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ON SUPERSOLVABLE GROUPS WHOSE MAXIMAL SUBGROUPS OF THE SYLOW SUBGROUPS ARE SUBNORMAL

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Abstract. A finite group G is called an MSN*-group if it is supersolvable, and all maximal subgroups of the Sylow subgroups of G are subnormal in G. A group G is called a minimal non-MSN*-group if every proper subgroup of G is an MSN*-group but G itself is not. In this paper, we obtain a complete classification of minimal non-MSN*-groups.

1. Introduction

Only finite groups are considered in this paper and our notation is standard.

Let $\mathfrak{F}$ be a class of groups. A group G is called a minimal non-$\mathfrak{F}$-group or $\mathfrak{F}$-critical group if all subgroups but G itself belong to $\mathfrak{F}$. The characterization of minimal non-$\mathfrak{F}$-groups plays a critical role in analyzing the structure of groups with certain group theory property. It is important to obtain a detailed knowledge of minimal non-$\mathfrak{F}$-groups so that some deep insights into what makes a group belong to $\mathfrak{F}$ may turn to be achievable. Moreover, when proving that a group belongs to $\mathfrak{F}$, researchers can benefit from such descriptions of the minimal non-$\mathfrak{F}$-groups by induction or minimal counterexample. Many meaningful results on this topic have been obtained, and they have indeed pushed forward the development of group theory. For example, Schmidt [10] determined the structure of minimal non-nilpotent groups, and Doerk [3] determined the structure of minimal non-supersolvable groups. Ballester-Bolinches and Esteban-Romero [1] provided a complete classification of minimal non-supersolvable groups, which is shown to be useful and hence is exploited in solving the problem studied in this paper.

Srinivasan [12] studied groups in which all maximal subgroups of the Sylow subgroups are normal, and proved that such groups are supersolvable. Later, Walls

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[13] called such groups MNP-groups and investigated the structure of the MNP-groups. Recently, Guo et al. [4] determined the complete classification of minimal non-MNP-groups (those groups which are not MNP-groups but whose proper subgroups are all MNP-groups).

On the other hand, Srinivasan [12] also proved that a group G is solvable but not necessarily supersolvable if all maximal subgroups of the Sylow subgroups are subnormal in G. The alternating group A_4 is such an example. Guo et al. [5] called such groups MSN-groups and gave a characterization of minimal non-MSN-groups (defined similarly as minimal non-MNP-groups above). Unfortunately, a complete classification of such groups is still unknown.

Naturally, we consider imposing some weaker conditions on MSN-groups. It is well known that CLT-groups (a group is said to be CLT if it possesses subgroups of every possible order, i.e., it satisfies the converse of Lagrange's theorem) are solvable (see [8]) and that supersolvable groups must be CLT (see [7]). Therefore, we investigate MSN-groups with the CLT-property as well as supersolvable MSN-groups. Specific definitions are as follows.

Definition 1.1. A CLT-group G is called a CMSN-group if all maximal subgroups of the Sylow subgroups of G are subnormal in G.

Definition 1.2. A supersolvable group G is called an MSN*-group if all maximal subgroups of the Sylow subgroups of G are subnormal in G.

It is clear that MSN*-groups must be CMSN-groups, but CMSN-groups need not to be MSN*-groups. For instance, $G = A_4 \times C_2$ is a non-supersolvable CMSN-group, where A_4 is the alternating group and C_2 is cyclic of order 2. It is interesting that they are equivalent under the assumption on minimal non-$\mathfrak{F}$-groups although these two subgroups are different.

A group G is said to be a *minimal non-MSN*-group* (respectively, a *minimal non-CMSN-group*) if every proper subgroup of G is an MSN*-group (respectively, a CMSN-group) but G itself is not. In this paper, the minimal non-MSN*-groups (i.e., minimal non-CMSN-groups) are classified completely.

2. Preliminary results

We collect some definitions and lemmas which will be used in the sequel.

Definition 2.1 ([6, Definition 1.4])**.** Let α be an automorphism of a group G. Then α is a semi-power automorphism of G if there exist elements $a_1, a_2, \ldots, a_n$ which generate G such that α maps a_i into a power of a_i for all $i \in \{1, 2, \ldots, n\}$.

By a result of [15, Theorem 2.7] and its Remark, the following lemma is true.

Lemma 2.1. *A group G is an MSN-group if and only if $G = H \rtimes K$, where H is a nilpotent normal Hall subgroup of G, K is a group whose Sylow subgroups are cyclic and the maximal subgroups of its Sylow subgroups are normal in G.*

Based on Lemma 2.1, the following result follows easily by applying [6, Lemma 2.2].

Lemma 2.2. *A group G is an MSN*-group if and only if $G = H \rtimes K$, where H is a nilpotent normal Hall subgroup of G, K is a group whose Sylow subgroups are cyclic and the maximal subgroups of its Sylow subgroups are normal in G, and every element of K induces a semi-power automorphism of order dividing a prime in $H/\Phi(H)$.*

Lemma 2.3. *Let G be a supersolvable minimal non-MSN*-group. Then $|\pi(G)|$ is 2, where $\pi(G)$ is the set of all primes dividing the order of G.*

Proof. Let $\{P_1, P_2, \ldots, P_s\}$ be a Sylow system of G. By the hypothesis, there exists some P_i $(1 \leqslant i \leqslant s)$ and a maximal subgroup P^* of P_i such that P^* is not subnormal in G. Assume $s \geqslant 3$. If P_i is non-cyclic, then P_iP_j $(j \neq i)$ are MSN*-groups. By Lemma 2.2, P_i is normal in P_iP_j and also normal in G. Thus, P^* is subnormal in G, a contradiction. Hence P_i is cyclic. In this case, by Lemma 2.2 again, P^* is normal in P_iP_j $(j \neq i)$, and so P^* is normal in G, a contradiction. Thus, $|\pi(G)| = 2$. □

Lemma 2.4 ([9, 13.4.3])**.** *Let α be a power automorphism of an abelian group A. If A is a p-group of finite exponent, then there is a positive integer l such that $a^\alpha = a^l$ for all a in A. If α is nontrivial and has order prime to p, then α is fixed-point-free.*

Lemma 2.5 ([5, Lemma 2.9])**.** *If a q-group G of order q^{n+1} has a unique non-cyclic maximal subgroup, then G is isomorphic to one of the following groups:*

(I) $C_{q^n} \times C_q = \langle a, b \mid a^{q^n} = b^q = 1, [a, b] = 1\rangle$, *where* $n \geqslant 2$*;*
(II) $M_{q^{n+1}} = \langle a, b \mid a^{q^n} = b^q = 1, b^{-1}ab = a^{1+q^{n-1}}\rangle$, *where* $n \geqslant 2$ *and* $n \geqslant 3$ *if* $q = 2$.

Lemma 2.6 ([14, Chapter 3, Theorem 1.1])**.** *A group G is supersolvable if and only if all subgroups of G are CLT-groups.*

3. Main results

In this section, we give the complete classification of minimal non-MSN*-groups.

Theorem 3.1. *The minimal non-MSN*-groups are exactly the groups of the following types:*

(I) $G = \langle x, y \mid x^p = y^{q^n} = 1, y^{-1}xy = x^r\rangle$, *where* $r^q \not\equiv 1 \pmod p$, $r^{q^2} \equiv 1 \pmod p$, $q \mid p-1$, $n \geqslant 2$ *with* $0 < r < p$.
(II) $G = \langle x, y \mid x^{pq} = y^q = 1, y^{-1}xy = x^r\rangle$, *where* $p \equiv 1 \pmod q$, $r \equiv 1 \pmod q$, $r^q \equiv 1 \pmod p$ *with* $1 < r < p$.
(III) $G = \langle x, y \mid x^{4p} = 1, y^2 = x^{2p}, y^{-1}xy = x^{-1}\rangle$ *with* $p > 2$.
(IV) $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^r, [x, z] = 1, [y, z] = 1\rangle$, *where* $n \geqslant 3$, $p \equiv 1 \pmod q$, $r \not\equiv 1 \pmod p$, $r^q \equiv 1 \pmod p$ *with* $1 < r < p$.
(V) $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^r, [x, z] = 1, z^{-1}yz = y^{1+q^{n-2}}\rangle$, *where* $n \geqslant 3$ *and* $n \geqslant 4$ *if* $q = 2$, $p \equiv 1 \pmod q$, $r \not\equiv 1 \pmod p$, $r^q \equiv 1 \pmod p$ *with* $1 < r < p$.

(VI) $G = P \rtimes Q$, *where* $Q = \langle y \rangle$ *is cyclic of order* $q^n > 1$, *with* $q \nmid p-1$, *and* P *is an irreducible* Q*-module over the field of* p *elements with kernel* $\langle y^q \rangle$ *in* Q.

(VII) $G = P \rtimes Q$, *where* P *is a non-abelian special* p*-group of rank* $2m$, *the order of* p *modulo* q *being* $2m$, $Q = \langle y \rangle$ *is cyclic of order* $q^n > 1$, y *induces an automorphism in* P *such that* $P/\Phi(P)$ *is a faithful and irreducible* Q*-module, and* y *centralizes* $\Phi(P)$. *Furthermore,* $|P/\Phi(P)| = p^{2m}$ *and* $|P'| \leqslant p^m$.

(VIII) $G = P \rtimes Q$, *where* $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ *is an elementary abelian* p*-group of order* p^q, $Q = \langle y \rangle$ *is cyclic of order* q^n, q *is the highest power of* q *dividing* $p-1$ *and* $n > 1$. *Define* $a_j^y = a_{j+1}$ *for* $0 \leqslant j < q-1$ *and* $a_{q-1}^y = a_0^i$, *where* i *is a primitive* q*-th root of unity modulo* p.

Proof. Assume that G is a minimal non-MSN*-group. Since every proper subgroup of G is an MSN*-group, G is supersolvable or minimal non-supersolvable by the definition. By applying a result of [1, Theorems 9, 10] and Lemma 2.3, $|\pi(G)|$ is 2 or 3, and G has a unique normal Sylow subgroup.

We first consider the case of G with $|\pi(G)| = 2$, and assume $G = PQ$ with $P \trianglelefteq G$ and $Q \ntrianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$, and $Q \in \mathrm{Syl}_q(G)$. Since all the Sylow q-subgroups are conjugate in G, we only consider the case that Q acts on P.

There are four situations, as follows.

(1) Assume that $P = \langle x \rangle$ and $Q = \langle y \rangle$, with $|x| = p^m$, $|y| = q^n$, and $p > q$.

In this case, G is metacyclic, $y^{-1}xy = x^r$ with $r^{q^n} \equiv 1 \pmod{p^m}$, $q \mid p-1$, $0 < r < p^m$, and $(p^m, q^n(r-1)) = 1$. Since $\langle y^q \rangle$ is not subnormal in G, it follows that $(y^q)^{-1}xy^q = x^{r^q} \neq x$. So $r^q \not\equiv 1 \pmod{p^m}$. The subnormality of $\langle y^{q^2} \rangle$ in $\langle x \rangle \langle y^q \rangle$ implies that $\langle y^{q^2} \rangle$ is normal in $\langle x \rangle \langle y^q \rangle$ by Lemma 2.2. So $(y^{q^2})^{-1}xy^{q^2} = x^{r^{q^2}} = x$. Hence $r^{q^2} \equiv 1 \pmod{p^m}$ and y induces a power automorphism of order q^2 on P. Surely, y^q induces a power automorphism of order q on P and every proper subgroup of $\langle y^q \rangle$ is normal in G. If $x^p \neq 1$, then by Lemma 2.4, $\langle x^p \rangle \langle y^q \rangle \neq \langle x^p \rangle \times \langle y^q \rangle$. Lemma 2.2 implies that $\langle y^q \rangle$ is normal in $\langle x^p \rangle \langle y \rangle$. Thus, $\langle x^p \rangle \langle y^q \rangle = \langle x^p \rangle \times \langle y^q \rangle$, a contradiction. So $x^p = 1$ and G is of type (I).

(2) Assume that P is cyclic and Q is non-cyclic.

Clearly, if $p < q$, then $Q \trianglelefteq G$, a contradiction. Hence $p > q$.

If Q has two non-cyclic maximal subgroups Q_1 and Q_2, then by Lemma 2.2, $PQ_1 = P \times Q_1$ and $PQ_2 = P \times Q_2$. Hence $Q = Q_1Q_2$ is normal in G, a contradiction. Therefore, every maximal subgroup of Q is cyclic or Q has a unique non-cyclic maximal subgroup. Thus, Q is the elementary abelian group of order q^2, the quaternion group Q_8 or one of the types of Lemma 2.5.

Case 1. Assume $P = \langle z \rangle$, $Q = \langle a^q = b^q = 1, [a,b] = 1 \rangle$. If $z^p \neq 1$, then $\langle z^p \rangle Q = \langle z^p \rangle \times Q$ by Lemma 2.2. If the actions of a and b on P by conjugation are both trivial, then G is nilpotent, a contradiction. Therefore, we may assume that the action of a on P by conjugation is non-trivial. By applying Lemma 2.4, $\langle z^p \rangle \langle a \rangle \neq \langle z^p \rangle \times \langle a \rangle$, a contradiction. Hence $z^p = 1$. If $C_G(P) = P$, then $G/C_G(P)$ is an elementary abelian group of order q^2. However, $G/C_G(P) \lesssim \mathrm{Aut}(P)$, and $\mathrm{Aut}(P)$ is cyclic, a contradiction. Hence either a or b is contained in $C_G(P)$. Let $C_G(P) = \langle x \rangle$, $y^{-1}xy = x^r$ with $1 < r < p$, where $x = zb$ is a generator of

$C_G(P)$, $|x| = pq$, $y = a$. Then we have that $p \equiv 1 \pmod{q}$, $r \equiv 1 \pmod{q}$ and $r^q \equiv 1 \pmod{p}$. So G is of type (II).

Case 2. Assume $Q = Q_8 = \langle a, b \mid a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1}\rangle$ and $P = \langle z\rangle$. If $z^p \neq 1$, then $\langle z^p\rangle Q = \langle z^p\rangle \times Q$ by Lemma 2.2. Using the same argument as in Case 1, we obtain the same contradiction. Hence $z^p = 1$. Since $P\langle a\rangle$ is an MSN*-group, $\langle a^2\rangle \leqslant C_G(P)$ by Lemma 2.2. If $C_G(P) = P \times \langle a^2\rangle$, then $G/C_G(P)$ is an elementary abelian group of order 4. Again, there is a contradiction as in Case 1. So $C_G(P)$ has an element of order 4 and $C_G(P) = \langle x\rangle$ is a cyclic group of order $4p$. Surely, G has an element y of order 4 such that $y \neq x^p$. Now we let $y^{-1}xy = x^r$ where $r \not\equiv 1 \pmod{4p}$. Since $(y^2)^{-1}xy^2 = x^{r^2} = x$, we have $r^2 \equiv 1 \pmod{4p}$. By computations, $G = \langle x, y \mid x^{4p} = 1,\ y^2 = x^{2p},\ y^{-1}xy = x^{-1}\rangle$. So G is of type (III).

Case 3. Assume that $P = \langle x\rangle$ and Q is of type (I) with $|Q| = q^n$ in Lemma 2.5. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1,\ [y, z] = 1\rangle$, where $n \geqslant 3$. Then Q has maximal subgroups $H = \langle y\rangle$, $K_0 = \langle y^q, z\rangle$ and $K_s = \langle y^q, zy^s\rangle = \langle zy^s\rangle$ with $s = 1, \dots, q-1$, where K_0 is the unique non-cyclic maximal subgroup of Q. Lemma 2.2 implies $PK_0 = P \times K_0$. By the hypothesis, $PH \neq P \times H$ and y induces a power automorphism of order q on P. Surely, $z \in Z(G)$. Further, by similar arguments as in Case 1, we have $x^p = 1$. Hence $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1,\ y^{-1}xy = x^r,\ [x, z] = 1,\ [y, z] = 1\rangle$, where $p \equiv 1 \pmod{q}$, $r \not\equiv 1 \pmod{p}$, $r^q \equiv 1 \pmod{p}$ with $1 < r < p$. So G is of type (IV).

Case 4. Assume that $P = \langle x\rangle$ and Q is of type (II) with $|Q| = q^n$ in Lemma 2.5. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1,\ z^{-1}yz = y^{1+p^{n-2}}\rangle$, where $n \geqslant 3$ and $n \geqslant 4$ if $p = 2$. Similarly as above, y induces a power automorphism of order q on P and $\langle z\rangle \leqslant C_G(P)$. Further, we can prove that $x^p = 1$, $y^{-1}xy = x^r$, where $p \equiv 1 \pmod{q}$, $r \not\equiv 1 \pmod{p}$, $r^q \equiv 1 \pmod{p}$ with $1 < r < p$. So G is of type (V).

(3) Assume that P is non-cyclic and $Q = \langle y\rangle$ is a cyclic subgroup of G with $|y| = q^n$.

If G is supersolvable, we can assume that $1 \trianglelefteq \cdots \trianglelefteq R \trianglelefteq P \trianglelefteq \cdots \trianglelefteq G$ is a chief series of G. By Maschke's theorem [9, 8.1.2], there exists a subgroup H of P such that $P/\Phi(P) = R/\Phi(P) \times H/\Phi(P)$, where $|H/\Phi(P)| = p$ and $H/\Phi(P) \trianglelefteq G/\Phi(P)$. Clearly, $H \trianglelefteq G$, $H \not\leq R$ and $1 \trianglelefteq H \trianglelefteq P \trianglelefteq G$ is a normal series of G. By applying Schreier's refinement theorem [9, 3.1.2], there exists a maximal subgroup K of P such that K is normal in G and $K \neq R$. By the minimality of G, RQ and KQ are both MSN*-groups of G. Lemma 2.2 implies that $\langle y^q\rangle$ is normal in G, so G is an MSN*-group, a contradiction. Therefore, G is minimal non-supersolvable.

Case 1. If G is also a minimal non-nilpotent group and P is abelian, by applying [2, Theorem 3], G is of one of the types (VI)–(VII).

Case 2. If G is not a minimal non-nilpotent group and P is abelian, by applying [1, Theorems 9, 10], we assume that $G = PQ$, where $P = \langle a_0, a_1, \dots, a_{q-1}\rangle$ is an elementary abelian p-group of order p^q, $Q = \langle y\rangle$ is cyclic of order q^n, q^f is the highest power of q dividing $p-1$ and $n > f \geqslant 1$. Define $a_j^y = a_{j+1}$ for $0 \leqslant j < q-1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q^f-th root of unity modulo p.

Considering a maximal group $P\langle y^q\rangle$ of G, by Lemma 2.2, $\langle y^{q^2}\rangle \leqslant C_G(P)$. Hence, $a_0^{i^q} = a_0^{y^{q^2}} = a_0$. Thus $i^q \equiv 1 \pmod p$, i.e., $f = 1$, G is of type (VIII).

Case 3. If G is not a minimal non-nilpotent group and P is non-abelian, by applying [1, Theorems 9, 10], we may assume that $G = PQ$ such that $P = \langle a_0, a_1\rangle$ is an extraspecial group of order p^3 with exponent p, $Q = \langle y\rangle$ is a cyclic group of order 2^n with 2^f the largest power of 2 dividing $p-1$ and $n > f \geqslant 1$, and $a_0^y = a_1$ and $a_1^y = a_0^i x$, where $x \in \langle [a_0, a_1]\rangle$ and i is a primitive 2^f-th root of unity modulo p.

Since $P\langle y^2\rangle$ and $\Phi(P)\langle y\rangle$ are MSN*-groups, we have that $\langle y^4\rangle \leqslant C_G(P)$ and $\langle y^2\rangle \leqslant C_G(\Phi(P))$ by Lemma 2.2. However, $a_0^{y^4} = a_0^{i^2} x^{i+1} \neq a_0$, a contradiction. Therefore, G is not of the type as above.

(4) Assume that P and Q are both non-cyclic.

Using similar arguments as in Situation (3), we easily have that G is minimal non-supersolvable. By the same arguments as in Situation (2), Q is the elementary abelian group of order q^2, the quaternion group Q_8 or one of the two types of Lemma 2.5.

Case 1. Let Q be an elementary abelian group of order q^2. By applying [1, Theorems 9, 10], none of them satisfies a minimal non-MSN*-group.

Case 2. Let $Q = Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$. For maximal subgroups $P\langle a\rangle$ and $P\langle b\rangle$ of G, we have $\Phi(Q) = \langle a^2\rangle \leqslant C_G(P)$ by Lemma 2.2. By examining Type 6 and Type 7 in [1, Theorems 9, 10], none of them satisfies a minimal non-MSN*-group.

Case 3. Let Q be as in Lemma 2.5 (I) with $|Q| = q^n$. Namely, $Q = \langle a, b \mid a^{q^{n-1}} = b^q = 1, [a, b] = 1\rangle$ where $n \geqslant 3$. By similar arguments as Case 2 in Situation (3), none of the types 6–10 in [1, Theorems 9, 10] satisfies a minimal non-MSN*-group.

Case 4. Let Q be as in Lemma 2.5 (II) with $|Q| = q^n$. Namely, $Q = \langle a, b \mid a^{q^{n-1}} = b^q = 1, b^{-1}ab = a^{1+q^{n-2}}\rangle$, where $n \geqslant 3$ and $n \geqslant 4$ if $q = 2$. It is clear that $\Phi(Q) = \langle a^q\rangle \leqslant C_G(P)$. By examining types 6–10 in [1, Theorems 9, 10], none of them satisfies a minimal non-MSN*-group.

We next consider the case of G with $|\pi(G)| = 3$.

Lemma 2.3 implies that G is minimal non-supersolvable. By types 11–12 in [1, Theorems 9, 10], we may assume that $G = PQR$ with $P \trianglelefteq G$, Q is neither cyclic nor normal in G, and $R \ntrianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $R \in \mathrm{Syl}_r(G)$. However, by Lemma 2.2, Q is normal in both PQ and QR, and so Q is normal in $G = \langle P, Q, R\rangle$, a contradiction. Hence none of them satisfies a minimal non-MSN*-group.

Conversely, it is clear that a group satisfying one of the types (I)–(VIII) is a minimal non-MSN*-group. □

Corollary 3.2. *The following statements for a group G are equivalent:*

(1) G is a minimal non-MSN*-group.
(2) G is a minimal non-CMSN-group.
(3) G is exactly of one type of Theorem 3.1.

Proof. From Theorem 3.1, it suffices to prove that a minimal non-CMSN-group is a minimal non-MSN*-group.

Let G be a minimal non-CMSN-group. For each maximal subgroup M of G, all subgroups of M are CLT-groups by the minimality of G. By Lemma 2.6, M is supersolvable, so G is a minimal non-MSN*-group. □

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