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ON THE SELF-CONJUGATENESS OF DIFFERENTIAL FORMS ON BOUNDED DOMAINS

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Abstract. Suppose Ω is a bounded domain in $\mathbb{R}^n$ with boundary Γ and let $\mathcal{W}$ be a non-homogeneous differential form harmonic in Ω and Hölder-continuous in $\Omega \cup \Gamma$. In this paper we study and obtain some necessary and sufficient conditions for the self-conjugateness of $\mathcal{W}$ in terms of its boundary value $\mathcal{W}|_\Gamma = \omega$.

1. Introduction

As is well known, a k-vector in $\mathbb{R}^n$ can be interpreted as a directed k-dimensional volume. Such entities were first considered by H. Grassmann in the second half of the 19th century. He thus created an algebraic structure which is now commonly known as the exterior algebra. At about the same time, Sir W. Hamilton invented his quaternion algebra which a. o. enabled him to represent rotations in three dimensional space. In his 1878 paper, W. K. Clifford united both systems into a single geometric algebra, later named after him.

Clifford analysis offers a function theory, which is a higher dimensional generalization of classical complex analysis in $\mathbb{R}^2$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way) to Euclidean space $\mathbb{R}^n$ ($n \geq 3$). The theory is centred around the concept of monogenic functions, which constitute the kernel of a first order vector valued, rotation invariant, differential operator called the Dirac operator, which factorizes the Laplacian. The best general reference here is [5]; see also the brief review [9] of the content of this book.

On the other hand, the world-renowned theory of differential forms provides also a generalization in $\mathbb{R}^n$ of holomorphic functions of one complex variable. Although Clifford analysis seems to be truly appropriate to study differential forms by using Clifford algebras in a very beautiful way, this has been mentioned so far only in a

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few papers. For an overview of the main operator identities and properties of these objects in the Clifford analysis context we refer to [11] and the references quoted there.

In [6] the authors compare the language of differential forms and that of Clifford algebra valued multi-vector fields and shown that the spaces of smooth differential forms on the one hand, and smooth multi-vector functions (multi-vector fields) on the other are isomorphic in a natural way. Moreover the action of the operator $d - d^*$, where d and d^* are the differential and codifferential operator respectively, on the space of smooth k-forms is identified with the action (on the right) of the Dirac operator, which plays the role of the Cauchy–Riemann operator on the space of smooth k-vector fields. Meanwhile the action of the operator $d + d^*$ is identified with the action (on the left) of the Dirac operator. An extensive treatment of the Clifford algebras of differential forms, an elegant fashion of standard physics techniques, can be found in [13, 18].

In the present paper necessary and sufficient conditions are formulated such that a harmonic (in the real sense) differential form in a domain Ω is really a self-conjugate differential form there. Here again the full use of the isomorphism between the smooth differential forms on the one hand and smooth multi-vector functions on the other is the key point.

2. Harmonic and self-conjugate differential forms. Statement of the problem

We will follow here the notations and conventions carried out in [6].

Denoting by $\Lambda^k\mathbb{R}^n$ the space of alternating real-valued k-forms ($0 \le k \le n$), the well known Grassmann algebra over $\mathbb{R}^n$ is the associative algebra

$$\Lambda\mathbb{R}^n := \bigoplus_{k=0}^{n} \Lambda^k\mathbb{R}^n$$

endowed with the exterior multiplication $\wedge$.

A basis for $\Lambda^k\mathbb{R}^n$ is obtained as follows. Let $\{dx^1, dx^2, \ldots, dx^n\}$ be a basis for the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$. If $A = \{i_1, \ldots, i_k\} \subset M = \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$, set

$$dx^A := dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

and

$$dx^{\emptyset} := 1.$$

Then for each $k = 0, 1, \ldots, n$, the set

$$\{dx^A : A \subset M,\ |A| := \operatorname{card}(A) = k\}$$

is a basis for $\Lambda^k\mathbb{R}^n$.

Note that in particular

$$dx^i \wedge dx^i = 0, \quad i = 1, \ldots, n;$$

$$dx^i \wedge dx^j + dx^j \wedge dx^i = 0, \quad 1 \le i \ne j \le n.$$

The following linear operators on $\Lambda^k\mathbb{R}^n$ play a fundamental role:

$$dx^j : \Lambda^k\mathbb{R}^n \to \Lambda^{k+1}\mathbb{R}^n : dx^j[dx^A] := dx^j \wedge (dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k})$$

and

$$\widehat{dx^j} : \Lambda^k\mathbb{R}^n \to \Lambda^{k-1}\mathbb{R}^n : \widehat{dx^j}[dx^A] = \sum_{r=1}^{k}(-1)^r \delta_{j\, i_r} dx^{A\setminus\{i_r\}},$$

where

$$dx^{A\setminus\{i_r\}} := dx^{i_1} \wedge \cdots \wedge dx^{i_{r-1}} \wedge dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_k}.$$

Here and in what follows we use the same symbol to denote the multiplication operator dx^j and the differential form dx^j itself. The way in which this operator acts on dx^A justifies such abuse of notation. These operators are then extended to $\Lambda\mathbb{R}^n$ by linearity.

Direct computation shows that

$$dx^j \wedge dx^r + dx^r \wedge dx^j = 0, \quad \widehat{dx^j} \wedge \widehat{dx^r} + \widehat{dx^r} \wedge \widehat{dx^j} = 0,$$
$$\widehat{dx^j} \wedge dx^r + dx^r \wedge \widehat{dx^j} = -\delta_{rj}\mathcal{I},$$

where $\mathcal{I}$ is the identity operator.

As usual (see [12]), a k-form in an open domain Ω of $\mathbb{R}^n$ is a map

$$\omega_k : \Omega \mapsto \Lambda^k\mathbb{R}^n, \quad x \mapsto \sum_{|A|=k} \omega_{k,A}(x) dx^A,$$

where for each A, $\omega_{k,A}$ is a real-valued function in Ω.

Such a map is said to belong to some class of functions on Ω if each of its components belongs to that class. In particular, we denote by $C^1(\Omega, \Lambda^k\mathbb{R}^n)$ the space of smooth k-forms in Ω and by $C^{0,\alpha}(\Gamma, \Lambda^k\mathbb{R}^n)$ $(C(\Gamma, \Lambda^k\mathbb{R}^n))$ the space of Hölder continuous (continuous) k-forms in Γ.

Furthermore, consider the fundamental linear operators on $C^1(\Omega, \Lambda^k\mathbb{R}^n)$, the exterior derivative and the co-derivative d and d^*, respectively:

$$d := \sum_{j=1}^{n} dx^j \frac{\partial}{\partial x_j}, \quad d^* := \sum_{j=1}^{n} \widehat{dx^j} \frac{\partial}{\partial x_j}.$$

It is easy to see that $d^2 = 0$, ${d^*}^2 = 0$, and $dd^* + d^*d = -\Delta$, the Laplacian in $\mathbb{R}^n$.

The kernels of the exterior derivative d and the co-derivative d^* consist of the so-called closed k-forms and co-closed k-forms, respectively. A smooth k-form in Ω, which is at the same time closed and co-closed, is called harmonic in Ω (in the sense of Hodge), i.e, a smooth k-form ω_k is said to be harmonic in Ω if and only if it satisfies in Ω the Hodge–de Rham system

$$\begin{cases} d\omega_k = 0, \\ d^*\omega_k = 0. \end{cases}$$

Note that if ω_k is harmonic in an open domain Ω, then automatically ω_k is also harmonic in the real sense, i.e., satisfies the Laplace equation $\Delta\omega_k = 0$ in Ω.

These definitions can be directly extended to non-homogeneous differential forms: $C^1(\Omega, \Lambda\mathbb{R}^n)$ will denote $\sum_{k=0}^{n} C^1(\Omega, \Lambda^k\mathbb{R}^n)$.

If $\mathcal{W} = \sum_{k=0}^{n} \omega_k \in C^1(\Omega, \Lambda\mathbb{R}^n)$, where $\omega_k \in C^1(\Omega, \Lambda^k\mathbb{R}^n)$ is a k-form, we consider the action of the exterior derivative d and the co-derivative d^* as

$$d\mathcal{W} = \sum_{0}^{n} d\omega_k$$

and

$$d^*\mathcal{W} = \sum_{0}^{n} d^*\omega_k,$$

respectively.

Following [10], a non-homogeneous differential form $\mathcal{W} \in C^1(\Omega, \Lambda\mathbb{R}^n)$ is said to be self-conjugate if

$$d\mathcal{W} = d^*\mathcal{W}$$

in Ω, i.e., if

$$d^*\mathcal{W}_1 = 0; \quad d\mathcal{W}_{k-1} = d^*\mathcal{W}_{k+1} \ (k = 1, \dots, n-1); \quad d\mathcal{W}_{n-1} = 0.$$

Since $dd^* + d^*d = -\Delta$, we have that any self-conjugate differential form is also real harmonic.

Note that if $\mathcal{W} = \omega_k$, then it is self-conjugate if and only if it is harmonic in the sense of Hodge.

The main problem that we address in the remainder of the paper is: Given a real harmonic differential form $\mathcal{W}$ in $\Omega \subset \mathbb{R}^n$, being Hölder continuous in $\Omega \cup \Gamma$, under what condition on $\mathcal{W}|_\Gamma = \omega$ is this differential form self-conjugate in Ω?

The next two sections contain some basic notions and results coming from Clifford analysis that we shall need in order to reach the goal of the present article.

3. Clifford algebras and multi-vectors

The real Clifford algebra associated with $\mathbb{R}^n$ endowed with the Euclidean metric is the minimal enlargement of $\mathbb{R}^n$ to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity, and such that $\underline{x}^2 = -|\underline{x}|^2$, for any $\underline{x} \in \mathbb{R}^n$.

It thus follows that if $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$, then we must have that $e_ie_j + e_je_i = -2\delta_{ij}$. Every element $a \in \mathbb{R}_{0,n}$ is of the form $a = \sum_{A\subseteq N} a_A e_A$, $N = \{1, \dots, n\}$, $a_A \in \mathbb{R}$, where $e_\emptyset := e_0 = 1$, $e_{\{j\}} = e_j$, and $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$ for $A = \{\alpha_1, \dots, \alpha_k\}$, where $\alpha_j \in \{1, \dots, n\}$ and $\alpha_1 < \cdots < \alpha_k$, or still as $a = \sum_{k=0}^{n} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector ($k = 0, 1, \dots, n$).

If we denote the space of k-vectors by $\mathbb{R}_{0,n}^k$, then $\mathbb{R}_{0,n} = \sum_{k=0}^{n} \oplus \mathbb{R}_{0,n}^k$, leading to the identification of $\mathbb{R}^n$ with $\mathbb{R}_{0,n}^1$.

For a 1-vector $\underline{x}$ and a k-vector Y_k, their product $\underline{x}Y_k$ splits into a $(k-1)$-vector and a $(k+1)$-vector, namely:

$$\underline{x}Y_k = [\underline{x}Y_k]_{k-1} + [\underline{x}Y_k]_{k+1},$$

where

$$[\underline{x}Y_k]_{k-1} = \frac{1}{2}(\underline{x}Y_k - (-1)^k Y_k \underline{x})$$

and

$$[\underline{x}Y_k]_{k+1} = \frac{1}{2}(\underline{x}Y_k + (-1)^k Y_k \underline{x}).$$

The inner and outer products between $\underline{x}$ and Y_k are then defined by

$$\underline{x} \bullet Y_k := [\underline{x}Y_k]_{k-1} \quad \text{and} \quad \underline{x} \wedge Y_k := [\underline{x}Y_k]_{k+1}.$$

Notice also that

$$[\underline{x}Y_k]_{k-1} = (-1)^{k+1}[Y_k\underline{x}]_{k-1},$$
$$[\underline{x}Y_k]_{k+1} = (-1)^{k}[Y_k\underline{x}]_{k+1}.$$

For further properties concerning inner and outer products between multi-vectors, we refer to [14].

Conjugation in $\mathbb{R}_{0,n}$ is defined by $\overline{a} := \sum_A a_A \overline{e_A}$, where

$$\overline{e_A} = (-1)^k e_{i_k} \cdots e_{i_2} e_{i_1}, \quad \text{if } e_A = e_{i_1} e_{i_2} \cdots e_{i_k}.$$

In particular for a 1-vector $\underline{x}$ we have:

$$\overline{\underline{x}} = -\underline{x}.$$

The natural isomorphism

$$\Theta : \mathbb{R}_{0,n}^{(k)} \mapsto \Lambda^k \mathbb{R}^n, \quad \sum_{|A|=k} Y_{k,A} e_A \mapsto \sum_{|A|=k} Y_{k,A} dx^A$$

was introduced and applied for example in [6, 3].

The following identities can be easily verified (see [3]).

Lemma 3.1. *Let $\underline{x}$ be a vector and Y_k a k-vector. Then:*

(1)
$$\Theta(e_j \bullet Y_k) = \sum_{|A|=k} Y_{k,A} \widehat{dx^j}[dx^A],$$

or, more generally,

$$\Theta(\underline{x} \bullet Y_k) = \sum_{j=1}^{n} \sum_{|A|=k} x_j Y_{k,A} \widehat{dx^j}[dx^A];$$

(2)
$$\Theta(e_j \wedge Y_k) = \sum_{|A|=k} Y_{k,A} dx^j \wedge dx^A,$$

or, more generally,

$$\Theta(\underline{x} \wedge Y_k) = \sum_{j=1}^{n} \sum_{|A|=k} x_j Y_{k,A} dx^j \wedge dx^A.$$

Of course one can extend the action of the isomorphism Θ by linearity to the whole $\mathbb{R}_{0,n}$.

4. Clifford analysis and harmonic multi-vector fields

From now on, Ω stands for a Jordan domain, i.e. a bounded oriented connected open subset of $\mathbb{R}^n$, the boundary of which is a compact topological surface to be denoted by Γ. We shall assume Γ to be smooth or piecewise smooth. This smoothness hypothesis on Γ can be relaxed, but for simplicity we shall not use this possibility in any essential way.

Let f be an $\mathbb{R}_{0,n}$-valued function in Ω, say

$$f(\underline{x}) = \sum_A f_A(\underline{x}) e_A, \ \underline{x} \in \Omega,$$

all f_A thus being real valued.

We will denote by $C^1(\Omega, \mathbb{R}_{0,n})$ the space of 1-time continuously differentiable $\mathbb{R}_{0,n}$-valued functions in Ω.

We say that f is right (resp. left) monogenic in Ω if $F\mathcal{D} = 0$ (resp. $\mathcal{D}F = 0$) in Ω, where $\mathcal{D}$ denotes the Dirac operator in $\mathbb{R}^n$:

$$\mathcal{D} = \sum_{j=1}^{n} e_j \partial_{x_j}.$$

An important example of a function which is both right and left monogenic is the fundamental solution of the Dirac operator, given by

$$E(\underline{x}) = \frac{1}{A_n} \frac{\overline{\underline{x}}}{|\underline{x}|^n}, \quad \underline{x} \in \mathbb{R}^n \setminus \{0\}.$$

Hereby A_n stands for the surface area of the unit sphere in $\mathbb{R}^n$.

The function $E(\underline{x})$ plays the same role in Clifford analysis as the Cauchy kernel does in complex analysis. For this reason it is also called the Cauchy kernel in $\mathbb{R}^n$.

Let $0 < k \leq n-1$ be fixed. Then the space of C^1-functions from Ω into $\mathbb{R}^k_{0,n}$, called k-vector fields, is denoted by $C^1(\Omega, \mathbb{R}^k_{0,n})$.

Notice that for $f_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ a straightforward calculation leads to $\overline{\mathcal{D} f_k} = \overline{f_k}\ \overline{\mathcal{D}}$ with $\overline{\mathcal{D}} = -\mathcal{D}$ and $\overline{f_k} = (-1)^{\frac{k(k+1)}{2}} f_k$. It thus follows that for an element in $C^1(\Omega, \mathbb{R}^k_{0,n})$ the notions of left and right monogenicity coincide.

Consequently, we will call $f_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ harmonic in Ω if either $\mathcal{D} f_k = 0$ or $f_k \mathcal{D} = 0$ in Ω.

Moreover we notice that through the isomorphism Θ, for $f_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ and $\omega_k = \Theta f_k \in C^1(\Omega, \Lambda^k \mathbb{R}^n)$ we have $\mathcal{D} \wedge f_k \longleftrightarrow d\omega_k$ and $\mathcal{D} \bullet f_k \longleftrightarrow d^*\omega_k$ and are equivalent $(0 < k < n-1)$

$$\mathcal{D} f_k = 0 \Longleftrightarrow \begin{cases} d\omega_k & = 0 \\ d^*\omega_k & = 0, \end{cases}$$

i.e. f_k harmonic in Ω is equivalent to saying that $\omega_k = \Theta f_k$ is a harmonic k-form in Ω.

More generally, let $\mathcal{W} = \sum_0^n \omega_k$ be a non-homogeneous differential form and consider the $\mathbb{R}_{0,n}$-valued function $F = \sum_0^n f_k$, where $\Theta f_k = \omega_k$; then $\mathcal{W}$ is a self-conjugate differential form in Ω if and only if F is a right monogenic function in Ω.

5. Criteria for monogenicity

Let us assume that f belongs to the Hölder space $C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$, $0 < \alpha < 1$. The Cauchy transform C_Γ and the Hilbert transform H_Γ of f are defined respectively by

$$C_\Gamma f(\underline{x}) = \int_\Gamma E(\underline{y} - \underline{x})\underline{\nu}(\underline{y}) f(\underline{y})\, d\underline{y}, \quad \underline{x} \in \mathbb{R}^n \setminus \Gamma,$$
$$H_\Gamma f(\underline{x}) = \int_\Gamma E(\underline{y} - \underline{x})\underline{\nu}(\underline{y}) f(\underline{y})\, d\underline{y}), \quad \underline{x} \in \Gamma,$$

where $\underline{\nu}(\underline{y}) = \sum_{j=1}^n e_j \nu_j(\underline{y})$ is the unit normal vector on Γ at the point $\underline{y}$.

It should be noticed that the last integral is taken in the principal value sense. Moreover, $C_\Gamma f$ is left monogenic in $\mathbb{R}^m \setminus \Gamma$. If f is additionally left monogenic in Ω, then by Cauchy's integral formula (see for instance [5], but it may be found in many other sources) we have

$$f(\underline{x}) = C_\Gamma f(\underline{x}), \quad \underline{x} \in \Omega. \tag{5.1}$$

In [7] the authors give a condensed account of results connected to the Hilbert transform on the smooth boundary of a bounded domain in Euclidean spaces.

Let us now formulate important properties of $C_\Gamma f$ and $H_\Gamma f$.

(A) $H_\Gamma f \in C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$.

(B) (Sokhotski–Plemelj formula) For $\underline{z} \in \Gamma$,

$$\lim_{\Omega \ni \underline{x} \to \underline{z}} C_\Gamma f(\underline{x}) = \frac{1}{2}\left[H_\Gamma f(\underline{z}) + f(\underline{z})\right].$$

A great number of original papers have been devoted to this subject. For the proof along classical lines we refer the reader to the pioneer work [15], whose author proved, in 1965, that the Cauchy transform has Hölder-continuous limit values for any Hölder-continuous densities and he obtained Plemelj–Sokhotski-type formulae.

We highlight the following important point. The claimed smoothness of the boundary Γ for the validity of the items (A) and (B) has been known for many years but subsequent developments put the study of the above-mentioned items in the context of weaker restrictions on the boundary (see for instance [4, Theorem 1]). An optimal generalization of both assertions can be found in [1, Theorem 6].

We have assumed that $f \in C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$ and hence integrals are understood in the Riemann sense (proper or improper). If now $f \in L_p(\Gamma, \mathbb{R}_{0,n})$ then one has to understand $C_\Gamma f$ as a Lebesgue integral, and the necessary changes can be easily made. For example, the (non-tangential) limits in (B) exist almost everywhere on Γ with respect to the surface Lebesgue measure. An Lp formulation of (A) follows from standard Calderón–Zygmund theory and recalling that $C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$ is dense in $L_p(\Gamma, \mathbb{R}_{0,n})$ by classical arguments. For a thorough treatment we refer the reader to [16].

The following theorem is basic in our next considerations. Its proof may be found in [2], but we include it here for the sake of completeness.

Theorem 5.1. *Let $F : \Omega \cup \Gamma \to \mathbb{R}_{0,n}$ be a function such that $F|_\Gamma = f$ belongs to $C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$. Then, the following are equivalent:*

(i) *F is left monogenic in Ω;*
(ii) *F is harmonic in Ω and $H_\Gamma f = f$.*

Proof. Suppose that F is left monogenic in Ω. From (5.1) we have $F(\underline{x}) = C_\Gamma f(\underline{x})$ for $\underline{x} \in \Omega$. Now (B) yields

$$f(\underline{x}) = H_\Gamma f(\underline{x}) + f(\underline{x}), \quad \underline{x} \in \Sigma.$$

Consequently, $H_\Gamma f(\underline{x}) = 0$ for all $\underline{x} \in \Sigma$.

Conversely, assume that F is harmonic in Ω and $H_\Gamma f = f$. Let us define

$$G(\underline{x}) = \begin{cases} C_\Gamma f(\underline{x}), & \underline{x} \in \Omega, \\ f(\underline{x}), & \underline{x} \in \Gamma. \end{cases}$$

The function G is left monogenic in Ω and hence harmonic in Ω. By (A) and (B), G is also continuous on $\Omega \cup \Gamma$. As $F - G$ is harmonic in Ω and $(F - G)|_\Gamma = 0$ it follows that $F(\underline{x}) = C_\Gamma f(\underline{x})$ for $\underline{x} \in \Omega$. □

Remark 5.2. As was mentioned before, we can extend the scope of the items (i) and (ii) to the much larger class of Lebesgue p-integrable functions (all formulas have to be reinterpreted), which makes it possible to carry our results with Lp-data. In this sense, our approach generalizes and strengthens the standard result on the necessary and sufficient condition for the possibility to extend a given L_2-function from the surface Γ to an L_2-monogenic function in the domain Ω, see [7, Section 5], [12, Section 5] and [17, Chapter 3]. This goes back as far as [19, Theorem 95], where the very particular case of the half-plane is considered.

6. Self-conjugate forms revisited

In this section we state and prove our main theorem, which gives a solution to the question asked at the end of Section 2. We retain the hypotheses on smoothness of Γ for ease of comprehension.

Theorem 6.1. *Let $\mathcal{W} : \Omega \cup \Gamma \to \Lambda\mathbb{R}^n$ be a non-homogeneous differential form such that $\mathcal{W}|_\Gamma = \omega$ belongs to $C^{0,\alpha}(\Gamma, \Lambda\mathbb{R}^n)$. Then, the following are equivalent:*

(i) *$\mathcal{W}$ is a self-conjugate form in Ω;*
(ii) *$\mathcal{W}$ is harmonic in Ω and*

$$\frac{2}{A_n} \int_\Gamma \sum_{j,r} \frac{z_j - \zeta_j}{|z - \zeta|^n} \nu_r(\zeta)(dz^j \wedge \widehat{dz^r} + \widehat{dz^j} \wedge dz^r)\omega(\zeta, dz)\, d\zeta = \omega(z, dz), \quad z \in \Gamma.$$

Proof. Theorem 5.1 gives a direct and brief proof. Indeed, by Theorem 5.1 and keeping in mind the isomorphism Θ the proof is then reduced to verify the equality

$$\Theta[H_\Gamma f] = \frac{2}{A_n}\int_\Gamma \sum_{j,r} \frac{z_j-\zeta_j}{|z-\zeta|^n}\nu_r(\zeta)(dz^j \wedge \widehat{dz^r} + \widehat{dz^j} \wedge dz^r)\omega(\zeta, dz)\, d\zeta,$$

where $\Theta[f] = \omega$, and this follows as a direct consequence of Lemma 3.1. □

Moreover, we obtain the following simple but important corollary.

Corollary 6.2. *Let $\mathcal{W}_k : \Omega \cup \Gamma \to \Lambda^k\mathbb{R}^n$ be a k-differential form such that $\mathcal{W}_k|_\Gamma = \omega_k$ belongs to $C^{0,\alpha}(\Gamma, \Lambda^k\mathbb{R}^n)$. Then, the following are equivalent:*

(i) *$\mathcal{W}_k$ is harmonic in the sense of Hodge in Ω;*
(ii) *$\mathcal{W}_k$ is real harmonic in Ω and*

$$\frac{2}{A_n}\int_\Gamma \sum_{j,r} \frac{z_j-\zeta_j}{|z-\zeta|^n}\nu_r(\zeta)(dz^j \wedge \widehat{dz^r} + \widehat{dz^j} \wedge dz^r)\omega_k(\zeta, dz)\, d\zeta = \omega_k(z, dz), \quad z \in \Gamma.$$

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