S}).$$

*If $\varphi : \mathfrak{T} \to \mathfrak{S}$ is a bijective Lie $n$-multiplicative mapping then $\varphi$ is almost additive.*

*Proof.* Indeed, let $\varphi : \mathfrak{T} \to \mathfrak{S}$ be a bijective Lie $n$-multiplicative mapping, $T_2, \ldots, T_n \in \mathfrak{T}$, and $Z \in \mathcal{Z}(\mathfrak{T})$; we have

$$p_n(\varphi(Z), \varphi(T_2), \ldots, \varphi(T_n)) = \varphi(p_n(Z, T_2, \ldots, T_n)) = \varphi(0) = 0.$$

Since $T_2, \ldots T_n$ are arbitrary and $\varphi$ is surjective it follows that $\varphi(Z) \in \mathcal{Z}(\mathfrak{S})$. □

**Proposition 5.1.** *For any prime ring $\mathfrak{R}$, the statement*

$$\text{If } p_2(r, \mathfrak{R}) \subseteq \mathcal{Z}(\mathfrak{R}) \text{ then } r \in \mathcal{Z}(\mathfrak{R})$$

*holds true.*

*Proof.* See [5, Lemma 3]. □

**Theorem 5.1.** *Let $\mathfrak{T}$ be a triangular $n$-matrix unital ring and $\mathfrak{S}$ be a prime ring. Then any bijective Lie $n$-multiplicative mapping from $\mathfrak{T}$ onto $\mathfrak{S}$ is almost additive.*

**Corollary 5.2** (Xiaofei Qi and Jinchuan Hou [6])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra and $\mathcal{V}$ any algebra over $\mathcal{R}$. Assume that $\Phi : \mathcal{U} \to \mathcal{V}$ is a Lie multiplicative isomorphism, that is, $\Phi$ satisfies*

$$\Phi(ST - TS) = \Phi(S)\Phi(T) - \Phi(T)\Phi(S) \quad \forall S, T \in \mathcal{U}.$$

*Then $\Phi(S + T) = \Phi(S) + \Phi(T) + Z_{S,T}$ for all $S, T \in \mathcal{U}$, where $Z_{S,T}$ is an element in the centre $\mathcal{Z}(\mathcal{V})$ of $\mathcal{V}$ depending on $S$ and $T$.*

*Proof.* This is consequence of our Theorem 4.1 for $n = 2$. □

## 6. Application in nest algebras

A *nest* $\mathcal{N}$ is a totally ordered set of closed subspaces of a Hilbert space $\mathcal{H}$ such that $\{0\}, \mathcal{H} \in \mathcal{N}$, and $\mathcal{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. The *nest algebra* associated to $\mathcal{N}$ is the set $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators over a complex Hilbert space $\mathcal{H}$.

We recall the standard result ([1, Proposition 16]) that says that we can view $\mathcal{T}(\mathcal{N})$ as a triangular algebra $\begin{pmatrix} A & M \\ & B \end{pmatrix}$ where $A, B$ are themselves nest algebras.

**Proposition 6.1.** *If $N \in \mathcal{N} \setminus \{0, H\}$ and $E$ is the orthonormal projection onto $N$, then $E\mathcal{N}E$ and $(1-E)\mathcal{N}(1-E)$ are nest, $\mathcal{T}(E\mathcal{N}E) = E\mathcal{T}(\mathcal{N})E$, and $\mathcal{T}((1-E)\mathcal{N}(1-E)) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$. Furthermore,*

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(E\mathcal{N}E) & E\mathcal{T}(\mathcal{N})(1-E) \\ & \mathcal{T}((1-E)\mathcal{N}(1-E)) \end{pmatrix}.$$

We refer the reader to [2] for the general theory of nest algebras.

**Corollary 6.1.** *Let $P_n$ be an increasing sequence of finite dimensional subspaces such that their union is dense in $\mathcal{H}$. Consider $\mathcal{P} = \{\{0\}, P_n, n \geq 1, \mathcal{H}\}$ a nest and $\mathcal{T}(\mathcal{P})$ the set consists of all operators which have a block upper triangular matrix with respect to $\mathcal{P}$. If a mapping $\varphi : \mathcal{T}(\mathcal{P}) \longrightarrow \mathcal{T}(\mathcal{P})$ satisfies*

$$\varphi(p_2([f, g]) = p_2(\varphi(f), \varphi(g))$$

*for all $f, g \in \mathcal{T}(\mathcal{P})$, then $\varphi$ is almost additive.*

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*B. L. M. Ferreira* ✉
Universidade Tecnológica Federal do Paraná, Avenida Professora Laura Pacheco Bastos, 800, 85053-510, Guarapuava, Brazil
brunoferreira@utfpr.edu.br

*H. Guzzo Jr.*
Universidade de São Paulo, Instituto de Matemática e Estatística, Rua do Matão, 1010, 05508-090, São Paulo, Brazil
guzzo@ime.usp.br

*Received: January 18, 2018*
*Accepted: July 17, 2018*