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FINITE-DIMENSIONAL HOPF ALGEBRAS OVER THE KAC–PALJUTKIN ALGEBRA H_8

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Abstract. Let H_8 be the Kac–Paljutkin algebra [Trudy Moskov. Mat. Obšč. **15** (1966), 224–261], which is the neither commutative nor cocommutative semisimple eight dimensional Hopf algebra. All simple Yetter–Drinfel'd modules over H_8 are given, and finite-dimensional Nichols algebras over H_8 are determined completely. It turns out that they are all of diagonal type. In fact, they are of Cartan types A_1, A_2, $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$, respectively. By the way, we calculate Gelfand–Kirillov dimensions for some Nichols algebras. As an application, we complete the classification of the finite-dimensional Hopf algebras over H_8 according to the lifting method.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The question of classification of all Hopf algebras over $\mathbb{K}$ of a given dimension up to isomorphism was posed by Kaplansky in 1975 [40]. Some progress has been made but, in general, it is a difficult question for lack of standard methods. One breakthrough is the so-called *lifting method* introduced by Andruskiewitsch and Schneider in 1998 [3], under the assumption that the coradical is a Hopf subalgebra.

We describe the procedure for the lifting method briefly. Let H be a Hopf algebra whose coradical H_0 is a Hopf subalgebra. The associated graded Hopf algebra of H is isomorphic to $R\#H_0$, where $R = \oplus_{n\in\mathbb{N}_0} R(n)$ is a braided Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of Yetter–Drinfel'd modules over H_0, # stands for the Radford *biproduct* or *bosonization* of R with H_0. As explained in [14], to classify finite-dimensional Hopf algebras H whose coradical is isomorphic to H_0 we have to deal with the following questions:

(a) Determine all Yetter–Drinfel'd modules V over H_0 such that the Nichols algebra $\mathfrak{B}(V)$ has finite dimension; find an efficient set of relations for $\mathfrak{B}(V)$.

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(b) If $R = \oplus_{n\in\mathbb{N}_0} R(n)$ is a finite-dimensional Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$ with $V = R(1)$, decide if $R \simeq \mathfrak{B}(V)$. Here $V = R(1)$ is a braided vector space called the *infinitesimal braiding*.

(c) Given V as in (a), classify all H such that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#H_0$ (lifting).

A *lifting* of $V \in {}^{H}_{H}\mathcal{YD}$ is a Hopf algebra L such that $\operatorname{gr} L = \mathfrak{B}(V)\#H$, where $\operatorname{gr} L$ is the graded Hopf algebra associated to the coradical filtration. In other words [16, Proposition 2.4], L is a lifting of V iff there is an epimorphism of Hopf algebras $\phi : \mathcal{T}(V) \coloneqq T(V)\#H \to L$ such that $\phi_{|H} = \mathrm{id}_H$ and

$$\phi_{|H\oplus V\#H} : H \oplus V\#H \to L_1 \text{ is an isomorphism of Hopf bimodules.} \tag{1.1}$$

Such ϕ is called a *lifting map*. If emphasis on H is needed, then we say that L is a lifting of V over H.

The lifting method was extensively used in the classification of finite-dimensional pointed Hopf algebras such as [15], [12], [25], [23], [2], [1], [9], [8] and so on. It is also effective to study finite-dimensional copointed Hopf algebras ([16], [27], [22]). We note that there are very few classification results on finite-dimensional Hopf algebras whose coradical is neither a group algebra nor the dual of a group algebra, some exceptions being [19], [26], [11]. It should be mentioned that [11] constructed Hopf algebras with the Chevalley property over a semisimple Hopf algebra H that is Morita-equivalent to a group algebra $\mathbb{K}G$ (in the sense of ${}^{H}_{H}\mathcal{YD} \simeq {}^{\mathbb{K}G}_{\mathbb{K}G}\mathcal{YD}$ as braided tensor categories). It doesn't cover our case since H_8 can be obtained from a group algebra by a 2-pseudo-cocycle twist but not by a 2-cocycle twist [45].

Here we would like to initiate a project for the study of Hopf algebras whose coradicals are low-dimensional neither commutative nor cocommutative semisimple Hopf algebras by running procedures of the lifting method. One important step is to study the Nichols algebras over those low-dimensional semisimple Hopf algebras. Nichols algebras were studied first by Nichols [44]. These are connected graded braided Hopf algebras [4] generated by primitive elements, and all primitive elements are of degree one. In the past decades, the study of Nichols algebras was mainly focused on categories of Yetter–Drinfel'd modules over group algebras. Under the assumption that the base field has characteristic 0, the classification of finite-dimensional Nichols algebras over abelian groups was completely solved in [30, 31] by using Lie theoretic structures, and the result of the classification played an important role later in the significant work [15]. The problem of classifying finite-dimensional Nichols algebras over non-abelian groups is difficult in general for lack of systematic method; for related works please refer to [12], [24], [29], [32], [33], [36], [35], etc.

In this paper, we mainly focus on the Kac–Paljutkin algebra H_8. The structure of our paper is as follows. In Section 2, we recall the fundamental notions related to Yetter–Drinfel'd modules, Nichols algebras and Gelfand–Kirillov dimension. In section 3, we construct all the simple left Yetter–Drinfel'd modules over H_8 according to Radford's method. In section 4, we get all the possible finite-dimensional Nichols algebras from Yetter–Drinfel'd modules over H_8. It turns out that they are of Cartan types A_1, A_2, $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$. Here is our first main result.

Theorem A. *Let $M \in {}^{H_8}_{H_8}\mathcal{YD}$. Then the Nichols algebra $\mathfrak{B}(M)$ is finite-dimensional iff M is isomorphic to one of the following Yetter–Drinfel'd modules:*

(1) $\Omega_1(n_1, n_2, n_3, n_4) := \bigoplus_{j=1}^4 M\langle b_j, g_j\rangle^{\oplus n_j}$ *with* $\sum_{j=1}^4 n_j \geq 1$, $(b_1, g_1) = (\mathrm{i}, x)$, $(b_2, g_2) = (-\mathrm{i}, x)$, $(b_3, g_3) = (\mathrm{i}, y)$ *and* $(b_4, g_4) = (-\mathrm{i}, y)$, *the infinitesimal braiding is of type* $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2+n_3+n_4}$.

(2) $\Omega_2(n_1, n_2) := M\langle \mathrm{i}, x\rangle^{\oplus n_1} \oplus M\langle -\mathrm{i}, x\rangle^{\oplus n_2} \oplus M\langle (xy, x)\rangle$, $n_1 + n_2 \geq 0$, *the infinitesimal braiding is of type* $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.

(3) $\Omega_3(n_1, n_2) := M\langle \mathrm{i}, y\rangle^{\oplus n_1} \oplus M\langle -\mathrm{i}, y\rangle^{\oplus n_2} \oplus M\langle (y, xy)\rangle$, $n_1 + n_2 \geq 0$, *the infinitesimal braiding is of type* $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.

(4) $\Omega_4(n_1, n_2) := M\langle \mathrm{i}, x\rangle^{\oplus n_1} \oplus M\langle \mathrm{i}, y\rangle^{\oplus n_2} \oplus W^{1,-1}$, $n_1 + n_2 \geq 0$, *the infinitesimal braiding is of type* $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.

(5) $\Omega_5(n_1, n_2) := M\langle -\mathrm{i}, x\rangle^{\oplus n_1} \oplus M\langle -\mathrm{i}, y\rangle^{\oplus n_2} \oplus W^{-1,-1}$, $n_1 + n_2 \geq 0$, *the infinitesimal braiding is of type* $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.

(6) $\Omega_6 := M\langle (xy, x)\rangle \oplus M\langle (y, xy)\rangle$, *the infinitesimal braiding is of type* $A_2 \times A_2$.

(7) $\Omega_7 := W^{1,-1} \oplus W^{-1,-1}$, *the infinitesimal braiding is of type* $A_2 \times A_2$.

Remark 1.1. We point out which of the Yetter–Drinfel'd modules have a principal realization and which not, since the liftings are known when there is a principal realization and not otherwise [5, Subsection 2.2]. Let (h) and (δ_h) be dual bases of H_8 and H_8^*, and $b \in \{\pm 1, \pm \mathrm{i}\}$. Define $\chi_b := \delta_1 + \delta_{xy} + b^2(\delta_x + \delta_y) + b(\delta_z + \delta_{zxy}) + b^3(\delta_{zx} + \delta_{zy}) \in \mathrm{Alg}(H_8, \mathbb{K})$, then (g, χ_b) is a *YD-pair* [7] iff $\mathbb{K}_g^{\chi_b} \simeq M\langle b, g\rangle$ is a one-dimensional Yetter–Drinfel'd module. $M\langle (g_1, g_2)\rangle$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ and $W^{b_1,-1}$ for $b_1 = \pm 1$ don't have a principal realization. So only $\Omega_1(n_1, n_2, n_3, n_4)$ has a principal realization.

In section 5, according to the lifting method, we give a classification for finite-dimensional Hopf algebras over H_8. Here is the second main result.

Theorem B. *Let H be a finite-dimensional Hopf algebra over H_8 such that its infinitesimal braiding is in ${}^{H_8}_{H_8}\mathcal{YD}$. Then H is isomorphic to either of:*

(1) $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$, *see Definition 5.4;*
(2) $\mathfrak{B}[\Omega_2(n_1, n_2)]\#H_8$, *see Proposition 5.10;*
(3) $\mathfrak{A}_4(n_1, n_2; I_4)$, *see Definition 5.18;*
(4) $\mathfrak{A}_6(\lambda)$, *see Definition 5.11;*
(5) $\mathfrak{A}_7(I_7)$, *see Definition 5.15.*

$\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ comprises two 16-dimensional nonisomorphic nonpointed self-dual Hopf algebras with coradical H_8 described in [19] as special cases. Except for the case (2), the remaining four families of Hopf algebras contain non-trivial lifting relations.

2. Preliminaries

2.1. **Conventions.** Let H be a Hopf algebra over $\mathbb{K}$, with antipode S. We will use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the comultiplication ([43]). Let ${}^H_H\mathcal{YD}$ be the category of left *Yetter–Drinfel'd modules* over H. A left Yetter–Drinfel'd module M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, ρ) satisfying

$$\rho(h \cdot m) = h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \quad \forall m \in M,\, h \in H, \tag{2.1}$$

where $\rho(m) = m_{(-1)} \otimes m_{(0)}$. The category ${}^H_H\mathcal{YD}$ is a braided monoidal category. The braiding $c \in \operatorname{End}_{\mathbb{K}}(M \otimes M)$ of M is defined by $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, and the inverse braiding is defined by $c^{-1}(v \otimes w) = w_{(0)} \otimes \left(S^{-1}(w_{(-1)}) \cdot v\right)$.

Definition 2.1 ([14, Definition 2.1])**.** Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $R = \bigoplus_{n\geq 0} R(n) \in {}^H_H\mathcal{YD}$ is called the *Nichols algebra* of V if

(i) $\mathbb{K} \simeq R(0)$, $V \simeq R(1) \in {}^H_H\mathcal{YD}$.
(ii) $R(1) = \mathcal{P}(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$.
(iii) R is generated as an algebra by $R(1)$.

In this case, R is denoted by $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$.

Remark 2.2. The Nichols algebra $\mathfrak{B}(V)$ is completely determined by the braiding. More precisely, as proved in [49] and noted in [14],

$$\mathfrak{B}(V) = \mathrm{K} \oplus V \oplus \bigoplus_{n=2}^{\infty} V^{\otimes n} / \ker \mathfrak{S}_n = T(V)/\ker \mathfrak{S},$$

where $\mathfrak{S}_{n,1} \in \operatorname{End}_{\mathbb{K}}\left(V^{\otimes(n+1)}\right)$, $\mathfrak{S}_n \in \operatorname{End}_{\mathbb{K}}\left(V^{\otimes n}\right)$,

$$\mathfrak{S}_{n,1} := \mathrm{id} + c_n + c_n c_{n-1} + \cdots + c_n c_{n-1} \cdots c_1 = \mathrm{id} + c_n \mathfrak{S}_{n-1,1},$$
$$\mathfrak{S}_1 := \mathrm{id}, \quad \mathfrak{S}_2 := \mathrm{id} + c, \quad \mathfrak{S}_n := (\mathfrak{S}_{n-1} \otimes \mathrm{id})\mathfrak{S}_{n-1,1}.$$

Lemma 2.3 ([28, Theorem 2.2], [6, Remark 1.4])**.** *Let $M_1, M_2 \in {}^H_H\mathcal{YD}$ be both finite-dimensional and assume $c_{M_1,M_2} c_{M_2,M_1} = \mathrm{id}_{M_2 \otimes M_1}$. Then $\mathfrak{B}(M_1 \oplus M_2) \simeq \mathfrak{B}(M_1) \otimes \mathfrak{B}(M_2)$ as graded vector spaces and* $\operatorname{GKdim} \mathfrak{B}(M_1 \oplus M_2) = \operatorname{GKdim} \mathfrak{B}(M_1) + \operatorname{GKdim} \mathfrak{B}(M_2)$.

Proposition 2.4 ([46, Radford biproduct])**.** *Let H be a Hopf algebra and $A \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. Then $A\#H$ is a Hopf algebra with*

$$\Delta(a\#h) = \sum \left[a_{(1)}\#(a_{(2)})_{(-1)} h_{(1)}\right] \otimes \left[(a_{(2)})_{(0)}\#h_{(2)}\right], \tag{2.2}$$

$$S(a\#h) = \sum \left(1\#S_H(h) S_H(a_{(-1)})\right) \left(S_A(a_{(0)})\#1\right), \tag{2.3}$$

$$(a\#h)(a'\#h') = \sum a(h_{(1)} \cdot a')\#h_{(2)} h', \quad a, a' \in A, h, h' \in H. \tag{2.4}$$

The map $\iota : H \to A\#H$ given by $\iota(h) = 1\#h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : A\#H \to H$ given by $\pi(a\#h) = \varepsilon_A(a)h$ for all $a \in A$, $h \in H$ is a surjective Hopf algebra map such that $\pi \circ \iota = \mathrm{id}_H$. Moreover, it holds that $A = (A\#H)^{\mathrm{co}\,\pi}$.

Conversely, let B be a Hopf algebra with bijective antipode and $\pi : B \to H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \to B$ such that $\pi \circ \iota = \mathrm{id}_H$. Then $A = B^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $B \simeq A\#H$ as Hopf algebras.

2.2. **GK-dimension.** Let A be a finitely generated algebra over a field $\mathbb{K}$, and assume $a_1, \ldots, a_m$ generate A. Set V to be the span of $a_1, \ldots, a_m$, and denote V^n the span of all monomials in the a_i's of length n. As a_i's generate A one has $A = \bigcup_{k=0}^{\infty} A_k$, where $A_k = \mathbb{K} + V + V^2 + \cdots + V^k$. The function $d_V(n) = \dim A_n$ is the growth function of A. The *Gelfand–Kirillov dimension* of a $\mathbb{K}$-algebra A is $\mathrm{GKdim}\, A = \overline{\lim} \log_n d_V(n)$. $\mathrm{GKdim}\, A$ does not depend on the choice of V. Suppose that $\mathrm{GKdim}\, A < \infty$. We say that a finite-dimensional subspace $V \subseteq A$ is *GK-deterministic* if

$$\mathrm{GKdim}\, A = \lim_{n\to\infty} \log_n \dim \sum_{0\le j\le n} V^j.$$

Clearly, if V is a GK-deterministic subspace of A, then any finite-dimensional subspace of A containing V is GK-deterministic. Let A and B be two algebras. Then

$$\mathrm{GKdim}(A \otimes B) \le \mathrm{GKdim}\, A + \mathrm{GKdim}\, B,$$

but the equality does not hold in general. However, it does hold when A or B has a GK-deterministic subspace, see [41, Proposition 3.11]. The Gelfand–Kirillov dimension is a useful tool in ring theory and Hopf algebraic theories. We shall not discuss in detail its importance but we refer the reader to [41] as a basic reference and [51, 50, 18, 6] for additional information related with Hopf algebras.

3. Simple Yetter–Drinfel'd modules of H_8

Recall that the neither commutative nor cocommutative semisimple 8-dimensional Hopf algebra H_8 in [42] is constructed as an extension of $\mathbb{K}[C_2 \times C_2]$ by $\mathbb{K}[C_2]$. A basis for H_8 is given by $\{1, x, y, xy = yx, z, xz, yz, xyz\}$ with the relations

$$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad zx = yz, \quad zy = xz.$$

The coalgebra structure and the antipode are defined by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = \varepsilon(y) = 1, \quad S(x) = x, \quad S(y) = y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \quad \varepsilon(z) = 1, \quad S(z) = z.$$

The automorphism group of H_8 is the Klein four-group [48]. These automorphisms are given in Table 1; they are going to be used in Corollary 5.3.

Denote a set of central orthogonal idempotents of H_8 as

$$e_1 = \frac{1}{8}(1 + x)(1 + y)(1 + z), \quad e_2 = \frac{1}{8}(1 + x)(1 + y)(1 - z),$$

$$e_3 = \frac{1}{8}(1 - x)(1 - y)(1 + \mathrm{i}\, z), \quad e_4 = \frac{1}{8}(1 - x)(1 - y)(1 - \mathrm{i}\, z),$$

	1	x	y	z
$\tau_1 = \mathrm{id}$	1	x	y	z
τ_2	1	x	y	xyz
τ_3	1	y	x	$\frac{1}{2}(z + xz + yz - xyz)$
τ_4	1	y	x	$\frac{1}{2}(-z + xz + yz + xyz)$

TABLE 1. Hopf algebra automorphisms of H_8.

$$e_5 = \frac{1-xy}{2}, \quad e_j e_k = \delta_{jk}, \quad j, k = 1, \ldots, 5; \ \mathrm{i} = \sqrt{-1};$$

and denote idempotents $e_5' = \frac{1}{4}(1-xy)(1+z)$, $e_5'' = \frac{1}{4}(1-xy)(1-z)$. Then

$$\begin{aligned} H_8 &= H_8e_1 \oplus H_8e_2 \oplus H_8e_3 \oplus H_8e_4 \oplus H_8e_5 \\ &= H_8e_1 \oplus H_8e_2 \oplus H_8e_3 \oplus H_8e_4 \oplus (H_8e_5' + H_8e_5''), \end{aligned}$$

where $H_8e_5' \simeq H_8e_5''$ as left H_8-modules, via $e_5' \mapsto xe_5''$, $xe_5' \mapsto e_5''$.

Definition 3.1. Denote $V_1(b) := \mathbb{K}\{v \mid x \cdot v = b^2 v,\ y \cdot v = b^2 v,\ z \cdot v = bv,\ b \in \{\pm 1, \pm \mathrm{i}\}\}$, where v is a vector. Let $V_2 \simeq H_8e_5'$ as left H_8-modules; the actions of the generators are given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3.2. *All simple left modules of H_8 are classified by $V_1(b), V_2, b \in \{\pm 1, \pm \mathrm{i}\}$.*

Remark 3.3. The result was also obtained in [20] under a different basis (thanks to referee for reminding us about this fact).

In the remaining part of the article, $V_1(b)$ and V_2 always mean simple left H_8-modules.

Lemma 3.4 ([47, Proposition 2])**.** *Let H be a bialgebra over a field $\mathbb{K}$ and suppose S is the antipode of H.*

(1) *If $L \in {}_H\mathcal{M}$, then $L \otimes H \in {}_H\mathcal{YD}^H$; the module and comodule actions are given by*

$$h \cdot (\ell \otimes a) = h_{(2)} \cdot \ell \otimes h_{(3)} a S^{-1}(h_{(1)}),\ \rho(\ell \otimes h) = (\ell \otimes h_{(1)}) \otimes h_{(2)}, \ \ \forall h, a \in H,\ \ell \in L.$$

Let $M \in {}_H\mathcal{YD}^H$.

(2) *Suppose that $L \in {}_H\mathcal{M}$ and $p : M \longrightarrow L$ is a map of left H-modules. Then the linear map $f : M \longrightarrow L \otimes H$ defined by $f(m) = p(m_{(0)}) \otimes m_{(1)}$ for all $m \in M$ is a map of Yetter–Drinfel'd H-modules, where $L \otimes H$ has the structure described in part* (1)*. Furthermore* $\ker f$ *is the largest Yetter–Drinfel'd H-submodule, indeed the largest subcomodule, contained in* $\ker p$.

(3) *M is isomorphic to a Yetter–Drinfel'd submodule of some $L \otimes H$ described above.*

Similarly, according to Radford's method, any simple left Yetter–Drinfel'd module over H_8 could be constructed by the submodule of the tensor product of a left module V of H_8 and H_8 itself, where the module and comodule structures are given by

$$\begin{aligned} h\cdot(\ell\boxtimes g) &= (h_{(2)}\cdot\ell)\boxtimes h_{(1)}gS(h_{(3)}),\\ \rho(\ell\boxtimes h) &= h_{(1)}\otimes(\ell\boxtimes h_{(2)}),\quad \forall h,g\in H_8,\ \ell\in V. \end{aligned}\tag{3.1}$$

Here we use $\boxtimes$ instead of $\otimes$ to avoid confusion by using too many symbols of the tensor product. We are going to construct all simple left Yetter–Drinfel'd modules over H_8 in this way. Keeping in mind that H_8 is semisimple, it is possible to do so. In fact, it is much easier than making use of the fact that ${}^{H_8}_{H_8}\mathcal{YD}\simeq {}_{\mathcal{D}(H_8^{\mathrm{cop}})}\mathcal{M}$. The following is a list of useful formulae for looking for simple objects of ${}^{H_8}_{H_8}\mathcal{YD}$.

Lemma 3.5.

$$\begin{aligned} (\mathrm{id}^{\otimes 2}\otimes S)\Delta^{(2)}(z) = \frac{1}{4}[&(1+y)z\otimes z\otimes z(1+x)+(1-y)z\otimes xz\otimes z(1+x)\\ &+(1+y)z\otimes yz\otimes z(1-x)+(y-1)z\otimes xyz\otimes z(1-x)], \end{aligned}$$

$$\begin{aligned} z_{(2)}\otimes z_{(1)}S(z_{(3)}) = \frac{1}{4}[&z\otimes(1+x)(1+y)+xz\otimes(1+x)(1-y)\\ &+yz\otimes(1-x)(1+y)+xyz\otimes(1-x)(1-y)], \end{aligned}\tag{3.2}$$

$$\begin{aligned} z_{(2)}\otimes z_{(1)}xS(z_{(3)}) = \frac{1}{4}[&z\otimes(1+x)(1+y)+xz\otimes(1+x)(y-1)\\ &+yz\otimes(1-x)(1+y)+xyz\otimes(x-1)(1-y)], \end{aligned}\tag{3.3}$$

$$\begin{aligned} z_{(2)}\otimes z_{(1)}yS(z_{(3)}) = \frac{1}{4}[&z\otimes(1+x)(1+y)+xz\otimes(1+x)(1-y)\\ &+yz\otimes(x-1)(1+y)+xyz\otimes(x-1)(1-y)], \end{aligned}\tag{3.4}$$

$$\begin{aligned} z_{(2)}\otimes z_{(1)}xyS(z_{(3)}) = \frac{1}{4}[&z\otimes(1+x)(1+y)+xz\otimes(1+x)(y-1)\\ &+yz\otimes(x-1)(1+y)+xyz\otimes(1-x)(1-y)], \end{aligned}\tag{3.5}$$

$$z_{(2)}\otimes z_{(1)}zS(z_{(3)}) = \frac{1}{2}[z\otimes(1+y)z+xyz\otimes x(y-1)z],\tag{3.6}$$

$$z_{(2)}\otimes z_{(1)}xzS(z_{(3)}) = \frac{1}{2}[z\otimes(1+y)z+xyz\otimes x(1-y)z],\tag{3.7}$$

$$z_{(2)}\otimes z_{(1)}yzS(z_{(3)}) = \frac{1}{2}[z\otimes x(1+y)z+xyz\otimes(y-1)z],\tag{3.8}$$

$$z_{(2)}\otimes z_{(1)}xyzS(z_{(3)}) = \frac{1}{2}[z\otimes x(1+y)z+xyz\otimes(1-y)z].\tag{3.9}$$

Definition 3.6. Define $M\langle b,g\rangle \coloneqq \mathbb{K}\{v\boxtimes g\mid v\in V_1(b)\}$, where $b\in\{\pm1,\pm\mathrm{i}\}$ and $g\in\{1,x,y,xy\}$.

Lemma 3.7. *There are eight pairwise non-isomorphic one dimensional Yetter–Drinfel'd modules over* H_8 *as* $M\langle b,g\rangle$ *with* $(b,g)\in\{(\pm1,1),(\pm1,xy),(\pm\mathrm{i},x),(\pm\mathrm{i},y)\}$. *The actions and coactions are given by*

$$x\cdot(v\boxtimes g)=b^2(v\boxtimes g),\quad y\cdot(v\boxtimes g)=b^2(v\boxtimes g),\quad z\cdot(v\boxtimes g)=b(v\boxtimes g),$$

$$\rho(v \boxtimes g) = g \otimes (v \boxtimes g), \quad v \boxtimes g \in M\langle b, g\rangle, \quad v \in V_1(b).$$

Proof. Let $v \in V_1(b)$. Then

$$z \cdot (v \boxtimes 1) \xlongequal{(3.2)} \frac{bv}{4} \boxtimes [1 + x + b^2(1-x)][1 + y + b^2(1-y)], \tag{3.10}$$

$$z \cdot (v \boxtimes xy) \xlongequal{(3.5)} \frac{bv}{4} \boxtimes [1 + x + b^2(x-1)][1 + y + b^2(y-1)], \tag{3.11}$$

$$z \cdot (v \boxtimes x) \xlongequal{(3.3)} \frac{bv}{4} \boxtimes [1 + x + b^2(1-x)][1 + y + b^2(y-1)], \tag{3.12}$$

$$z \cdot (v \boxtimes y) \xlongequal{(3.4)} \frac{bv}{4} \boxtimes [1 + x + b^2(x-1)][1 + y + b^2(1-y)]. \tag{3.13}$$

so

$$z \cdot (v \boxtimes 1) = bv \boxtimes 1, \quad z \cdot (v \boxtimes xy) = bv \boxtimes xy, \quad \text{when } b = \pm 1;$$
$$z \cdot (v \boxtimes x) = bv \boxtimes x, \quad z \cdot (v \boxtimes y) = bv \boxtimes y, \quad \text{when } b = \pm \mathrm{i}.$$

Now it is easy to see that $M\langle b, g\rangle$ defined above is a one-dimensional Yetter–Drinfel'd module by Radford's method and the eight one-dimensional Yetter–Drinfel'd modules are pairwise non-isomorphic by observations on their actions and coactions. □

Definition 3.8. Let $(g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}$ and denote three vector spaces as

$$M\langle(1, xy)\rangle \coloneqq \mathbb{K}\{v \boxtimes 1, v \boxtimes xy \mid v \in V_1(\mathrm{i})\},$$
$$M\langle(x, y)\rangle \coloneqq \mathbb{K}\{v \boxtimes x, v \boxtimes y \mid v \in V_1(1)\},$$
$$M\langle(g_1, g_2)\rangle \coloneqq \mathbb{K}\{(v_1 + v_2) \boxtimes g_1, (v_1 - v_2) \boxtimes g_2 \mid v_1, v_2 \in V_2\}.$$

Lemma 3.9. *There are six pairwise non-isomorphic two-dimensional simple Yetter–Drinfel'd modules over H_8 as below, where the action and coaction are given by formulae* (3.1)*.*

(1) $M\langle(1, xy)\rangle$*, the actions of generators on* $(v \boxtimes 1, v \boxtimes xy)$ *are given by*

$$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

(2) $M\langle(x, y)\rangle$*, the actions of generators on* $(v \boxtimes x, v \boxtimes y)$ *are given by*

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(3) $M\langle(g_1, g_2)\rangle$*, where* $(g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}$*; the actions of generators on the row vector* $((v_1 + v_2) \boxtimes g_1, (v_1 - v_2) \boxtimes g_2)$ *are given by*

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Since the coactions are easy to see, we can focus on their structures as left H_8-modules. Parts (1) and (2) of the lemma can be checked by formulae (3.10) to (3.13). Let $v_1, v_2 \in V_2$. Then

$$
\begin{aligned}
z \cdot (v_1 \boxtimes 1) &\overset{(3.2)}{=\!=} \frac{1}{2}[v_1 \boxtimes (x+y) + v_2 \boxtimes (x-y)],\\
z \cdot (v_2 \boxtimes 1) &\overset{(3.2)}{=\!=} \frac{1}{2}[v_1 \boxtimes (-x+y) + v_2 \boxtimes (-x-y)],\\
z \cdot (v_1 \boxtimes xy) &\overset{(3.5)}{=\!=} \frac{1}{2}[v_1 \boxtimes (x+y) + v_2 \boxtimes (y-x)],\\
z \cdot (v_2 \boxtimes xy) &\overset{(3.5)}{=\!=} \frac{1}{2}[v_1 \boxtimes (x-y) + v_2 \boxtimes (-x-y)],\\
z \cdot (v_1 \boxtimes y) &\overset{(3.4)}{=\!=} \frac{1}{2}[v_1 \boxtimes (1+xy) + v_2 \boxtimes (1-xy)],\\
z \cdot (v_2 \boxtimes y) &\overset{(3.4)}{=\!=} \frac{1}{2}[v_1 \boxtimes (-1+xy) + v_2 \boxtimes (-1-xy)],\\
z \cdot (v_1 \boxtimes x) &\overset{(3.3)}{=\!=} \frac{1}{2}[v_1 \boxtimes (1+xy) + v_2 \boxtimes (-1+xy)],\\
z \cdot (v_2 \boxtimes x) &\overset{(3.3)}{=\!=} \frac{1}{2}[v_1 \boxtimes (1-xy) + v_2 \boxtimes (-1-xy)].
\end{aligned}
$$

So we have

$$
\begin{aligned}
z \cdot [(v_1+v_2) \boxtimes 1] &= (v_1 - v_2) \boxtimes y, & z \cdot [(v_1-v_2) \boxtimes y] &= (v_1 + v_2) \boxtimes 1,\\
z \cdot [(v_1+v_2) \boxtimes x] &= (v_1 - v_2) \boxtimes 1, & z \cdot [(v_1-v_2) \boxtimes 1] &= (v_1 + v_2) \boxtimes x,\\
z \cdot [(v_1+v_2) \boxtimes xy] &= (v_1 - v_2) \boxtimes x, & z \cdot [(v_1-v_2) \boxtimes x] &= (v_1 + v_2) \boxtimes xy,\\
z \cdot [(v_1+v_2) \boxtimes y] &= (v_1 - v_2) \boxtimes xy, & z \cdot [(v_1-v_2) \boxtimes xy] &= (v_1 + v_2) \boxtimes y.
\end{aligned}
$$

Part (3) is immediate to check. The six two-dimensional Yetter–Drinfel'd modules are pairwise non-isomorphic since they are pairwise non-isomorphic as comodules. □

Lemma 3.10. *Let $b_1, b_2 \in \{\pm 1\}$ and $v \in V_1(b_2)$, and denote*

$$w_1^{b_1,b_2} := v \boxtimes (1 + \mathrm{i}b_1 y)z, \quad w_2^{b_1,b_2} := v \boxtimes x(1 - \mathrm{i}b_1 y)z.$$

Then $W^{b_1,b_2} = \mathbb{K}w_1^{b_1,b_2} \oplus \mathbb{K}w_2^{b_1,b_2}$ is a family of four pairwise non-isomorphic two-dimensional simple Yetter–Drinfel'd modules over H_8 with the actions of generators on the row vector $(w_1^{b_1,b_2}, w_2^{b_1,b_2})$ and coactions given by

$$
x \mapsto \begin{pmatrix} 0 & -\mathrm{i}b_1 \\ \mathrm{i}b_1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -\mathrm{i}b_1 \\ \mathrm{i}b_1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} \frac{(1-\mathrm{i}b_1)b_2}{2} & \frac{(1-\mathrm{i}b_1)b_2}{2} \\ \frac{-(1-\mathrm{i}b_1)b_2}{2} & \frac{(1-\mathrm{i}b_1)b_2}{2} \end{pmatrix},
$$

$$
\begin{aligned}
\rho\left(w_1^{b_1,b_2}\right) &= \frac{(1+y)z}{2} \otimes w_1^{b_1,b_2} + \frac{(1-y)z}{2} \otimes w_2^{b_1,b_2},\\
\rho\left(w_2^{b_1,b_2}\right) &= \frac{x(1+y)z}{2} \otimes w_2^{b_1,b_2} + \frac{x(1-y)z}{2} \otimes w_1^{b_1,b_2}.
\end{aligned}
$$

Proof. It is straightforward by the definition of Yetter–Drinfel'd module. When $b_2 \neq b_2'$, $W^{b_1,b_2} \not\simeq W^{b_1,b_2'}$ since we will see that their braidings are different in Proposition 4.1. As explained in the following remark, W^{b_1,b_2} has another basis $\{p_1, p_2\}$ with $p_1 \in V_1(b_2)$ and $p_2 \in V_1(-b_1b_2\mathrm{i})$. So $W^{b_1,b_2} \not\simeq W^{b_1',b_2}$ if $b_1 \neq b_1'$. □

Remark 3.11. (1) Let $M = \mathbb{K}\{v\boxtimes z, v\boxtimes xz, v\boxtimes yz, v\boxtimes xyz \mid v \in V_1(b)\}$, $b \in \{\pm 1\}$. z acts on elements of M as

$$z\cdot(v\boxtimes z) \xlongequal{(3.6)} \frac{bv}{2}\boxtimes(1-x+y+xy)z,$$
$$z\cdot(v\boxtimes xz) \xlongequal{(3.7)} \frac{bv}{2}\boxtimes(1+x+y-xy)z,$$
$$z\cdot(v\boxtimes yz) \xlongequal{(3.8)} \frac{bv}{2}\boxtimes(-1+x+y+xy)z,$$
$$z\cdot(v\boxtimes xyz) \xlongequal{(3.9)} \frac{bv}{2}\boxtimes(1+x-y+xy)z.$$

Then $M \simeq W^{1,b} \oplus W^{-1,b}$ as Yetter–Drinfel'd modules over H_8.

(2) Let $f_{jk} := \frac{1}{4}[1+(-1)^j x][1+(-1)^k y]$, $j,k = 0,1$. Denote

$$p_1 = w_1^{b_1,b_2} + \mathrm{i}b_1 w_2^{b_1,b_2}, \quad p_2 = w_1^{b_1,b_2} - \mathrm{i}b_1 w_2^{b_1,b_2}.$$

Then $W^{b_1,b_2} = \mathbb{K}p_1 \oplus \mathbb{K}p_2$ with the actions of generators on the row vector (p_1,p_2) and coactions given by

$$x \mapsto \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad y \mapsto \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad z \mapsto \begin{pmatrix}b_2 & 0\\ 0 & -\mathrm{i}b_1b_2\end{pmatrix},$$
$$\rho(p_1) = [f_{00} - \mathrm{i}b_1 f_{11}]\, z\otimes p_1 + [f_{10} + \mathrm{i}b_1 f_{01}]\, z\otimes p_2,$$
$$\rho(p_2) = [f_{00} + \mathrm{i}b_1 f_{11}]\, z\otimes p_2 + [f_{10} - \mathrm{i}b_1 f_{01}]\, z\otimes p_1.$$

According to [42, Remark 2.14], H_8 is presented by generators x, y, w, where the expressions containing z are replaced by

$$w = \left(f_{00} + \sqrt{\mathrm{i}}f_{10} + \frac{1}{\sqrt{\mathrm{i}}}f_{01} + \mathrm{i}f_{11}\right)z, \quad w^2 = 1,$$
$$wx = yw, \quad S(w) = \left(\frac{1+\mathrm{i}}{2}x + \frac{1-\mathrm{i}}{2}y\right)w,$$
$$\Delta(w) = \left(\frac{1}{2}(1+xy)\otimes 1 + \frac{1+\mathrm{i}}{4}(1-xy)\otimes x + \frac{1-\mathrm{i}}{4}(1-xy)\otimes y\right)(w\otimes w).$$

Let $a+1 = \pm\sqrt{2}$. We define

$$w_1^{(1)} := (v_1 + \mathrm{i}av_2)\boxtimes\frac{\mathrm{i}}{2}\left[(x+y) + \sqrt{\mathrm{i}}(x-y)\right]w$$
$$+ (av_1 - \mathrm{i}v_2)\boxtimes\frac{1}{2}\left[(x+y) - \sqrt{\mathrm{i}}(x-y)\right]w,$$
$$w_2^{(1)} := (v_1 + \mathrm{i}av_2)\boxtimes\frac{\mathrm{i}}{2}\left[(1+xy) + \sqrt{\mathrm{i}}(1-xy)\right]w$$
$$- (av_1 - \mathrm{i}v_2)\boxtimes\frac{1}{2}\left[(1+xy) - \sqrt{\mathrm{i}}(1-xy)\right]w,$$

$$w_1^{(2)} \coloneqq (v_1 - \mathrm{i}av_2) \boxtimes \frac{\mathrm{i}}{2}\left[(x+y) + \sqrt{\mathrm{i}}(x-y)\right] w + (av_1 + \mathrm{i}v_2) \boxtimes \frac{1}{2}\left[(x+y) - \sqrt{\mathrm{i}}(x-y)\right] w,$$

$$w_2^{(2)} \coloneqq (v_1 - \mathrm{i}av_2) \boxtimes \frac{\mathrm{i}}{2}\left[(1+xy) + \sqrt{\mathrm{i}}(1-xy)\right] w - (av_1 + \mathrm{i}v_2) \boxtimes \frac{1}{2}\left[(1+xy) - \sqrt{\mathrm{i}}(1-xy)\right] w.$$

Lemma 3.12. *Let $a+1 = \pm\sqrt{2}$. There are four pairwise non-isomorphic simple Yetter–Drinfel'd modules W_1^a and W_2^a over H_8 as follows:*

(1) *Let $W_1^a = \mathbb{K}w_1^{(1)} \oplus \mathbb{K}w_2^{(1)}$. Then W_1^a is a two-dimensional simple Yetter–Drinfel'd module over H_8 with actions given by*

$$\begin{cases} x \cdot w_1^{(1)} = -w_1^{(1)}, \\ y \cdot w_1^{(1)} = w_1^{(1)}, \\ z \cdot w_1^{(1)} = \frac{1}{2}(1-\mathrm{i})(a+1)w_2^{(1)}, \\ w \cdot w_1^{(1)} = \frac{1}{2\sqrt{\mathrm{i}}}(1-\mathrm{i})(a+1)w_2^{(1)}, \end{cases} \qquad \begin{cases} x \cdot w_2^{(1)} = w_2^{(1)}, \\ y \cdot w_2^{(1)} = -w_2^{(1)}, \\ z \cdot w_2^{(1)} = \frac{1}{2}(1+\mathrm{i})(a+1)w_1^{(1)}, \\ w \cdot w_2^{(1)} = \frac{\sqrt{\mathrm{i}}}{2}(1+\mathrm{i})(a+1)w_1^{(1)}, \end{cases}$$

and coactions given by

$$\rho\left(w_1^{(1)}\right) = \frac{1}{2}(x+y)w \otimes w_1^{(1)} + \frac{\sqrt{\mathrm{i}}}{2}(x-y)w \otimes w_2^{(1)},$$
$$\rho\left(w_2^{(1)}\right) = \frac{1}{2}(1+xy)w \otimes w_2^{(1)} + \frac{\sqrt{\mathrm{i}}}{2}(1-xy)w \otimes w_1^{(1)}.$$

(2) *Let $W_2^a = \mathbb{K}w_1^{(2)} \oplus \mathbb{K}w_2^{(2)}$. Then W_2^a is a two-dimensional simple Yetter–Drinfel'd module over H_8 with actions given by*

$$\begin{cases} x \cdot w_1^{(2)} = w_1^{(2)}, \\ y \cdot w_1^{(2)} = -w_1^{(2)}, \\ z \cdot w_1^{(2)} = \frac{1}{2}(1-\mathrm{i})(a+1)w_2^{(2)}, \\ w \cdot w_1^{(2)} = \frac{\sqrt{\mathrm{i}}}{2}(1-\mathrm{i})(a+1)w_2^{(2)}, \end{cases} \qquad \begin{cases} x \cdot w_2^{(2)} = -w_2^{(2)}, \\ y \cdot w_2^{(2)} = w_2^{(2)}, \\ z \cdot w_2^{(2)} = \frac{1}{2}(1+\mathrm{i})(a+1)w_1^{(2)}, \\ w \cdot w_2^{(2)} = \frac{1}{2\sqrt{\mathrm{i}}}(1+\mathrm{i})(a+1)w_1^{(2)}, \end{cases}$$

and coactions given by

$$\rho\left(w_1^{(2)}\right) = \frac{1}{2}(x+y)w \otimes w_1^{(2)} + \frac{\sqrt{\mathrm{i}}}{2}(x-y)w \otimes w_2^{(2)},$$
$$\rho\left(w_2^{(2)}\right) = \frac{1}{2}(1+xy)w \otimes w_2^{(2)} + \frac{\sqrt{\mathrm{i}}}{2}(1-xy)w \otimes w_1^{(2)}.$$

Proof. It is straightforward to check by the definition of Yetter–Drinfel'd module. Actually, $M \simeq \bigoplus_{a+1=\pm\sqrt{2}}(W_1^a \oplus W_2^a)$ as Yetter–Drinfel'd modules over H_8, where $M = \mathbb{K}\{v_j \boxtimes z, v_j \boxtimes xz, v_j \boxtimes yz, v_j \boxtimes xyz \mid v_j \in V_2,\ j = 1,2\}$.

Since $\sqrt{\mathrm{i}} = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}$, $\frac{1}{2}\sqrt{2}\sqrt{\mathrm{i}}(1-\mathrm{i}) = 1$. Denote $a+1 = b\sqrt{2}$, $b = \pm 1$, $p_1^{(1)} = \sqrt{\mathrm{i}}w_1^{(1)} + w_2^{(1)}$, $p_2^{(1)} = -\sqrt{\mathrm{i}}w_1^{(1)} + w_2^{(1)}$; then $W_1^a = \mathbb{K}p_1^{(1)} \oplus \mathbb{K}p_2^{(1)}$ with actions on the row vector $\left(p_1^{(1)}, p_2^{(1)}\right)$ given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}. \tag{3.14}$$

Let $p_1^{(2)} = w_1^{(2)} + \frac{1}{\sqrt{\mathrm{i}}} w_2^{(2)}$, $p_2^{(2)} = w_1^{(2)} - \frac{1}{\sqrt{\mathrm{i}}} w_2^{(2)}$; then $W_2^a = \mathbb{K}p_1^{(2)} \oplus \mathbb{K}p_2^{(2)}$ with actions on the row vector $\left(p_1^{(2)}, p_2^{(2)}\right)$ also given by (3.14). Now we can observe that $W_1^{-1+\sqrt{2}}$ is isomorphic to $W_1^{-1-\sqrt{2}}$ as modules (or comodules) under a suitably chosen base, but they are not isomorphic as modules and comodules simultaneously. So $W_1^{-1+\sqrt{2}} \not\simeq W_1^{-1-\sqrt{2}}$ as Yetter–Drinfel'd modules. For the same reason, we have $W_2^{-1+\sqrt{2}} \not\simeq W_2^{-1-\sqrt{2}}$ and $W_1^a \not\simeq W_2^a$. □

Obviously, any module in Lemma 3.9 is not isomorphic to any one of modules in Lemmas 3.10 and 3.12 as comodules. As H_8-modules, $W^{b_1,b_2} \simeq V_1(b_2) \oplus V_1(-b_1 b_2 \mathrm{i})$, and $W_1^a \simeq W_2^a \simeq V_2$. So Yetter–Drinfel'd modules in Lemmas 3.9, 3.10 and 3.12 are pairwise non-isomorphic. Keeping in mind that H_8 is semisimple, now we are arriving at

Theorem 3.13. *All the simple Yetter–Drinfel'd modules over H_8 are classified by*

- *Eight pairwise non-isomorphic simple Yetter–Drinfel'd modules of one-dimension:*

$$M\langle b, g\rangle, \quad (b,g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\}.$$

- *Fourteen pairwise non-isomorphic simple Yetter–Drinfel'd modules of two-dimension:*

$$M\langle(1, xy)\rangle,\ M\langle(x,y)\rangle,\ M\langle(g_1, g_2)\rangle,\ W^{b_1,b_2},\ W_1^a,\ W_2^a,$$

where $(g_1, g_2) \in \{(1,y), (x,1), (xy,x), (y,xy)\}$, $b_1, b_2 \in \{\pm 1\}$, $a+1 = \pm\sqrt{2}$.

Remark 3.14. Jun Hu and Yinhuo Zhang investigated $\mathcal{D}(H)$-modules in [37] and [38] by using Radford's construction [47]. In particular, they constructed all simple modules of $\mathcal{D}(H_8)$ under a different basis of H_8.

4. Nichols algebras in ${}^{H_8}_{H_8}\mathcal{YD}$

In this section, we try to determine all the finite-dimensional Nichols algebras generated by Yetter–Drinfel'd modules over H_8. As a byproduct, we calculate Gelfand–Kirillov dimensions for some Nichols algebras.

We begin by studying the Nichols algebras of simple Yetter–Drinfel'd modules.

Proposition 4.1. *Given a simple Yetter–Drinfel'd module M over H_8,* $\dim \mathfrak{B}(M)$ *(*$\operatorname{GKdim} \mathfrak{B}(M)$ *for some cases) is presented in Table 2. Moreover,*

$$(1)\ \ \mathfrak{B}(M\langle b, g\rangle) = \begin{cases} \mathbb{K}[p], & \text{if } (b,g) \in \{(\pm 1, 1), (\pm 1, xy)\}, \\ \mathbb{K}[p]/(p^2) = \bigwedge \mathbb{K}p, & \text{if } (b,g) \in \{(\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\}. \end{cases}$$

(2) *Both braidings of* $M\langle(g_1, g_2)\rangle$ *for* $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ *and* $W^{b_1,-1}$ *for* $b_1 = \pm 1$ *are of Cartan type* A_2*, so their corresponding Nichols algebras are isomorphic to an algebra which is generated by* p_1*,* p_2 *satisfying relations* $p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0$*,* $p_1^2 = p_2^2 = 0$*.*

$M \in {}^{H_8}_{H_8}\mathcal{YD}$	condition	$\dim \mathfrak{B}(M)$	$\mathrm{GKdim}\, \mathfrak{B}(M)$
$M\langle b, g\rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	1
	$(b, g) \in \{(\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\}$	2	0
$M\langle(1, xy)\rangle$		∞	2
$M\langle(x, y)\rangle$		∞	2
$M\langle(g_1, g_2)\rangle$	$(g_1, g_2) \in \{(1, y), (x, 1)\}$	∞	∞
	$(g_1, g_2) \in \{(xy, x), (y, xy)\}$	8	0
W^{b_1,b_2}	$b_1 = \pm 1,\ b_2 = -1$	8	0
	$b_1 = \pm 1,\ b_2 = 1$	∞	∞
W_1^a, W_2^a	$a + 1 = \pm\sqrt{2}$	∞	

Table 2. Nichols algebras of simple Yetter–Drinfel'd modules over H_8.

Proof. ◇ Because $c(p \otimes p) = g \cdot p \otimes p = \begin{cases} p \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ -p \otimes p, & \text{if } (b, g) \in \{(\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\} \end{cases}$ under the assumption that $M\langle b, g\rangle = \mathbb{K}p$, part (1) is obvious.

◇ As for part (2), we only give a proof for the case $W^{b_1,-1}$ for $b_1 = \pm 1$. Let $p_1 = w_1^{b_1,b_2} + \mathrm{i}b_1 w_2^{b_1,b_2}$ and $p_2 = w_1^{b_1,b_2} - \mathrm{i}b_1 w_2^{b_1,b_2}$; then the braiding of W^{b_1,b_2} is given by

$$c(p_1 \otimes p_1) = b_2 p_1 \otimes p_1, \qquad c(p_2 \otimes p_2) = b_2 p_2 \otimes p_2,$$
$$c(p_1 \otimes p_2) = -b_2 p_2 \otimes p_1, \qquad c(p_2 \otimes p_1) = b_2 p_1 \otimes p_2.$$

When $b_2 = 1$, $\mathrm{GKdim}\, \mathfrak{B}\left(W^{b_1,1}\right) = \infty$ according to [6, Lemma 2.8]. When $b_2 = -1$, the braiding is of type A_2. As discussed in [10], the Nichols algebra $\mathfrak{B}\left(W^{b_1,-1}\right)$ is generated by p_1, p_2 with relations $p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0$, $p_1^2 = p_2^2 = 0$. So $\dim\left(\mathfrak{B}\left(W^{b_1,-1}\right)\right) = 8$.

◇ Let $p_1 = v \boxtimes 1$, $p_2 = v \boxtimes xy \in M\langle(1, xy)\rangle$; then $c(p_j \otimes p_k) = p_k \otimes p_j$, where $j, k = 1, 2$. If we view $M\langle(1, xy)\rangle = \mathbb{K}p_1 \oplus \mathbb{K}p_2$ as braided vector spaces, then $\mathrm{GKdim}\, \mathfrak{B}\left(M\langle(1, xy)\rangle\right) = \mathrm{GKdim}\, \mathfrak{B}(\mathbb{K}p_1) + \mathrm{GKdim}\, \mathfrak{B}(\mathbb{K}p_2) = 2$ by Lemma 2.3. Similarly, $\mathrm{GKdim}\, \mathfrak{B}\left(M\langle(x, y)\rangle\right) = 2$.

◇ Let $p_1 = (v_1 + v_2) \boxtimes 1$, $p_2 = (v_1 - v_2) \boxtimes y \in M\langle(1, y)\rangle$. The braiding is given by

$$c(p_1 \otimes p_1) = p_1 \otimes p_1, \qquad c(p_1 \otimes p_2) = p_2 \otimes p_1,$$
$$c(p_2 \otimes p_1) = -p_1 \otimes p_2, \qquad c(p_2 \otimes p_2) = p_2 \otimes p_2.$$

By [6, Lemma 2.8], $\mathrm{GKdim}\,\mathfrak{B}\left(M\langle(1,y)\rangle\right)=\infty$. For the same reason, we obtain $\mathrm{GKdim}\,\mathfrak{B}\left(M\langle(x,1)\rangle\right)=\infty$.

$\diamond$ Let $\theta=\frac{1}{2}(\mathrm{i}-1)(a+1)$. Then

$$c\left(w_1^{(1)}\otimes w_1^{(1)}\right)=-\theta w_2^{(1)}\otimes w_2^{(1)},\quad c\left(w_1^{(1)}\otimes w_2^{(1)}\right)=\theta w_1^{(1)}\otimes w_2^{(1)},$$
$$c\left(w_2^{(1)}\otimes w_1^{(1)}\right)=-\theta w_2^{(1)}\otimes w_1^{(1)},\quad c\left(w_2^{(1)}\otimes w_2^{(1)}\right)=\theta w_1^{(1)}\otimes w_1^{(1)},$$
$$c\left(\mathrm{i}w_1^{(1)}\otimes w_1^{(1)}+w_2^{(1)}\otimes w_2^{(1)}\right)=-\mathrm{i}\theta\left(\mathrm{i}w_1^{(1)}\otimes w_1^{(1)}+w_2^{(1)}\otimes w_2^{(1)}\right),$$
$$c\left(-\mathrm{i}w_1^{(1)}\otimes w_1^{(1)}+w_2^{(1)}\otimes w_2^{(1)}\right)=\mathrm{i}\theta\left(-\mathrm{i}w_1^{(1)}\otimes w_1^{(1)}+w_2^{(1)}\otimes w_2^{(1)}\right).$$

By induction,

$$\mathfrak{S}_{2n-1,1}\left(\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}\right)=\frac{(1+\theta)[1-(-\theta^2)^n]}{1+\theta^2}\left(\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}\right),$$

$$\mathfrak{S}_{2n,1}\left(\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}\otimes w_1^{(1)}\right)$$
$$=\frac{1-\theta+(-1)^n\theta^{2n+1}(1+\theta)}{1+\theta^2}\left(\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}\otimes w_1^{(1)}\right).$$

It means that $\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}$ is an eigenvector of $\mathfrak{S}_{2n-1}$ and $\left(w_1^{(1)}\otimes w_2^{(1)}\right)^{\otimes n}\otimes w_1^{(1)}$ is an eigenvector of $\mathfrak{S}_{2n}$ both with nonzero eigenvalue. So $\dim\mathfrak{B}\left(W_1^a\right)=\infty$. And $\dim\mathfrak{B}\left(W_2^a\right)=\infty$ is similar to prove. □

Proposition 4.2. (1) $\mathfrak{B}\left[M\langle b,g\rangle\oplus M\left\langle b',g'\right\rangle\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(M\left\langle b',g'\right\rangle\right)$ *for* (b,g), $(b',g')\in\{(\pm1,1),\ (\pm1,xy),\ (\pm\mathrm{i},x),\ (\pm\mathrm{i},y)\}$.

(2) *When* $(b,g)\in\{(\pm1,1),(\pm1,xy)\}$ *the following holds:*

$$\mathfrak{B}\left[M\langle b,g\rangle\oplus M\langle(1,xy)\rangle\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(M\langle(1,xy)\rangle\right),$$
$$\mathfrak{B}\left[M\langle b,g\rangle\oplus M\langle(x,y)\rangle\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(M\langle(x,y)\rangle\right).$$

(3) $\mathfrak{B}\left[M\langle b,g\rangle\oplus M\langle(g_1,g_2)\rangle\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(M\langle(g_1,g_2)\rangle\right)$ *for the following cases:*
   (a) $(b,g)=(\pm\mathrm{i},x)$, $(g_1,g_2)=(xy,x)$;
   (b) $(b,g)=(\pm\mathrm{i},y)$, $(g_1,g_2)=(y,xy)$;
   (c) $(b,g)=(\pm1,1)$, $(g_1,g_2)\in\{(xy,x),(y,xy)\}$.

(4) $\mathfrak{B}\left[M\langle b,g\rangle\oplus W^{b_1,-1}\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(W^{b_1,-1}\right)$ *for the following cases:*
   (a) $(b,g)\in\{(1,1),(1,xy)\}$, $b_1=\pm1$;
   (b) $(b,g)\in\{(\mathrm{i},x),(\mathrm{i},y)\}$, $b_1=1$;
   (c) $(b,g)\in\{(-\mathrm{i},x),(-\mathrm{i},y)\}$, $b_1=-1$.

(5) $\mathfrak{B}\left[M\langle(xy,x)\rangle\oplus M\langle(y,xy)\rangle\right]\simeq\mathfrak{B}\left(M\langle(xy,x)\rangle\right)\otimes\mathfrak{B}\left(M\langle(y,xy)\rangle\right)$.

(6) $\mathfrak{B}\left(W^{1,-1}\oplus W^{-1,-1}\right)\simeq\mathfrak{B}\left(W^{1,-1}\right)\otimes\mathfrak{B}\left(W^{-1,-1}\right)$.

(7) $\mathrm{GKdim}\,\mathfrak{B}\left[M\langle b,g\rangle\oplus M\langle(g_1,g_2)\rangle\right]=\infty$ *for* $(b,g)=(\pm\mathrm{i},x)$, $(g_1,g_2)=(y,xy)$ *or* $(b,g)=(\pm\mathrm{i},y)$, $(g_1,g_2)=(xy,x)$.

(8) $\operatorname{GKdim}\mathfrak{B}\left[M\langle b,g\rangle\oplus W^{b_1,-1}\right]=\infty$ *for* $(b,g)\in\{(\mathrm{i},x),(\mathrm{i},y)\}$, $b_1=-1$ *or* $(b,g)\in\{(-\mathrm{i},x),(-\mathrm{i},y)\}$, $b_1=1$.

(9) $\dim\mathfrak{B}\left((M\langle(g_1,g_2)\rangle)^{\oplus 2}\right)=\infty$ *for* $(g_1,g_2)\in\{(xy,x),(y,xy)\}$.

(10) $\dim\mathfrak{B}\left(W^{b_1,-1}\oplus W^{b_1,-1}\right)=\infty$ *for* $b_1=\pm1$.

Proof. Parts (1)–(6) are direct results of Lemma 2.3. We only prove some cases as a byproduct in the following.

◇ Let $p_1=(v_1+v_2)\boxtimes g_1$, $p_2=(v_1-v_2)\boxtimes g_2\in M\langle(g_1,g_2)\rangle$, where $(g_1,g_2)\in\{(xy,x),(y,xy)\}$. Let $p=v\boxtimes g\in M\langle b,g\rangle$. Then

$$c(p\otimes p_1)=\begin{cases}-p_1\otimes p, & \text{if } g\in\{y,xy\}\\ \ \ p_1\otimes p, & \text{if } g\in\{1,x\},\end{cases}\qquad c(p\otimes p_2)=\begin{cases}-p_2\otimes p, & \text{if } g\in\{x,xy\}\\ \ \ p_2\otimes p, & \text{if } g\in\{1,y\},\end{cases}$$

$$c(p_1\otimes p)=\begin{cases}b^2p\otimes p_1, & \text{if } g_1=y\\ \ \ p\otimes p_1, & \text{if } g_1=xy,\end{cases}\qquad c(p_2\otimes p)=\begin{cases}b^2p\otimes p_2, & \text{if } g_2=x\\ \ \ p\otimes p_2, & \text{if } g_2=xy.\end{cases}$$

∘ When $(g_1,g_2)=(y,xy)$ and $(b,g)=(\pm\mathrm{i},x)$,

$$c(p\otimes p_1)=p_1\otimes p,\qquad c(p\otimes p_2)=-p_2\otimes p,$$
$$c(p_1\otimes p)=-p\otimes p_1,\qquad c(p_2\otimes p)=p\otimes p_2.$$

The generalized Dynkin diagram is given by Figure 1. According to [31], $\dim\mathfrak{B}[M\langle\pm\mathrm{i},x\rangle\oplus M\langle(y,xy)\rangle]=\infty$.

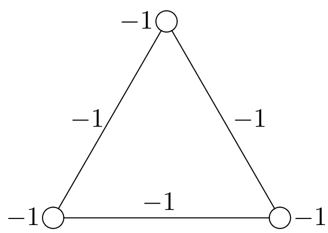

FIGURE 1

∘ When $(g_1,g_2)=(xy,x)$ and $(b,g)=(\pm\mathrm{i},y)$, the generalized Dynkin diagram associated to the braiding is given by Figure 1. According to [31], $\dim\mathfrak{B}[M\langle\pm\mathrm{i},y\rangle\oplus M\langle(xy,x)\rangle]=\infty$. We thus finish part (7).

∘ As for cases listed in part (6), $\mathfrak{B}\left[M\langle b,g\rangle\oplus M\langle(g_1,g_2)\rangle\right]\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(M\langle(g_1,g_2)\rangle\right)$ by Lemma 2.3.

◇ Let $p=v\boxtimes g\in M\langle b,g\rangle$, where $(b,g)\in\{(\pm1,1),(\pm1,xy),(\pm\mathrm{i},x),(\pm\mathrm{i},y)\}$. Then

$$c(p\otimes w_1^{b_1,-1})=\begin{cases}w_1^{b_1,-1}\otimes p, & \text{if } (b,g)\in\{(\pm1,1),(\pm1,xy)\}\\ \mathrm{i}b_1w_2^{b_1,-1}\otimes p, & \text{if } (b,g)\in\{(\pm\mathrm{i},x),(\pm\mathrm{i},y)\},\end{cases}$$

$$c(p\otimes w_2^{b_1,-1})=\begin{cases}w_2^{b_1,-1}\otimes p, & \text{if } (b,g)\in\{(\pm1,1),(\pm1,xy)\}\\ -\mathrm{i}b_1w_1^{b_1,-1}\otimes p, & \text{if } (b,g)\in\{(\pm\mathrm{i},x),(\pm\mathrm{i},y)\},\end{cases}$$

$$c(w_1^{b_1,-1}\otimes p)=\begin{cases} bp\otimes w_1^{b_1,-1}, & \text{if } (b,g)\in\{(\pm 1,1),(\pm 1,xy)\}\\ bp\otimes w_2^{b_1,-1}, & \text{if } (b,g)\in\{(\pm \mathrm{i},x),(\pm \mathrm{i},y)\},\end{cases}$$

$$c(w_2^{b_1,-1}\otimes p)=\begin{cases} bp\otimes w_2^{b_1,-1}, & \text{if } (b,g)\in\{(\pm 1,1),(\pm 1,xy)\}\\ -bp\otimes w_1^{b_1,-1}, & \text{if } (b,g)\in\{(\pm \mathrm{i},x),(\pm \mathrm{i},y)\}.\end{cases}$$

- In case $(b,g)\in\{(1,1),(1,xy)\}$, according to Lemma 2.3 we have
$$\mathfrak{B}\left(M\langle b,g\rangle\oplus W^{b_1,-1}\right)\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(W^{b_1,-1}\right).$$
- In case $(b,g)\in\{(\pm\mathrm{i},x),(\pm\mathrm{i},y)\}$, if $\mathrm{i}b_1b=-1$, according to Lemma 2.3 we have $\mathfrak{B}\left(M\langle b,g\rangle\oplus W^{b_1,-1}\right)\simeq\mathfrak{B}\left(M\langle b,g\rangle\right)\otimes\mathfrak{B}\left(W^{b_1,-1}\right)$. If $\mathrm{i}b_1b=1$, the generalized Dynkin diagram associated to the braiding of $M\langle b,g\rangle\oplus W^{b_1,-1}$ is given by Figure 1. Now we finish parts (4) and (8).

◇ As for $(g_1,g_2)\in\{(xy,x),(y,xy)\}$, $\dim\mathfrak{B}\left((M\langle(g_1,g_2)\rangle)^{\oplus 2}\right)=\infty$ by [31], since the generalized Dynkin diagram associated to the braiding is given by Figure 2.

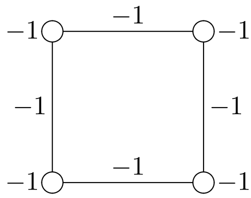

FIGURE 2

◇ As for $W^{b_1,-1}\oplus W^{b_1',-1}$ with b_1 and b_1' in $\{\pm1\}$. Let $p_1=w_1^{b_1,-1}+\mathrm{i}b_1'w_2^{b_1,-1}$, $p_2=w_1^{b_1,-1}-\mathrm{i}b_1'w_2^{b_1,-1}$, $p_1'=w_1^{b_1',-1}+\mathrm{i}b_1'w_2^{b_1',-1}$ and $p_2'=w_1^{b_1',-1}-\mathrm{i}b_1'w_2^{b_1',-1}$. Then

$$c(p_1\otimes p_1')=-p_1'\otimes p_1,\qquad c(p_2\otimes p_2')=-p_2'\otimes p_2,$$
$$c(p_1\otimes p_2')=p_2'\otimes p_1,\qquad c(p_2\otimes p_1')=-p_1'\otimes p_2.$$

When $b_1=b_1'$, the generalized Dynkin diagram associated to the braiding is given by Figure 2. By [31], $\dim\mathfrak{B}\left(W^{b_1,-1}\oplus W^{b_1,-1}\right)=\infty$. This finishes (10).

When $b_1=-b_1'$, we have $p_2=w_1^{b_1,-1}+\mathrm{i}b_1w_2^{b_1,-1}$, $p_1=w_1^{b_1,-1}-\mathrm{i}b_1w_2^{b_1,-1}$, $p_2'=w_1^{b_1',-1}+\mathrm{i}b_1w_2^{b_1',-1}$, $p_1'=w_1^{b_1',-1}-\mathrm{i}b_1w_2^{b_1',-1}$, and

$$c(p_2'\otimes p_2)=-p_2\otimes p_2',\qquad c(p_1'\otimes p_1)=-p_1\otimes p_1',$$
$$c(p_2'\otimes p_1)=p_1\otimes p_2',\qquad c(p_1'\otimes p_2)=-p_2\otimes p_1'.$$

By Lemma 2.3, we have
$$\mathfrak{B}\left(W^{b_1,-1}\oplus W^{-b_1,-1}\right)\simeq\mathfrak{B}\left(W^{b_1,-1}\right)\otimes\mathfrak{B}\left(W^{-b_1,-1}\right).$$

This finishes (6). □

Proposition 4.3. *The following equalities hold for* $b_1 = \pm 1$*:*

$$\dim \mathfrak{B}\left(M\langle(xy,x)\rangle \oplus W^{b_1,-1}\right) = \infty = \dim \mathfrak{B}\left(M\langle(y,xy)\rangle \oplus W^{b_1,-1}\right).$$

Proof. We only prove $\dim \mathfrak{B}\left(M\langle(xy,x)\rangle \oplus W^{b_1,-1}\right) = \infty$ because the rest is similar to prove. Let $p_1' = w_1^{b_1,-1} + \mathrm{i}b_1 w_2^{b_1,-1}$ and $p_2' = w_1^{b_1,-1} - \mathrm{i}b_1 w_2^{b_1,-1}$. Then

$$\begin{aligned} c(p_1 \otimes p_1') &= p_1' \otimes p_1, & c(p_1 \otimes p_2') &= p_2' \otimes p_1, \\ c(p_2 \otimes p_1') &= p_1' \otimes p_2, & c(p_2 \otimes p_2') &= -p_2' \otimes p_2, \\ c(p_1' \otimes p_1) &= p_2 \otimes p_2', & c(p_1' \otimes p_2) &= \mathrm{i}b_1 p_1 \otimes p_2', \\ c(p_2' \otimes p_1) &= p_2 \otimes p_1', & c(p_2' \otimes p_2) &= -\mathrm{i}b_1 p_1 \otimes p_1'. \end{aligned}$$

Suppose $\mathfrak{B}\left(M\langle(xy,x)\rangle \oplus W^{b_1,-1}\right)$ is finite-dimensional; then according to [34, Theorem 7.2(3)], $\mathrm{ad}(M(xy,x))\left(W^{b_1,-1}\right) = (\mathrm{id}-c^2)\left(M(xy,x) \otimes W^{b_1,-1}\right)$ is irreducible. Denote

$$\begin{aligned} A &= (\mathrm{id} - c^2)(p_1 \otimes p_1') = p_1 \otimes p_1' - p_2 \otimes p_2'; \\ B &= (\mathrm{id} - c^2)(p_1 \otimes p_2') = p_1 \otimes p_2' - p_2 \otimes p_1'; \\ C &= (\mathrm{id} - c^2)(p_2 \otimes p_1') = p_2 \otimes p_1' - \mathrm{i}b_1 p_1 \otimes p_2'; \\ D &= (\mathrm{id} - c^2)(p_2 \otimes p_2') = p_2 \otimes p_2' + \mathrm{i}b_1 p_1 \otimes p_1'. \end{aligned}$$

If $a_1A + a_2B + a_3C + a_4D = 0$ for parameters $a_j \in \mathbb{K}$, $j = 1, \ldots, 4$, then $a_1A + a_4D = 0$ and $a_2B + a_3C = 0$. Hence $a_1 = a_4 = 0$, and $a_2 = a_3 = 0$. So A, B, C, D are linearly independent. This is a contradiction since $(\mathrm{id} - c^2)\left(M(xy,x) \otimes W^{b_1,-1}\right)$ is irreducible and there aren't any 4-dimensional irreducible Yetter–Drinfel'd modules over H_8. □

Remark 4.4. According to Propositions 4.1, 4.2, and 4.3, we calculate Nichols algebras over direct sum of two simple objects of ${}^{H_8}_{H_8}\mathcal{YD}$ in Table 3.

Proof of Theorem A. Firstly, we recall the fact that for any submodule $M_1 \subset M_2 \in {}^H_H\mathcal{YD}$, $\mathfrak{B}(M_1) \subset \mathfrak{B}(M_2)$. Then $\dim \mathfrak{B}(M_2) = \infty$ if $\dim \mathfrak{B}(M_1) = \infty$. The Nichols algebras $\mathfrak{B}(M)$ associated with M listed in Theorem A are finite-dimensional according to Lemma 2.3 and [31]. In fact, $\Omega_1(n_1, n_2, n_3, n_4)$ is of Cartan type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2+n_3+n_4}$; $\Omega_k(n_1, n_2)$ for $k = 2, 3, 4, 5$ is of Cartan type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$; Ω_k for $k = 6, 7$ is of Cartan type $A_2 \times A_2$. So let $M \in {}^{H_8}_{H_8}\mathcal{YD}$; then $\dim \mathfrak{B}(M) < \infty$ if and only if M is isomorphic to one of the modules in the list of Theorem A according to Table 2, Table 3, Propositions 4.1 & 4.2.

5. Hopf algebras over H_8

In this section, according to the lifting method, we determine the finite-dimensional Hopf algebra H with coradical H_8 such that its infinitesimal braiding is isomorphic to a Yetter–Drinfel'd module M over H_8. We begin by proving that H is generated by elements of degree one in Theorem 5.1. That is, $\mathrm{gr}\, H \simeq \mathfrak{B}(M)\#H_8$.

$M \in {}^{H_8}_{H_8}\mathcal{YD}$	condition	$\dim \mathfrak{B}(M)$	$\mathrm{GKdim}\, \mathfrak{B}(M)$
$M\langle b_1, g_1\rangle \oplus M\langle b_2, g_2\rangle$	(b_1, g_1), $(b_2, g_2) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	2
	$(b_1, g_1) \in \{(\pm 1, 1), (\pm 1, xy)\}$ $(b_2, g_2) \in \{(\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\}$	∞	1
	(b_1, g_1), $(b_2, g_2) \in \{(\pm \mathrm{i}, x), (\pm \mathrm{i}, y)\}$	4	0
$M\langle b, g\rangle \oplus M\langle (1, xy)\rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	3
$M\langle b, g\rangle \oplus M\langle (x, y)\rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	3
$M\langle (g_1, g_2)\rangle \oplus M\langle (g_1', g_2')\rangle$	$(g_1, g_2) = (g_1', g_2') = (xy, x)$ or (y, xy)	∞	
	$(g_1, g_2) = (xy, x)$, $(g_1', g_2') = (y, xy)$	64	0
$W^{b_1,-1} \oplus W^{b_1',-1}$	$b_1 = b_1' = \pm 1$	∞	
	$b_1 = 1$, $b_1' = -1$	64	0
$M\langle (g_1, g_2)\rangle \oplus W^{b_1,-1}$	$(g_1, g_2) = (xy, x)$ or (y, xy), $b_1 = \pm 1$	∞	
$M\langle b, g\rangle \oplus M\langle (g_1, g_2)\rangle$	$(b, g) = (\pm \mathrm{i}, x)$, $(g_1, g_2) = (xy, x)$	16	0
	$(b, g) = (\pm \mathrm{i}, x)$, $(g_1, g_2) = (y, xy)$	∞	
	$(b, g) = (\pm \mathrm{i}, y)$, $(g_1, g_2) = (xy, x)$	∞	
	$(b, g) = (\pm \mathrm{i}, y)$, $(g_1, g_2) = (y, xy)$	16	0
	$(b, g) \in \{(\pm 1, 1)\}$ $(g_1, g_2) \in \{(xy, x), (y, xy)\}$	∞	1
$M\langle b, g\rangle \oplus W^{b_1,-1}$	$(b, g) \in \{(1, 1), (1, xy)\}$, $b_1 = \pm 1$	∞	1
	$(b, g) \in \{(\mathrm{i}, x), (\mathrm{i}, y)\}$, $b_1 = 1$	16	0
	$(b, g) \in \{(\mathrm{i}, x), (\mathrm{i}, y)\}$, $b_1 = -1$	∞	
	$(b, g) \in \{(-\mathrm{i}, x), (-\mathrm{i}, y)\}$, $b_1 = -1$	16	0
	$(b, g) \in \{(-\mathrm{i}, x), (-\mathrm{i}, y)\}$, $b_1 = 1$	∞	

Table 3. Nichols algebras over the direct sum of two simple objects in ${}^{H_8}_{H_8}\mathcal{YD}$.

Theorem 5.1. *Let H be a finite-dimensional Hopf algebra over H_8 such that its infinitesimal braiding is isomorphic to a Yetter–Drinfel'd module over H_8. Then the diagram of H is a Nichols algebra, and consequently H is generated by the elements of degree one with respect to the coradical filtration.*

Proof. Since $\operatorname{gr} H \simeq R\#H_8$, with $R = \bigoplus_{n\geq 0} R(n)$ the diagram of H, we need to prove that R is a Nichols algebra. Actually we only need to prove that $R \simeq \mathfrak{B}(M)$ for some M in the list of Theorem A since R is finite-dimensional. Let $\mathcal{J} = \bigoplus_{n\geq 0} R(n)^*$ be the graded dual of R; then $\mathcal{J}$ is a graded Hopf algebra in ${}^{H_8}_{H_8}\mathcal{YD}$ with $\mathcal{J}(0) = \mathbb{K}1$. According to [13, Lemma 5.5], $R(1) = \mathcal{P}(R)$ if and only if $\mathcal{J}$ is generated as an algebra by $\mathcal{J}(1)$, that is, if $\mathcal{J}$ is itself a Nichols algebra.

Considering $\mathfrak{B}(M) \in {}^{H_8}_{H_8}\mathcal{YD}$ for M in the list of Theorem A, since $\mathfrak{B}(M) = T(M)/\mathcal{I}$, in order to show that $\mathcal{P}(\mathcal{J}) = \mathcal{J}(1)$ it is enough to prove that the relations that generate the ideal $\mathcal{I}$ hold in $\mathcal{J}$. This can be done by a case-by-case computation. We perform here three cases, and leave the rest to the reader.

Suppose $M = \Omega_1(n_1, n_2, n_3, n_4)$. A direct computation shows that the elements r in $\mathcal{J}$ representing the quadratic relations are primitive and they satisfy $c(r\otimes r) = r \otimes r$. As $\dim \mathcal{J} < \infty$, it must be $r = 0$ in $\mathcal{J}$ and hence there exists a projective algebra map $\mathfrak{B}(M) \to \mathcal{J}$, which implies that $\mathcal{P}(\mathcal{J}) = \mathcal{J}(1)$.

Suppose $M = \Omega_6$; then M is generated by elements $p_1 = (v_1 + v_2) \boxtimes xy$, $p_2 = (v_1 - v_2) \boxtimes x$, $p'_1 = (v_1 + v_2) \boxtimes y$, $p'_2 = (v_1 - v_2) \boxtimes xy$ and the ideal defining the Nichols algebra is generated by the elements p_1^2, p_2^2, ${p'_1}^2$, ${p'_2}^2$, $p_1p_2p_1p_2 + p_2p_1p_2p_1$, $p'_1p'_2p'_1p'_2 + p'_2p'_1p'_2p'_1$, $p_1p'_1 + p'_1p_1$, $p_1p'_2 + p'_2p_1$, $p_2p'_1 - p'_1p_2$, $p_2p'_2 + p'_2p_2$. We can check directly that all those generators of the defining ideal of $\mathfrak{B}(M)$ are primitive elements, or by using [21, Theorem 6]. It is enough to show that $c(r \otimes r) = r\otimes r$ for all generators given above for the defining ideal. Since $\rho(p_1) = xy \otimes p_1$, $\rho(p_2) = x \otimes p_2$, $\rho(p'_1) = y \otimes p'_1$, we have $\rho(p_1^2) = 1 \otimes p_1^2$, $\rho(p_1p_2p_1p_2 + p_2p_1p_2p_1) = 1\otimes (p_1p_2p_1p_2 + p_2p_1p_2p_1)$, $\rho(p_1p'_1 + p'_1p_1) = x\otimes (p_1p'_1 + p'_1p_1)$. It is easy to see that $c(r \otimes r) = r \otimes r$ holds for $r = p_1^2$, $p_1p_2p_1p_2 + p_2p_1p_2p_1$, and $p_1p'_1 + p'_1p_1$. We leave the rest to the reader.

Suppose $M = \Omega_4(n_1, n_2)$. Then M is generated by elements $p_1 = w_1^{1,-1} + \mathrm{i}w_2^{1,-1}$, $p_2 = w_1^{1,-1} - \mathrm{i}w_2^{1,-1}$, $\{X_j\}_{j=1,\dots,n_1}$, $\{Y_k\}_{k=1,\dots,n_2}$ with $\mathbb{K}X_j \simeq M\langle \mathrm{i}, x\rangle$, $\mathbb{K}Y_k \simeq M\langle \mathrm{i}, y\rangle$ and the ideal defining the Nichols algebra is generated by the elements p_1^2, p_2^2, $p_1p_2p_1p_2 + p_2p_1p_2p_1$, X_j^2, $\{X_{j_1}X_{j_2} + X_{j_2}X_{j_1}\}_{1\leq j_1<j_2\leq n_1}$, Y_k^2, $\{Y_{k_1}Y_{k_2} + Y_{k_2}Y_{k_1}\}_{1\leq k_1<k_2\leq n_2}$, $p_1Y_k - Y_kp_1$, $p_2Y_k + Y_kp_2$, $p_1X_j - X_jp_1$, $p_2X_j + X_jp_2$. We can check directly that all those generators of the defining ideal of $\mathfrak{B}(M)$ are primitive elements, or by using [21, Theorem 6]. It is enough to show that $c(r\otimes r) = r\otimes r$ for all generators given above for the defining ideal. Since $\rho(p_1) = (f_{00} - \mathrm{i}f_{11})z \otimes p_1 + (f_{10} + \mathrm{i}f_{01})z \otimes p_2$, $\rho(p_2) = (f_{00} + \mathrm{i}f_{11})z \otimes p_2 + (f_{10} - \mathrm{i}f_{01})z \otimes p_1$, $\rho(X_j) = x \otimes X_j$,

$$\begin{aligned}\rho(p_1p_2p_1p_2 + p_2p_1p_2p_1) = {} & [(f_{00} - \mathrm{i}f_{11})z(f_{00} + \mathrm{i}f_{11})z]^2 \otimes p_1p_2p_1p_2 \\ & + [(f_{00} + \mathrm{i}f_{11})z(f_{00} - \mathrm{i}f_{11})z]^2 \otimes p_2p_1p_2p_1 \\ & + [(f_{10} + \mathrm{i}f_{01})z(f_{10} - \mathrm{i}f_{01})z]^2 \otimes p_2p_1p_2p_1 \\ & + [(f_{10} - \mathrm{i}f_{01})z(f_{10} + \mathrm{i}f_{01})z]^2 \otimes p_1p_2p_1p_2\end{aligned}$$

$$= xy \otimes (p_1p_2p_1p_2 + p_2p_1p_2p_1),$$

$$\rho(p_1X_j - X_jp_1) = (f_{00} + \mathrm{i}f_{11})z \otimes (p_1X_j - X_jp_1) + (f_{10} - \mathrm{i}f_{01})z \otimes (p_2X_j + X_jp_2).$$

Because

$$\begin{aligned}(f_{10} - \mathrm{i}f_{01})z \cdot (p_1X_j - X_jp_1)\\ &= \frac{f_{10} - \mathrm{i}f_{01}}{2} \cdot [((1+y)z \cdot p_1)(z \cdot X_j) - ((1-y)z \cdot X_j)(xz \cdot p_1)]\\ &= (-\mathrm{i})(f_{10} - \mathrm{i}f_{01}) \cdot (p_1X_j - X_jp_1) = 0,\end{aligned}$$

$$(f_{00} + \mathrm{i}f_{11})z \cdot (p_1X_j - X_jp_1) = (-\mathrm{i})(f_{00} + \mathrm{i}f_{11}) \cdot (p_1X_j - X_jp_1) = p_1X_j - X_jp_1,$$

$$xy \cdot (p_1p_2p_1p_2 + p_2p_1p_2p_1) = p_1p_2p_1p_2 + p_2p_1p_2p_1,$$

$c(r \otimes r) = r \otimes r$ holds for $r = p_1p_2p_1p_2 + p_2p_1p_2p_1$ and $p_1X_j - X_jp_1$. We leave the rest to the reader. □

Lemma 5.2 ([13, Lemma 6.1]). *Let H be a Hopf algebra, $\psi : H \to H$ an automorphism of Hopf algebras, V, W Yetter–Drinfel'd modules over H.*

(1) *Let V^ψ be the same space underlying V but with action and coaction*

$$h \cdot_\psi v = \psi(h) \cdot v, \quad \rho^\psi(v) = \left(\psi^{-1} \otimes \mathrm{id}\right) \rho(v), \quad h \in H, v \in V.$$

Then V^ψ is also a Yetter–Drinfel'd module over H. If $T : V \to W$ is a morphism in ${}^H_H\mathcal{YD}$, then $T^\psi : V^\psi \to W^\psi$ also is. Moreover, the braiding $c : V^\psi \otimes W^\psi \to W^\psi \otimes V^\psi$ coincides with the braiding $c : V \otimes W \to W \otimes V$.

(2) *If R is an algebra (resp., a coalgebra, a Hopf algebra) in ${}^H_H\mathcal{YD}$, then R^ψ also is, with the same structural maps.*

(3) *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. Then the map $\Psi : R^\psi \# H \to R\# H$ given by $\Psi(r\#h) = r\#\psi(h)$ is an isomorphism of Hopf algebras.*

Corollary 5.3. (1) $[M\langle b\mathrm{i}, x\rangle]^{\tau_3} \simeq M\langle -b\mathrm{i}, y\rangle$, $b = \pm 1$.

(2) $[M\langle (xy, x)\rangle]^{\tau_3} \simeq M\langle (y, xy)\rangle$,
$\left(W^{b_1,-1}\right)^{\tau_3} \simeq W^{-b_1,-1}$ *with* $b_1 = \pm 1$.

(3) $\mathfrak{B}\left(\Omega_2(n_1, n_2)\right) \# H_8 \simeq \mathfrak{B}\left(\Omega_3(n_2, n_1)\right) \# H_8$,
$\mathfrak{B}\left(\Omega_4(n_1, n_2)\right) \# H_8 \simeq \mathfrak{B}\left(\Omega_5(n_2, n_1)\right) \# H_8$.

Let H be a lifting of $\mathfrak{B}\left(\Omega_1(n_1, n_2, n_3, n_4)\right) \# H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T\left(\Omega_1(n_1, n_2, n_3, n_4)\right) \# H_8 \to H$ [16, Proposition 2.4]. Denote

$$\begin{aligned} X_j &= (v \boxtimes x)\#1, \quad v \boxtimes x \in M\langle \mathrm{i}, x\rangle, \quad j = 1, \dots, n_1,\\ Y_k &= (v \boxtimes x)\#1, \quad v \boxtimes x \in M\langle -\mathrm{i}, x\rangle, \quad k = 1, \dots, n_2,\\ p_s &= (v \boxtimes y)\#1, \quad v \boxtimes y \in M\langle \mathrm{i}, y\rangle, \quad s = 1, \dots, n_3,\\ q_t &= (v \boxtimes y)\#1, \quad v \boxtimes y \in M\langle -\mathrm{i}, y\rangle, \quad t = 1, \dots, n_4. \end{aligned} \tag{5.1}$$

Definition 5.4. For $n_1, n_2, n_3, n_4 \in \mathbb{N}^{\geq 0}$ with $n_1 + n_2 + n_3 + n_4 \geq 1$, and $I_1 = \{(\lambda_{j,s})_{n_1 \times n_3}, (\theta_{k,t})_{n_2 \times n_4}\}$ with entries in $\mathbb{K}$, we denote by $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ the algebra $\left[T\left(\Omega_1(n_1, n_2, n_3, n_4)\right) \# H_8\right] / \mathcal{I}(I_1)$, where $\mathcal{I}(I_1)$ is the ideal generated by

$$X_j^2 = 0, \quad Y_k^2 = 0, \quad p_s^2 = 0, \quad q_t^2 = 0, \tag{5.2}$$

$$X_{j_1}X_{j_2} + X_{j_2}X_{j_1} = 0, \quad j_1, j_2 \in \{1, \dots, n_1\}, \tag{5.3}$$
$$Y_{k_1}Y_{k_2} + Y_{k_2}Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \dots, n_2\}, \tag{5.4}$$
$$p_{s_1}p_{s_2} + p_{s_2}p_{s_1} = 0, \quad s_1, s_2 \in \{1, \dots, n_3\}, \tag{5.5}$$
$$q_{t_1}q_{t_2} + q_{t_2}q_{t_1} = 0, \quad t_1, t_2 \in \{1, \dots, n_4\}, \tag{5.6}$$
$$X_jY_k + Y_kX_j = 0, \quad p_sq_t + q_tp_s = 0, \tag{5.7}$$
$$X_jq_t + q_tX_j = 0, \quad Y_kp_s + p_sY_k = 0, \tag{5.8}$$
$$Y_kq_t + q_tY_k = \theta_{k,t}(1 - xy), \quad X_jp_s + p_sX_j = \lambda_{j,s}(1 - xy). \tag{5.9}$$

Remark 5.5. It is easy to see that $\mathcal{I}(I_1)$ is a Hopf ideal, hence $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ is a Hopf algebra. In particular, when parameters in I_1 are all equal to zero, then $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1) \simeq \mathfrak{B}(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8$.

Proposition 5.6. (1) *Let H be a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_1(n_1, n_2, n_3, n_4)$. Then $H \simeq \mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$.*

(2) *$\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1) \simeq \mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1')$ iff there exist invertible matrices $(\alpha_{jj'})_{n_1\times n_1}$, $(\beta_{kk'})_{n_2\times n_2}$, $(\gamma_{ss'})_{n_3\times n_3}$, $(\eta_{tt'})_{n_4\times n_4}$ such that*

$$\begin{gathered}\sum_{j'=1}^{n_1}\sum_{k'=1}^{n_2} \alpha_{jj'}\beta_{kk'} = 0, \quad \sum_{j'=1}^{n_1}\sum_{t'=1}^{n_4} \alpha_{jj'}\eta_{tt'} = 0, \quad \sum_{j'=1}^{n_1}\sum_{s'=1}^{n_3} \alpha_{jj'}\gamma_{ss'}\lambda'_{j,s} = \lambda_{j,s},\\ \sum_{k'=1}^{n_2}\sum_{s'=1}^{n_3} \beta_{kk'}\gamma_{ss'} = 0, \quad \sum_{s'=1}^{n_3}\sum_{t'=1}^{n_4} \gamma_{ss'}\eta_{tt'} = 0, \quad \sum_{k'=1}^{n_2}\sum_{t'=1}^{n_4} \beta_{kk'}\eta_{tt'}\theta'_{k,t} = \theta_{k,t};\end{gathered} \tag{5.10}$$

or $n_1 = n_4$, $n_2 = n_3$ and there exist invertible matrices $(\alpha'_{jj'})_{n_1\times n_1}$, $(\beta'_{ks'})_{n_2\times n_2}$, $(\gamma'_{sk'})_{n_2\times n_2}$, $(\eta'_{tj'})_{n_1\times n_1}$ such that

$$\begin{gathered}\sum_{t'=1}^{n_1}\sum_{s'=1}^{n_2} \alpha'_{jt'}\beta'_{ks'} = 0, \quad \sum_{k'=1}^{n_2}\sum_{j'=1}^{n_1} \gamma'_{sk'}\eta'_{tj'} = 0, \quad \sum_{t'=1}^{n_1}\sum_{k'=1}^{n_2} \alpha'_{jt'}\gamma'_{sk'}\lambda'_{j,s} = \lambda_{j,s},\\ \sum_{t'=1}^{n_1}\sum_{j'=1}^{n_1} \alpha'_{jt'}\eta'_{tj'} = 0, \quad \sum_{s'=1}^{n_2}\sum_{k'=1}^{n_2} \beta'_{ks'}\gamma'_{sk'} = 0, \quad \sum_{s'=1}^{n_2}\sum_{j'=1}^{n_1} \beta'_{ks'}\eta'_{tj'}\theta'_{k,t} = \theta_{k,t}.\end{gathered} \tag{5.11}$$

Proof. (1) According to (5.1), the extra relations of generators in $T(\Omega_1(n_1, n_2, n_3, n_4))\# H_8$ besides H_8 are given by

$$\begin{aligned} xX_j &= -X_jx, & yX_j &= -X_jy, & zX_j &= \mathrm{i}X_jxz,\\ xY_k &= -Y_kx, & yY_k &= -Y_ky, & zY_k &= -\mathrm{i}Y_kxz,\\ xp_s &= -p_sx, & yp_s &= -p_sy, & zp_s &= \mathrm{i}p_sxz,\\ xq_t &= -q_tx, & yq_t &= -q_ty, & zq_t &= -\mathrm{i}q_txz, \end{aligned}$$

and their coproducts are given by

$$\begin{aligned} \Delta(X_j) &= X_j \otimes 1 + x \otimes X_j, & \Delta(Y_k) &= Y_k \otimes 1 + x \otimes Y_k,\\ \Delta(p_s) &= p_s \otimes 1 + y \otimes p_s, & \Delta(q_t) &= q_t \otimes 1 + y \otimes q_t. \end{aligned}$$

Let $p=(v\boxtimes g)\#1\in[M\langle b,g\rangle]\#1$, $p'=(v'\boxtimes g')\#1\in[M\langle b',g'\rangle]\#1$; then $\Delta[\phi(pp'+p'p)]=\phi(pp'+p'p)\otimes 1+gg'\otimes\phi(pp'+p'p)$. So $\phi(pp'+p'p)=0$ (when $gg'=1$) or $\phi(pp'+p'p)=\lambda(1-gg')$ (when $gg'\neq 1$) for some $\lambda\in\mathbb{K}$ related with p and p'. So ϕ keeps relations (5.2)–(5.7) and there exist $(\lambda_{j,s})_{n_1\times n_3}$, $(\mu_{j,t})_{n_1\times n_4}$, $(\zeta_{k,s})_{n_2\times n_3}$, $(\theta_{k,t})_{n_2\times n_4}$ with entries in $\mathbb{K}$ such that ϕ keeps relations (5.9) and $\phi(X_jq_t+q_tX_j)=\mu_{j,t}(1-xy)$, $\phi(Y_kp_s+p_sY_k)=\zeta_{k,s}(1-xy)$. Since $z(1-xy)=(1-xy)z$, so $\phi[z(X_jq_t+q_tX_j)]=\phi[(X_jq_t+q_tX_j)z]$. By direct calculation, we have $\phi[z(X_jq_t+q_tX_j)]=-\phi[(X_jq_t+q_tX_j)z]$, so $\mu_{j,t}=0$. Similarly, $\zeta_{k,s}=0$. Now we have $\mathcal{I}(I_1)\subseteq\ker\phi$. So there is a surjective Hopf algebra map from $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)$ to H. We can observe that any element of $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)$ can be expressed by a linear sum of $\{X_1^{\alpha_1}\cdots X_{n_1}^{\alpha_{n_1}}Y_1^{\beta_1}\cdots Y_{n_2}^{\beta_{n_2}}p_1^{\gamma_1}\cdots p_{n_3}^{\gamma_{n_3}}q_1^{\kappa_1}\cdots q_{n_4}^{\kappa_{n_4}}x^cy^dz^e\}$ for all parameters $\alpha_1,\dots,\alpha_{n_1}$, $\beta_1,\dots,\beta_{n_2}$, $\gamma_1,\dots,\gamma_{n_3}$, $\kappa_1,\dots,\kappa_{n_4}$, c, d, e in $\{0,1\}$. In fact, the set is a basis of $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)$ according to the Diamond Lemma [17]. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. That is, the ambiguities can be reduced to the same expression by different substitution rules. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation. By Theorem 5.1, we have $\operatorname{gr} H\simeq\mathfrak{B}(\Omega_1(n_1,n_2,n_3,n_4))\#H_8$. That is to say that $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)$ has the same dimension as H. So $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)$ is a lifting for all n_1, n_2, n_3, n_4.

(2) Suppose that $\Phi:\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1)\to\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1')$ is an isomorphism of Hopf algebras, where $I_1'=\{(\lambda'_{j,s})_{n_1\times n_3},(\theta'_{k,t})_{n_2\times n_4}\}$ and $\mathfrak{A}_1(n_1,n_2,n_3,n_4;I_1')$ is generated by x, y, z, X'_j, Y'_k, p'_s, q'_t.

When $\Phi|_{H_8}=\mathrm{id}$ or τ_1, $\Phi(X_j)$ is $(x,1)$-skew primitive, so $\Phi(X_j)\in\oplus_{j'=1}^{n_1}\mathbb{K}X'_{j'}\oplus_{k'=1}^{n_2}\mathbb{K}Y'_{k'}\oplus\mathbb{K}(1-x)$. $xX_j=-X_jx$ implies that $\Phi(X_j)$ doesn't contain the term of $1-x$. And $zX_j=\mathrm{i}X_jxz$, $zY'_{k'}=-\mathrm{i}Y'_{k'}xz$ implies that $\Phi(X_j)$ doesn't contain the terms of $\oplus_{k'=1}^{n_2}\mathbb{K}Y'_{k'}$. So there exists an invertible matrix $(\alpha_{jj'})_{n_1\times n_1}$ such that $\Phi(X_j)=\sum_{j'=1}^{n_1}\alpha_{jj'}X'_{j'}$. Similarly, $\Phi(Y_k)=\sum_{k'=1}^{n_2}\beta_{kk'}Y'_{k'}$, $\Phi(p_s)=\sum_{s'=1}^{n_3}\gamma_{ss'}p'_{s'}$, $\Phi(q_t)=\sum_{t'=1}^{n_4}\eta_{tt'}q'_{t'}$ with $(\beta_{kk'})_{n_2\times n_2}$, $(\gamma_{ss'})_{n_3\times n_3}$, $(\eta_{tt'})_{n_4\times n_4}$ invertible. In this case, Φ is an isomorphism of Hopf algebras if and only if the relations (5.10) hold.

When $\Phi|_{H_8}=\tau_3$ or τ_4, $\Phi(X_j)$ is $(y,1)$-skew primitive, so $\Phi(X_j)\in\oplus_{s'=1}^{n_3}\mathbb{K}p'_{s'}\oplus_{t'=1}^{n_4}\mathbb{K}q'_{t'}\oplus\mathbb{K}(1-y)$. Since $\Phi(xX_j)=-\Phi(X_jx)$, $\Phi(X_j)$ doesn't contain the term of $1-y$. And $\Phi(X_j)$ doesn't contain the terms of $\oplus_{s'=1}^{n_3}\mathbb{K}p'_{s'}$ because of $\Phi(zX_j)=\mathrm{i}\Phi(X_jxz)$. So $n_1=n_4$, and there exists an invertible matrix $(\alpha'_{jt'})_{n_1\times n_1}$ such that $\Phi(X_j)=\sum_{t'=1}^{n_1}\alpha'_{jt'}q'_{t'}$. Similarly, we have $n_2=n_3$ and $\Phi(Y_k)=\sum_{s'=1}^{n_2}\beta'_{ks'}p'_{s'}$, $\Phi(p_s)=\sum_{k'=1}^{n_2}\gamma'_{sk'}Y'_{k'}$, $\Phi(q_t)=\sum_{j'=1}^{n_1}\eta'_{tj'}X'_{j'}$ with $(\beta'_{ks'})_{n_2\times n_2}$, $(\gamma'_{sk'})_{n_2\times n_2}$, $(\eta'_{tj'})_{n_1\times n_1}$ invertible. In this case, Φ is an isomorphism of Hopf algebras if and only if the relations (5.11) hold. □

Lemma 5.7. *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle(y,xy)\rangle$. Then $H\simeq\mathfrak{B}[M\langle(y,xy)\rangle]\#H_8$.*

Proof. Let H be a lifting of $\mathfrak{B}\left(\Omega_2(n_1,n_2)\right)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi:T\left(M\langle(y,xy)\rangle\right)\#H_8\to H$ [16, Proposition 2.4]. Denote $p_1=$

$[(v_1+v_2)\boxtimes y]\#1 \in M\langle(y,xy)\rangle]\#1$ and $p_2=[(v_1-v_2)\boxtimes xy]\#1 \in M\langle(y,xy)\rangle]\#1$; then $\mathfrak{B}[M\langle(y,xy)\rangle]$ is generated by p_1, p_2 satisfying the relations

$$p_1^2=0, \quad p_2^2=0, \quad p_1p_2p_1p_2+p_2p_1p_2p_1=0.$$

It is easy to see that p_1^2, p_2^2, and $p_1p_2p_1p_2+p_2p_1p_2p_1$ are primitive, so those elements are in $\ker\phi$ since H is finite-dimensional. Hence there is a surjective map from $\mathfrak{B}[M\langle(y,xy)\rangle]\#H_8$ to H. By Theorem 5.1, $\operatorname{gr} H \simeq \mathfrak{B}[M\langle(y,xy)\rangle]\#H_8$, so $H\simeq \operatorname{gr} H$. □

Lemma 5.8. *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle(xy,x)\rangle$. Then $H\simeq\mathfrak{B}[M\langle(xy,x)\rangle]\#H_8$.*

Remark 5.9. Let $p_1=[(v_1+v_2)\boxtimes xy]\#1$, $p_2=[(v_1-v_2)\boxtimes x]\#1$ be a basis of $M\langle(xy,x)\rangle]\#1$. It is easy to see that p_1^2, p_2^2, and $p_1p_2p_1p_2+p_2p_1p_2p_1$ are primitive. The proof of the lemma is similar to that of Lemma 5.7. In fact, $\mathfrak{B}[M\langle(xy,x)\rangle]\#H_8$ is isomorphic to $\mathfrak{B}[M\langle(y,xy)\rangle]\#H_8$ by Corollary 5.3.

Proposition 5.10. *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_2(n_1,n_2)$. Then $H\simeq\mathfrak{B}[\Omega_2(n_1,n_2)]\#H_8$.*

Proof. Let H be a lifting of $\mathfrak{B}\left(\Omega_2(n_1,n_2)\right)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi: T\left(\Omega_2(n_1,n_2)\right)\#H_8\to H$ [16, Proposition 2.4]. Denote

$$\begin{gathered}p_1=[(v_1+v_2)\boxtimes xy]\#1, \quad p_2=[(v_1-v_2)\boxtimes x]\#1, \qquad v_1,v_2\in V_2,\\ X_j=(v\boxtimes x)\#1, \quad v\in V_1(\mathrm{i}), \qquad j=1,\dots,n_1,\\ Y_k=(v'\boxtimes x)\#1, \quad v'\in V_1(-\mathrm{i}), \qquad k=1,\dots,n_2.\end{gathered}$$

Let $\mathcal{I}$ be the ideal of relations of $\mathfrak{B}\left[\Omega_2(n_1,n_2)\right]\#H_8$; then $\mathcal{I}$ is generated by the relations

$$\begin{gathered}p_1^2=0, \quad p_2^2=0, \quad p_1p_2p_1p_2+p_2p_1p_2p_1=0,\\ X_{j_1}X_{j_2}+X_{j_2}X_{j_1}=0, \quad j_1,j_2\in\{1,\dots,n_1\},\\ Y_{k_1}Y_{k_2}+Y_{k_2}Y_{k_1}=0, \quad k_1,k_2\in\{1,\dots,n_2\},\\ X_j^2=0, \quad Y_k^2=0, \quad X_jY_k+Y_kX_j=0,\\ p_2X_j+X_jp_2=0, \quad p_2Y_k+Y_kp_2=0,\\ p_1X_j-X_jp_1=0, \quad p_1Y_k-Y_kp_1=0.\end{gathered}$$

We have the following formulae by direct calculation:

$$\begin{aligned}x(p_1X_j-X_jp_1)=-(p_1X_j-X_jp_1)x, \quad x(p_1Y_k-Y_kp_1)&=-(p_1Y_k-Y_kp_1),\\ \Delta[\phi(p_1X_j-X_jp_1)]&=\phi(p_1X_j-X_jp_1)\otimes 1+y\otimes\phi(p_1X_j-X_jp_1),\\ \Delta[\phi(p_1Y_k-Y_kp_1)]&=\phi(p_1Y_k-Y_kp_1)\otimes 1+y\otimes\phi(p_1Y_k-Y_kp_1),\\ \Delta[\phi(p_2X_j+X_jp_2)]&=\phi(p_2X_j+X_jp_2)\otimes 1+1\otimes\phi(p_2X_j+X_jp_2),\\ \Delta[\phi(p_2Y_k+Y_kp_2)]&=\phi(p_2Y_k+Y_kp_2)\otimes 1+1\otimes\phi(p_2Y_k+Y_kp_2).\end{aligned}$$

Together with Lemma 5.8 and Proposition 5.6, we can see $\mathcal{I} \subseteq \ker\phi$, so there is a surjective map from $\mathfrak{B}[\Omega_2(n_1,n_2)]\#H_8$ to H. By Theorem 5.1, $\operatorname{gr} H \simeq \mathfrak{B}[\Omega_2(n_1,n_2)]\#H_8$, so $H \simeq \operatorname{gr} H$. □

Definition 5.11. For $\lambda \in \mathbb{K}$, denote $\mathfrak{A}_6(\lambda)$ by the algebra $[T(\Omega_6)\#H_8]/\mathcal{I}(\lambda)$, where $\mathcal{I}(\lambda)$ is the ideal generated by the relations

$$p_1^2 = 0, \quad p_2^2 = 0, \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0, \tag{5.12}$$

$$q_1^2 = 0, \quad q_2^2 = 0, \quad q_1q_2q_1q_2 + q_2q_1q_2q_1 = 0, \tag{5.13}$$

$$p_1q_1 + q_1p_1 = \lambda(1-x), \quad p_2q_2 + q_2p_2 = \lambda(1-y),$$

$$p_1q_2 - q_2p_1 = 0, \quad p_2q_1 + q_1p_2 = 0.$$

Remark 5.12. In fact, $\mathcal{I}(\lambda)$ is a Hopf ideal, so $\mathfrak{A}_6(\lambda)$ is a Hopf algebra. In particular, when $\lambda = 0$, $\mathfrak{A}_6(0) \simeq \mathfrak{B}(\Omega_6)\#H_8$.

Proposition 5.13. (1) *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to Ω_6. Then $H \simeq \mathfrak{A}_6(\lambda)$.*

(2) *$\mathfrak{A}_6(\lambda) \simeq \mathfrak{A}_6(1)$ for $\lambda \neq 0$, and $\mathfrak{A}_6(1) \not\simeq \mathfrak{A}_6(0)$.*

Proof. (1) Let H be a lifting of $\mathfrak{B}(\Omega_6)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T(\Omega_6)\#H_8 \to H$. Denote $p_1 = [(v_1+v_2)\boxtimes y]\#1$, $p_2 = [(v_1-v_2)\boxtimes xy]\#1$, $q_1 = [(v_1+v_2)\boxtimes xy]\#1$, $q_2 = [(v_1-v_2)\boxtimes x]\#1$ in $[M\langle(y,xy)\rangle \oplus M\langle(xy,x)\rangle]\#1$. By Lemmas 5.7 and 5.8, the map ϕ keeps relations (5.12) and (5.13).

Since $\operatorname{gr} H \simeq \mathfrak{B}(\Omega_6)\#H_8$ and

$$\Delta[\phi(p_1q_1 + q_1p_1)] = \phi(p_1q_1 + q_1p_1) \otimes 1 + x \otimes \phi(p_1q_1 + q_1p_1),$$

$$\Delta[\phi(p_2q_2 + q_2p_2)] = \phi(p_2q_2 + q_2p_2) \otimes 1 + y \otimes \phi(p_2q_2 + q_2p_2),$$

$$\Delta[\phi(p_1q_2 - q_2p_1)] = \phi(p_1q_2 - q_2p_1) \otimes 1 + xy \otimes \phi(p_1q_2 - q_2p_1),$$

$$\Delta[\phi(p_2q_1 + q_1p_2)] = \phi(p_2q_1 + q_1p_2) \otimes 1 + 1 \otimes \phi(p_2q_1 + q_1p_2),$$

we have $\phi(p_1q_1 + q_1p_1) = \lambda_1(1-x)$, $\phi(p_2q_2 + q_2p_2) = \lambda_2(1-y)$, $\phi(p_1q_2 - q_2p_1) = \lambda_3(1-xy)$, $\phi(p_2q_1 + q_1p_2) = 0$ for some λ_1, λ_2 and λ_3 in $\mathbb{K}$. Since $z(p_1q_1 + q_1p_1) = (p_2q_2 + q_2p_2)z$ and $z(p_1q_2 - q_2p_1) = (p_2q_1 + q_1p_2)xz$, we have $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$. That is to say $\mathcal{I}(\lambda) \subseteq \ker\phi$. So there is a surjective map from $\mathfrak{A}_6(\lambda)$ to H. Now we only need to prove that $\dim \mathfrak{A}_6(\lambda) = \dim H$. In fact, $\mathfrak{A}_6(\lambda) \simeq \mathfrak{B}(\Omega_6) \otimes H_8$ as vector space by the Diamond Lemma. It suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation.

(2) When $\lambda \neq 0$, $\Phi : \mathfrak{A}_6(\lambda) \simeq \mathfrak{A}_6(1)$, by $\Phi|_{H_8} = \mathrm{id}$, $p_i \mapsto \frac{p_i}{\sqrt{\lambda}}$, $q_i \mapsto \frac{q_i}{\sqrt{\lambda}}$ for $i = 1, 2$. $\mathfrak{A}_6(1) \not\simeq \mathfrak{A}_6(0)$ can be proved similarly to the proof of the second part of Proposition 5.17. □

Lemma 5.14. *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $W^{b_1,-1}$, where $b_1 = \pm 1$. Then*

$$H \simeq \left[T\left(W^{b_1,-1}\right)\#H_8\right] \big/ \mathcal{I}(\lambda_1,\lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{K},$$

where $\mathcal{I}(\lambda_1,\lambda_2)$ *is a Hopf ideal generated by the relations*

$$p_1^2=\lambda_1(1-xy),\quad p_2^2=\mathrm{i}b_1\lambda_1(1-xy),\quad p_1p_2p_1p_2+p_2p_1p_2p_1=\lambda_2(1-xy). \tag{5.14}$$

Proof. Let H be a lifting of $\mathfrak{B}\left(W^{b_1,-1}\right)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi: T\left(W^{b_1,-1}\right)\#H_8\to H$. $p_1=\left(w_1^{b_1,-1}+\mathrm{i}b_1w_2^{b_1,-1}\right)\#1$, $p_2=\left(w_1^{b_1,-1}-\mathrm{i}b_1w_2^{b_1,-1}\right)\#1$,

$$\Delta(p_1)=[f_{00}-\mathrm{i}b_1f_{11}]\,z\otimes p_1+[f_{10}+\mathrm{i}b_1f_{01}]\,z\otimes p_2+p_1\otimes 1, \tag{5.15}$$
$$\Delta(p_2)=[f_{00}+\mathrm{i}b_1f_{11}]\,z\otimes p_2+[f_{10}-\mathrm{i}b_1f_{01}]\,z\otimes p_1+p_2\otimes 1. \tag{5.16}$$

By a straightforward computation, we have

$$\Delta[\phi(p_1^2)]=\frac{1}{2}(1+xy)\otimes\phi(p_1^2)+\frac{\mathrm{i}b_1}{2}(1-xy)\otimes\phi(p_2^2)+\phi(p_1^2)\otimes 1,$$
$$\Delta[\phi(p_2^2)]=\frac{1}{2}(1+xy)\otimes\phi(p_2^2)-\frac{\mathrm{i}b_1}{2}(1-xy)\otimes\phi(p_1^2)+\phi(p_2^2)\otimes 1.$$

So there exists a parameter $\lambda_1\in\mathbb{K}$ such that $\phi(p_1^2)=\lambda_1(1-xy)$ and $\phi(p_2^2)=\mathrm{i}b_1\lambda_1(1-xy)$.

$$\begin{aligned}\Delta[\phi(p_1p_2)]=&\frac{1}{2}(x+y)\otimes\phi(p_1p_2)+\frac{\mathrm{i}b_1}{2}(x-y)\otimes\phi(p_2p_1)+\phi(p_1p_2)\otimes 1\\&+\phi(p_2)\,[f_{00}-\mathrm{i}b_1f_{11}]\,z\otimes\phi(p_1)+\phi(p_2)\,[f_{10}+\mathrm{i}b_1f_{01}]\,z\otimes\phi(p_2)\\&+\phi(p_1)\,[f_{00}+\mathrm{i}b_1f_{11}]\,z\otimes\phi(p_2)+\phi(p_1)\,[f_{10}-\mathrm{i}b_1f_{01}]\,z\otimes\phi(p_1),\end{aligned}$$

$$\begin{aligned}\Delta[\phi(p_2p_1)]=&\frac{1}{2}(x+y)\otimes\phi(p_2p_1)+\frac{\mathrm{i}b_1}{2}(y-x)\otimes\phi(p_1p_2)+\phi(p_2p_1)\otimes 1\\&+[f_{00}+\mathrm{i}b_1f_{11}]\,z\phi(p_1)\otimes\phi(p_2)+[f_{10}-\mathrm{i}b_1f_{01}]\,z\phi(p_1)\otimes\phi(p_1)\\&+\phi(p_2)\,[f_{00}-\mathrm{i}b_1f_{11}]\,z\otimes\phi(p_1)+\phi(p_2)\,[f_{10}+\mathrm{i}b_1f_{01}]\,z\otimes\phi(p_2).\end{aligned}$$

Denote $\Delta[\phi(p_1p_2)]=B-A+E_1$ and $\Delta[\phi(p_2p_1)]=B+A+E_2$, where

$$\begin{aligned}A&=[f_{00}+\mathrm{i}b_1f_{11}]\,z\phi(p_1)\otimes\phi(p_2)+[f_{10}-\mathrm{i}b_1f_{01}]\,z\phi(p_1)\otimes\phi(p_1),\\B&=\phi(p_2)\,[f_{00}-\mathrm{i}b_1f_{11}]\,z\otimes\phi(p_1)+\phi(p_2)\,[f_{10}+\mathrm{i}b_1f_{01}]\,z\otimes\phi(p_2),\\E_2&=\frac{1}{2}(x+y)\otimes\phi(p_2p_1)+\frac{\mathrm{i}b_1}{2}(y-x)\otimes\phi(p_1p_2)+\phi(p_2p_1)\otimes 1,\\E_1&=\frac{1}{2}(x+y)\otimes\phi(p_1p_2)+\frac{\mathrm{i}b_1}{2}(x-y)\otimes\phi(p_2p_1)+\phi(p_1p_2)\otimes 1.\end{aligned}$$

We can obtain $A^2+B^2=0$, since

$$\begin{aligned}A^2&=-\frac{1}{2}(x+y)\phi(p_1^2)\otimes\phi(p_2^2)+\frac{\mathrm{i}b_1}{2}(1-xy)\phi(p_1^2)\otimes\phi(p_1^2)\\&=\mathrm{i}b_1\lambda_1^2(1-xy)\otimes(1-xy),\end{aligned}$$

$$\begin{aligned}B^2&=-\frac{1}{2}(x+y)\phi(p_2^2)\otimes\phi(p_1^2)+\frac{\mathrm{i}b_1}{2}(1-xy)\phi(p_2^2)\otimes\phi(p_2^2)\\&=-\mathrm{i}b_1\lambda_1^2(1-xy)\otimes(1-xy).\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&p_1(p_1p_2+p_2p_1)=(p_2p_1+p_1p_2)p_1, \quad p_2(p_1p_2+p_2p_1)=(p_2p_1+p_1p_2)p_2,\\
&p_1(p_1p_2-p_2p_1)=(p_2p_1-p_1p_2)p_1, \quad p_2(p_1p_2-p_2p_1)=(p_2p_1-p_1p_2)p_2,\\
&\quad (x+y)p_2(f_{00}-\mathrm{i}b_1f_{11})z=-p_2(f_{00}-\mathrm{i}b_1f_{11})z(x+y),\\
&\quad (x-y)p_2(f_{00}-\mathrm{i}b_1f_{11})z=p_2(f_{00}-\mathrm{i}b_1f_{11})z(x-y),\\
&\quad (x+y)p_2(f_{10}+\mathrm{i}b_1f_{01})z=-p_2(f_{10}+\mathrm{i}b_1f_{01})z(x+y),\\
&\quad (x-y)p_2(f_{10}+\mathrm{i}b_1f_{01})z=p_2(f_{10}+\mathrm{i}b_1f_{01})z(x-y),\\
&\quad (p_1p_2+p_2p_1)p_2(f_{00}-\mathrm{i}b_1f_{11})z=-p_2(f_{00}-\mathrm{i}b_1f_{11})z(p_1p_2+p_2p_1),\\
&\quad (p_1p_2+p_2p_1)p_2(f_{10}+\mathrm{i}b_1f_{01})z=-p_2(f_{10}+\mathrm{i}b_1f_{01})z(p_1p_2+p_2p_1),
\end{aligned}$$

we deduce $B(E_1+E_2)+(E_1+E_2)B=0$. Similarly, we have $A(E_2-E_1)+(E_2-E_1)A=0$.

$$\begin{aligned}
\Delta[\phi(p_1p_2p_1p_2+p_2p_1p_2p_1)] &= (B-A+E_1)^2+(B+A+E_2)^2\\
&= 2(A^2+B^2)+B(E_1+E_2)+(E_1+E_2)B\\
&\quad +A(E_2-E_1)+(E_2-E_1)A+E_1^2+E_2^2\\
&= E_1^2+E_2^2\\
&= \left(\frac{1}{2}(x+y)\otimes\phi(p_1p_2)+\frac{\mathrm{i}b_1}{2}(x-y)\otimes\phi(p_2p_1)+\phi(p_1p_2)\otimes 1\right)^2\\
&\quad +\left(\frac{1}{2}(x+y)\otimes\phi(p_2p_1)+\frac{\mathrm{i}b_1}{2}(y-x)\otimes\phi(p_1p_2)+\phi(p_2p_1)\otimes 1\right)^2\\
&= \frac{1}{2}(1+xy)\otimes[\phi(p_1p_2)]^2-\frac{1}{2}(1-xy)\otimes[\phi(p_2p_1)]^2+[\phi(p_1p_2)]^2\otimes 1\\
&\quad +\frac{1}{2}(1+xy)\otimes[\phi(p_2p_1)]^2-\frac{1}{2}(1-xy)\otimes[\phi(p_1p_2)]^2+[\phi(p_2p_1)]^2\otimes 1\\
&= xy\otimes\phi\left[(p_1p_2)^2+(p_2p_1)^2\right]+\phi\left[(p_1p_2)^2+(p_2p_1)^2\right]\otimes 1.
\end{aligned}$$

So there exists a parameter $\lambda_2\in\mathbb{K}$ such that $\phi(p_1p_2p_1p_2+p_2p_1p_2p_1)=\lambda_2(1-xy)$. Hence the map ϕ keeps relations (5.14), so there exists a surjective map from $\left[T(W^{b_1,-1})\#H_8\right]\big/\mathcal{I}(\lambda_1,\lambda_2)$ to H. Now we only need to prove that

$$\dim\left[T(W^{b_1,-1})\#H_8\right]\big/\mathcal{I}(\lambda_1,\lambda_2)=\dim H.$$

In fact, $\left[T(W^{b_1,-1})\#H_8\right]\big/\mathcal{I}(\lambda_1,\lambda_2)\simeq\mathfrak{B}(W^{b_1,-1})\otimes H_8$ as vector space by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation. □

Definition 5.15. Let $I_7=(\lambda_1,\dots,\lambda_5)$ with $\lambda_i\in\mathbb{K}$ $(i=1,\dots,5)$. Denote by $\mathfrak{A}_7(I_7)$ the algebra $\left[T(\Omega_7)\#H_8\right]\big/\mathcal{I}(I_7)$, where $\mathcal{I}(I_7)$ is the ideal generated by the relations

$$p_1^2 = \lambda_1(1-xy), \quad p_2^2 = \mathrm{i}\lambda_1(1-xy), \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = \lambda_2(1-xy), \tag{5.17}$$

$$q_1^2 = \lambda_3(1-xy), \quad q_2^2 = -\mathrm{i}\lambda_3(1-xy), \quad q_1q_2q_1q_2 + q_2q_1q_2q_1 = \lambda_4(1-xy), \tag{5.18}$$

$$p_1q_2 + q_2p_1 = 0, \quad p_2q_1 + q_1p_2 = 0, \tag{5.19}$$

$$p_1q_1 + q_1p_1 = \lambda_5(x+y-2), \quad p_2q_2 - q_2p_2 = -\mathrm{i}\lambda_5(x-y). \tag{5.20}$$

Remark 5.16. In fact, $\mathcal{I}(I_7)$ is a Hopf ideal, so $\mathfrak{A}_7(I_7)$ is a Hopf algebra. In particular, when $\lambda_i = 0$ for $i = 1, 2, \ldots, 5$, $\mathfrak{A}_7(I_7) \simeq \mathfrak{B}(\Omega_7)\#H_8 \simeq [\mathfrak{B}(W^{1,-1}) \otimes \mathfrak{B}(W^{-1,-1})]\#H_8$.

Proposition 5.17. (1) *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to Ω_7. Then $H \simeq \mathfrak{A}_7(I_7)$.*

(2) *$\mathfrak{A}_7(\lambda_1, \ldots, \lambda_5) \simeq \mathfrak{A}_7(\lambda_1', \ldots, \lambda_5')$ iff there exist nonzero parameters a, b, α, β in $\mathbb{K}$ such that*

$$\begin{aligned} &a^2\lambda_1' = \lambda_1, \quad b^2\lambda_1' = \lambda_1, \quad a^2b^2\lambda_2' = \lambda_2, \quad a\alpha\lambda_5' = \lambda_5, \\ &\alpha^2\lambda_3' = \lambda_3, \quad \beta^2\lambda_3' = \lambda_3, \quad \alpha^2\beta^2\lambda_4' = \lambda_4, \quad b\beta\lambda_5' = \lambda_5; \end{aligned} \tag{5.21}$$

or there exist nonzero parameters α_1, β_1 in $\mathbb{K}$ such that

$$\alpha_1^2\lambda_3' = \lambda_1, \quad \beta_1^2\lambda_1' = \lambda_3, \quad -\alpha_1^4\lambda_4' = \lambda_2, \quad -\beta_1^4\lambda_2' = \lambda_4, \quad \alpha_1\beta_1\lambda_5' = \lambda_5. \tag{5.22}$$

Proof. (1) Let H be a lifting of $\mathfrak{B}(\Omega_7)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T(\Omega_7)\#H_8 \to H$. Denote $p_1 = \left(w_1^{1,-1} + \mathrm{i}w_2^{1,-1}\right)\#1$, $p_2 = \left(w_1^{1,-1} - \mathrm{i}w_2^{1,-1}\right)\#1$, $q_1 = \left(w_1^{-1,-1} - \mathrm{i}w_2^{-1,-1}\right)\#1$, $q_2 = \left(w_1^{-1,-1} + \mathrm{i}w_2^{-1,-1}\right)\#1$. The coproducts of p_1, p_2, q_1, q_2 are given by (5.15) and (5.16). The relations of generators in $T(\Omega_7)\#H_8$ are given by

$$\begin{aligned} &xp_1 = p_1x, \quad yp_1 = p_1y, \quad xp_2 = -p_2x, \quad yp_2 = -p_2y, \\ &xq_1 = q_1x, \quad yq_1 = q_1y, \quad xq_2 = -q_2x, \quad yq_2 = -q_2y, \\ &zp_1 = -p_1z, \quad zp_2 = \mathrm{i}p_2xz, \quad zq_1 = -q_1z, \quad zq_2 = -\mathrm{i}q_2xz. \end{aligned}$$

By Lemma 5.14, the map ϕ keeps relations (5.17) and (5.18). It is only possible for $\phi(p_1q_2+q_2p_1) = 0$, $\phi(p_2q_1+q_1p_2) = 0$, since $x(p_1q_2+q_2p_1) = -(p_1q_2+q_2p_1)x$, $x(p_2q_1+q_1p_2) = -(p_2q_1+q_1p_2)x$, and

$$\Delta[\phi(p_1q_2+q_2p_1)] = \frac{1}{2}\left[(1+xy)+\mathrm{i}(1-xy)\right]\otimes\phi(p_1q_2+q_2p_1) + \phi(p_1q_2+q_2p_1)\otimes 1,$$

$$\Delta[\phi(p_2q_1+q_1p_2)] = \frac{1}{2}\left[(1+xy)-\mathrm{i}(1-xy)\right]\otimes\phi(p_2q_1+q_1p_2) + \phi(p_2q_1+q_1p_2)\otimes 1.$$

Similarly, the map ϕ keeps relation (5.20), since

$$z(p_1q_1+q_1p_1) = (p_1q_1+q_1p_1)z, \quad z(p_2q_2-q_2p_2) = -(p_2q_2-q_2p_2)z,$$

$$\Delta[\phi(p_1q_1+q_1p_1)] = \frac{x+y}{2}\otimes\phi(p_1q_1+q_1p_1)+\phi(p_1q_1+q_1p_1)\otimes 1 + \frac{\mathrm{i}(x-y)}{2}\otimes\phi(p_2q_2-q_2p_2),$$

$$\Delta[\phi(p_2q_2-q_2p_2)] = \frac{x+y}{2}\otimes\phi(p_2q_2-q_2p_2)+\phi(p_2q_2-q_2p_2)\otimes 1 - \frac{\mathrm{i}(x-y)}{2}\otimes\phi(p_1q_1+q_1p_1).$$

That is to say $\mathcal{I}(I_7)\subseteq\ker\phi$, so there exists a surjective map from $\mathfrak{A}_7(I_7)$ to H. By Theorem 5.1, $\operatorname{gr} H\simeq\mathfrak{B}(\Omega_7)\#H_8$. Now we only need to prove that $\dim\mathfrak{A}_7(I_7)=\dim H$. In fact, $\mathfrak{A}_7(I_7)\simeq\mathfrak{B}(W^{1,-1})\otimes\mathfrak{B}(W^{-1,-1})\otimes H_8$ as vector spaces by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation.

(2) Let $A=\mathfrak{A}_7(I_7)$, and $\Phi\in\operatorname{Aut}_{\mathrm{Hopf}}A$; then $\Phi|_{H_8}$ is given by Table 1. Denote $A_0'=\mathbb{K}1\oplus\mathbb{K}x\oplus\mathbb{K}y\oplus\mathbb{K}xy$, $A_0''=\mathbb{K}z\oplus\mathbb{K}xz\oplus\mathbb{K}yz\oplus\mathbb{K}xyz$, and $A_1'=p_1H_8\oplus p_2H_8\oplus q_1H_8\oplus q_2H_8$. Then the first two terms of coradical filtration of A are $A_0=H_8$ and $A_1=A_1'\oplus H_8$.

$$\begin{aligned}\Delta\Phi(p_1)&=\Phi\left((f_{00}-\mathrm{i}f_{11})z\right)\otimes\Phi(p_1)+\Phi\left((f_{10}+\mathrm{i}f_{01})z\right)\otimes\Phi(p_2)+\Phi(p_1)\otimes 1,\\&\in A_0''\otimes[\Phi(p_1)+\Phi(p_2)]+\Phi(p_1)\otimes 1;\\\Delta\Phi(p_2)&=\Phi\left((f_{00}+\mathrm{i}f_{11})z\right)\otimes\Phi(p_2)+\Phi\left((f_{10}-\mathrm{i}f_{01})z\right)\otimes\Phi(p_1)+\Phi(p_2)\otimes 1,\\&\in A_0''\otimes[\Phi(p_1)+\Phi(p_2)]+\Phi(p_2)\otimes 1.\end{aligned}$$

Denote $\Phi(p_1)=a_0+a_1+a_2$, $\Phi(p_2)=b_0+b_1+b_2$, where $a_0,\ b_0\in A_0'$, $a_1,\ b_1\in A_0''$, and $a_2,\ b_2\in A_1'$. Then $\Delta(a_0)=a_0\otimes 1$, which implies $a_0\in\mathbb{K}1$, and similarly $b_0\in\mathbb{K}1$. Hence $\Delta(a_1)=\Phi\left((f_{00}-\mathrm{i}f_{11})z\right)\otimes(a_0+a_1)+\Phi\left((f_{10}+\mathrm{i}f_{01})z\right)\otimes(b_0+b_1)+\Phi(a_1)\otimes 1$, which implies

$$a_1=-a_0\Phi\left((f_{00}-\mathrm{i}f_{11})z\right)-b_0\Phi\left((f_{10}+\mathrm{i}f_{01})z\right).$$

And similarly, $b_1=-b_0\Phi\left((f_{00}+\mathrm{i}f_{11})z\right)-a_0\Phi\left((f_{10}-\mathrm{i}f_{01})z\right)$.

$$\begin{aligned}\Phi(xp_1)=\Phi(p_1x)&\Rightarrow\Phi(x)a_1=a_1\Phi(x)\Rightarrow b_0\Phi\left((f_{10}+\mathrm{i}f_{01})z\right)=0\Rightarrow b_0=0,\\\Phi(zp_1)=-\Phi(p_1z)&\Rightarrow\Phi(z)a_1=-a_1\Phi(z)\Rightarrow a_0\Phi\left(f_{00}+\mathrm{i}f_{11}\right)=0\Rightarrow a_0=0.\end{aligned}$$

So $a_1=b_1=0$, hence $\Phi(p_1)=a_2\in A_1'$ and $\Phi(p_2)=b_2\in A_1'$.

Now suppose $\Phi:\mathfrak{A}_7(\lambda_1,\dots,\lambda_5)\to\mathfrak{A}_7(\lambda_1',\dots,\lambda_5')$ is an isomorphism. When $\Phi|_{H_8}=\mathrm{id}$, then by (5.15) and (5.16) $\Phi(p_1)=ap_1'+cq_2'$ for some $a,\ c\in\mathbb{K}$ and $\Phi(p_2)=bp_2'+dq_1'$ for some $b,\ d\in\mathbb{K}$, where p_1', etc. are generators of $\mathfrak{A}_7(\lambda_1',\dots,\lambda_5')$. $\Phi(xp_1)=x(ap_1'+cq_2')=(ap_1'-cq_2')x=\Phi(p_1x)=(ap_1'+cq_2')x$, so $c=0$ and similarly $d=0$. Similarly, we have $\Phi(q_1)=\alpha q_1'$, $\Phi(q_2)=\beta q_2'$ for some nonzero parameters α and β. In this case, Φ is an isomorphism of Hopf algebras if and only if the relations (5.21) hold.

When $\Phi|_{H_8}=\tau_2$, suppose $\Phi(p_1)=\alpha_1p_1'+\alpha_2p_2'+\alpha_3q_1'+\alpha_4q_2'$; then $a_2=a_4=0$ since $\Phi(xp_1)=\Phi(p_1x)$. By (5.15) and (5.16), we can obtain $\alpha_3=0$ and $\Phi(p_2)=$

$-\alpha_1 p_2'$. Similarly, $\Phi(q_1) = \beta_1 q_1'$, $\Phi(q_2) = -\beta_1 q_2'$ for some parameter β_1. Φ respects relations of A_1.

According to the defining relations of $\mathcal{I}(I_7)$, we have

$$\alpha_1^2 \lambda_1' = \lambda_1, \quad \alpha_1^4 \lambda_2' = \lambda_2, \quad \beta_1^2 \lambda_3' = \lambda_3, \quad \beta_1^4 \lambda_4' = \lambda_4, \quad \alpha_1 \beta_1 \lambda_5' = \lambda_5.$$

This is just a special case of (5.21).

When $\Phi|_{H_8} = \tau_3$, then $\Phi(p_1) \in \mathbb{K}q_1'$, $\Phi(p_2) \in \mathbb{K}q_2'$, $\Phi(q_1) \in \mathbb{K}p_1'$, $\Phi(q_2) \in \mathbb{K}p_2'$. According to (5.15) and (5.16), $\Phi(p_1) = \alpha_1 q_1'$, $\Phi(p_2) = -\mathrm{i}\alpha_1 q_2'$, $\Phi(q_1) = \beta_1 p_1'$, $\Phi(q_2) = \mathrm{i}\beta_1 p_2'$ for nonzero parameters α_1 and β_1. Φ respects relations of A_1. In this case, Φ is an isomorphism of Hopf algebras if and only if the relations (5.22) hold.

When $\Phi|_{H_8} = \tau_4$, then $\Phi(p_1) = \alpha_1 q_1'$, $\Phi(p_2) = \mathrm{i}\alpha_1 q_2'$, $\Phi(q_1) = \beta_1 p_1'$, $\Phi(q_2) = -\mathrm{i}\beta_1 p_2'$ for nonzero parameters α_1 and β_1. Φ respects relations of A_1. According to the defining relations of $\mathcal{I}(I_7)$, we obtain relations of parameters which exactly coincide with (5.22). □

Definition 5.18. For a set of parameters $I_4 = \{\lambda_1, \lambda_2, (\lambda_{j,k})_{n_1 \times n_2}\}$, denote by $\mathfrak{A}_4(n_1, n_2; I_4)$ the algebra $T[\Omega_4(n_1, n_2)]\#H_8/\mathcal{I}(I_4)$, where $\mathcal{I}(I_4)$ is the ideal generated by the relations

$$p_1^2 = \lambda_1(1-xy), \quad p_2^2 = \mathrm{i}\lambda_1(1-xy), \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = \lambda_2(1-xy), \tag{5.23}$$

$$X_{j_1}^2 = 0, \quad X_{j_1}X_{j_2} + X_{j_2}X_{j_1} = 0, \quad j_1, j_2 \in \{1, \dots, n_1\}, \tag{5.24}$$

$$Y_{k_1}^2 = 0, \quad Y_{k_1}Y_{k_2} + Y_{k_2}Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \dots, n_2\}, \tag{5.25}$$

$$X_jY_k + Y_kX_j = \lambda_{j,k}(1-xy), \tag{5.26}$$

$$p_1Y_k - Y_kp_1 = 0, \quad p_2Y_k + Y_kp_2 = 0, \quad p_1X_j - X_jp_1 = 0, \quad p_2X_j + X_jp_2 = 0. \tag{5.27}$$

Remark 5.19. In fact, $\mathcal{I}(I_4)$ is a Hopf ideal, so $\mathfrak{A}_4(n_1, n_2; I_4)$ is a Hopf algebra. In particular, when all the parameters in I_4 are zero, then $\mathfrak{A}_4(n_1, n_2; I_4) \simeq \mathfrak{B}[\Omega_4(n_1, n_2)]\#H_8$.

Proposition 5.20. (1) *Suppose H is a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_4(n_1, n_2)$. Then $H \simeq \mathfrak{A}_4(n_1, n_2; I_4)$.*

(2) *$\mathfrak{A}_4(n_1, n_2; I_4) \simeq \mathfrak{A}_4(n_1, n_2; I_4')$ iff there exist two invertible matrices $(\alpha_{js})_{n_1 \times n_1}$, $(\beta_{kt})_{n_2 \times n_2}$ and two nonzero parameters a, b, such that*

$$\begin{aligned} &\sum_{s=1}^{n_1} \sum_{t=1}^{n_2} \alpha_{js}\beta_{kt}\lambda_{j,k}' = \lambda_{j,k}, \quad j \in \{1, \dots, n_1\},\ k \in \{1, \dots, n_2\}; \\ &a^2\lambda_1' = \lambda_1, \quad b^2\lambda_1' = \lambda_1, \quad a^2b^2\lambda_2' = \lambda_2. \end{aligned} \tag{5.28}$$

Proof. (1) Let H be a lifting of $\mathfrak{B}\left(\Omega_4(n_1, n_2)\right)\#H_8$. Then there exists an epimorphism of Hopf algebras $\phi : T\left(\Omega_4(n_1, n_2)\right)\#H_8 \to H$. Denote

$$\begin{aligned} &p_1 = \left(w_1^{1,-1} + \mathrm{i}w_2^{1,-1}\right)\#1, \quad p_2 = \left(w_1^{1,-1} - \mathrm{i}w_2^{1,-1}\right)\#1, \\ &X_j = (v \boxtimes x)\#1, \quad Y_k = (v \boxtimes y)\#1, \quad v \in V_1(\mathrm{i}),\ j \in \{1, \dots, n_1\},\ k \in \{1, \dots, n_2\}. \end{aligned}$$

Then the relations of generators in $T[\Omega_4(n_1,n_2)]\#H_8$ are given by

$$\begin{aligned} xp_1 &= p_1x, & yp_1 &= p_1y, & xp_2 &= -p_2x, & yp_2 &= -p_2y,\\ zp_1 &= -p_1z, & xX_j &= -X_jx, & yX_j &= -X_jy, & zX_j &= \mathrm{i}X_jxz,\\ zp_2 &= \mathrm{i}p_2xz, & xY_k &= -Y_kx, & yY_k &= -Y_ky, & zY_k &= \mathrm{i}Y_kxz, \end{aligned}$$

and the coproducts of generators are given by

$$\begin{aligned} \Delta(X_j) &= X_j\otimes 1 + x\otimes X_j, \quad \Delta(Y_k) = Y_k\otimes 1 + y\otimes Y_k,\\ \Delta(p_1) &= (f_{00}-\mathrm{i}f_{11})z\otimes p_1 + (f_{10}+\mathrm{i}f_{01})z\otimes p_2 + p_1\otimes 1,\\ \Delta(p_2) &= (f_{00}+\mathrm{i}f_{11})z\otimes p_2 + (f_{10}-\mathrm{i}f_{01})z\otimes p_1 + p_2\otimes 1. \end{aligned}$$

As similarly proved in Proposition 5.6 and Lemma 5.14, the map ϕ keeps relations (5.23)–(5.26). Since $r=0$ in $\operatorname{gr} H$ for $r=\phi(p_1Y_k-Y_kp_1)$ and $\phi(p_2Y_k+Y_kp_2)$, r is an element of at most degree one. It is only possible for

$$\phi(p_1Y_k-Y_kp_1) = -\mu_k\left(-f_{10}+\mathrm{i}f_{01}\right)z, \quad \phi(p_2Y_k+Y_kp_2) = -\mu_k\left(f_{00}-\mathrm{i}f_{11}\right)z + \mu_k 1,$$

because of the following relations:

$$\begin{aligned} x\left(p_1Y_k-Y_kp_1\right) &= -\left(p_1Y_k-Y_kp_1\right)x, \quad z\left(p_1Y_k-Y_kp_1\right) = -\mathrm{i}\left(p_1Y_k-Y_kp_1\right)xz,\\ x\left(p_2Y_k+Y_kp_2\right) &= \left(p_2Y_k+Y_kp_2\right)x, \quad z\left(p_2Y_k+Y_kp_2\right) = \left(p_2Y_k+Y_kp_2\right)z,\\ \Delta(p_1Y_k-Y_kp_1) &= (p_1Y_k-Y_kp_1)\otimes 1 + (f_{00}+\mathrm{i}f_{11})z\otimes(p_1Y_k-Y_kp_1)\\ &\quad + (-f_{10}+\mathrm{i}f_{01})z\otimes(p_2Y_k+Y_kp_2),\\ \Delta(p_2Y_k+Y_kp_2) &= (p_2Y_k+Y_kp_2)\otimes 1 + (f_{00}-\mathrm{i}f_{11})z\otimes(p_2Y_k+Y_kp_2)\\ &\quad - (f_{10}+\mathrm{i}f_{01})z\otimes(p_1Y_k-Y_kp_1). \end{aligned}$$

Similarly, we get

$$\phi(p_1X_j-X_jp_1) = -\mu'_j\left(f_{10}-\mathrm{i}f_{01}\right)z, \quad \phi(p_2X_j+X_jp_2) = -\mu'_j\left(f_{00}-\mathrm{i}f_{11}\right)z + \mu'_j 1,$$

from the following formulae:

$$\begin{aligned} x\left(p_1X_j-X_jp_1\right) &= -\left(p_1X_j-X_jp_1\right)x, \quad z\left(p_1X_j-X_jp_1\right) = -\mathrm{i}\left(p_1X_j-X_jp_1\right)xz,\\ x\left(p_2X_j+X_jp_2\right) &= \left(p_2X_j+X_jp_2\right)x, \quad z\left(p_2X_j+X_jp_2\right) = \left(p_2X_j+X_jp_2\right)z,\\ \Delta\left(p_1X_j-X_jp_1\right) &= (f_{00}+\mathrm{i}f_{11})z\otimes\left(p_1X_j-X_jp_1\right) + \left(p_1X_j-X_jp_1\right)\otimes 1\\ &\quad + (f_{10}-\mathrm{i}f_{01})z\otimes\left(p_2X_j+X_jp_2\right),\\ \Delta\left(p_2X_j+X_jp_2\right) &= (f_{00}-\mathrm{i}f_{11})z\otimes\left(p_2X_j+X_jp_2\right) + \left(p_2X_j+X_jp_2\right)\otimes 1\\ &\quad + (f_{10}+\mathrm{i}f_{01})z\otimes\left(p_1X_j-X_jp_1\right). \end{aligned}$$

Since $\phi(X_jY_k+Y_kX_j)=\lambda_{j,k}(1-xy)$, $\phi[p_1(X_jY_k+Y_kX_j)]=\phi[(X_jY_k+Y_kX_j)p_1] \Rightarrow \mu'_j=\mu_k=0$. So ϕ keeps relations (5.27). Now we have $\mathcal{I}(I_7)\subseteq\ker\phi$, so there exists a surjective map from $\mathfrak{A}_4(n_1,n_2;I_4)$ to H. By Theorem 5.1, $\operatorname{gr} H\simeq \mathfrak{B}[\Omega_4(n_1,n_2)]\#H_8$. Now we only need to prove that $\dim\mathfrak{A}_4(n_1,n_2;I_4)=\dim H$. In fact, $\mathfrak{A}_4(n_1,n_2;I_4)\simeq\mathfrak{B}(W^{1,-1})\otimes\mathfrak{B}(M\langle\mathrm{i},x\rangle)^{\otimes n_1}\otimes\mathfrak{B}(M\langle\mathrm{i},y\rangle)^{\otimes n_2}\otimes H_8$ as vector spaces by the Diamond Lemma. By the Diamond Lemma, it suffices to show that all overlap ambiguities are resolvable. Calculating the ambiguities and showing that these are resoluble is a tedious but straightforward computation.

(2) Suppose $\Phi : \mathfrak{A}_4(n_1, n_2; I_4) \to \mathfrak{A}_4(n_1, n_2; I'_4)$ is an isomorphism of Hopf algebras. Similarly to the proof of Proposition 5.17, it is easy to see that $\Phi|_{H_8} \in \{\mathrm{id}, \tau_2\}$ and $\Phi(p_1) = ap'_1$, $\Phi(p_2) = bp'_2$. Since X_j is $(x, 1)$-skew primitive and $xX_j = -X_j x$, $\Phi(X_j) \in \oplus_{s=1}^{n_1} \mathbb{K}X'_s$. Similarly, we have $\Phi(Y_k) \in \oplus_{t=1}^{n_2} \mathbb{K}Y'_t$. Let $\Phi(X_j) = \sum_{s=1}^{n_1} \alpha_{js} X'_s$, $\Phi(Y_k) = \sum_{t=1}^{n_2} \beta_{kt} Y'_t$, where the matrices $(\alpha_{js})_{n_1 \times n_1}$ and $(\beta_{kt})_{n_2 \times n_2}$ are invertible. Then Φ is an isomorphism of Hopf algebras if and only if the relations (5.28) hold. □

Proof of Theorem B. Let H be a finite-dimensional Hopf algebra over H_8 such that its infinitesimal braiding $M \in {}^{H_8}_{H_8}\mathcal{YD}$; then M is in the list of Theorem A. We need to give a construction for any finite-dimensional Hopf algebra H over H_8 up to isomorphism such that its infinitesimal braiding is isomorphic to M. By Theorem 5.1, $\operatorname{gr} H \simeq \mathfrak{B}(M)\#H_8$. According to Corollary 5.3, up to isomorphism, $\operatorname{gr} H \simeq \mathfrak{B}(M)\#H_8$ for $M = \Omega_1(n_1, n_2, n_3, n_4)$, $\Omega_2(n_1, n_2)$, $\Omega_4(n_1, n_2)$, Ω_6, Ω_7. Propositions 5.6, 5.10, 5.20, 5.13 and 5.17 finish the proof.

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