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CONFORMAL AND KILLING VECTOR FIELDS ON REAL SUBMANIFOLDS OF THE CANONICAL COMPLEX SPACE FORM $\mathbb{C}^m$

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ABSTRACT. In this paper, we find a conformal vector field as well as a Killing vector field on a compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$. In particular, using immersion $\psi : M \to \mathbb{C}^m$ of a compact real submanifold M and the complex structure J of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$, we find conditions under which the tangential component of $J\psi$ is a conformal vector field as well as conditions under which it is a Killing vector field. Finally, we obtain a characterization of n-spheres in the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$.

1. Introduction

Conformal vector fields and Killing vector fields play a vital role in geometry of a Riemannian manifold (M, g) as well as in physics (cf. [13]). In geometry, these vector fields are used in characterizing spheres among compact or complete Riemannian manifolds (cf. [4]–[12]). A Killing vector field is said to be nontrivial if it is not parallel. The existence of a nontrivial Killing vector field on a compact Riemannian manifold constrains its geometry as well as its topology: it does not allow the Riemannian manifold (M, g) to have nonpositive Ricci curvature and if (M, g) is positively curved, its fundamental group has a cyclic subgroup (cf. [2]). In most of the cases, a conformal vector field or a Killing vector field on a Riemannian manifold (M, g) is derived through treating it as a submanifold of a Euclidean space. For example, a unit sphere S^n admits a conformal vector field that is tangential component of a constant vector field on the ambient Euclidean space R^{n+1}. Similarly, an odd dimensional unit sphere S^{2m-1} with unit normal vector field N as a hypersurface of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$ admits a Killing vector field $\xi = -JN$, where J is the canonical complex structure on $\mathbb{C}^m$. Therefore it is an interesting question to find a conformal vector field as well as a Killing vector field on a real submanifold of a canonical complex space form

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$(\mathbb{C}^m, J, \langle\, ,\rangle)$. A similar study is taken up in [1] for submanifolds in a Euclidean space. Given an n-dimensional real submanifold (M, g) of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$ with immersion $\psi : M \to \mathbb{C}^m$, we treat ψ as the position vector field of points on M in $\mathbb{C}^m$, and consequently we have the expression $J\psi = v + \overline{N}$, where v is the tangential component and $\overline{N}$ is the normal component of $J\psi$ on M. This gives a globally defined vector field v on the real submanifold M.

In this paper, we study the above mentioned question for real submanifolds of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$ and obtain conditions under which the vector field v is a conformal vector field (Theorems 3.1, 3.2) or a Killing vector field (Theorems 4.1, 4.3). We also use this vector field v to find a characterization of a sphere $S^n(c)$ of constant curvature c in the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$ (cf. Theorem 5.1). It is worth noting that the existence of the Killing vector field v not only restricts the geometry and topology of the real submanifold M but also has an influence on the dimensions of both the real submanifold and the ambient canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$ (cf. Corollary 4.2). Finally, at the end of this paper, we give an example of a real submanifold of $(\mathbb{C}^m, J, \langle\, ,\rangle)$ on which v is a nontrivial conformal vector field (that is, v is not Killing) and another example of a real submanifold on which v is nontrivial Killing vector field (that is, non-parallel).

2. Preliminaries

Let M be an immersed n-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$, J and $\langle\, ,\rangle$ being the canonical complex structure and the Euclidean metric on $\mathbb{C}^m$ respectively. We denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M, by $\Gamma(\upsilon)$ the space of sections of the normal bundle υ of M, and by $\overline{\nabla}$ and ∇ the Riemannian connections on $\mathbb{C}^m$ and on M respectively. Then we have the following Gauss and Weingarten equations for the real submanifold M (cf. [3]):

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.1}$$

$X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(\upsilon)$, where h is the second fundamental form, A_N is the Weingarten map with respect to the normal $N \in \Gamma(\upsilon)$, which is related to the second fundamental form h by

$$g(A_N X, Y) = \langle h(X,Y), N\rangle, \quad X, Y \in \mathfrak{X}(M),$$

and $\nabla^{\perp}$ is the connection in the normal bundle υ. The curvature tensor field R of the real submanifold M is given by

$$R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \quad X, Y, Z \in \mathfrak{X}(M).$$

The Ricci tensor field of the real submanifold M is given by

$$\operatorname{Ric}(X,Y) = ng(h(X,Y), H) - \sum_{i=1}^{n} g(h(X,e_i), h(Y,e_i)),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M and

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$$

is the mean curvature vector field of the real submanifold M.

The Ricci operator Q is a symmetric operator defined by

$$\mathrm{Ric}(X, Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M).$$

Let $\psi : M \to \mathbb{C}^m$ be the immersion of the real submanifold M. Then we set

$$J\psi = v + \overline{N},$$

where v is the tangential component and $\overline{N}$ is the normal component of $J\psi$.

Now, define skew symmetric tensors φ and G, and the tensors Ψ and F as follows:

$$JX = \varphi X + FX, \quad X \in \mathfrak{X}(M),$$
$$JN = \Psi N + GN, \quad N \in \Gamma(\upsilon),$$

where

$$\varphi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \quad F : \mathfrak{X}(M) \longrightarrow \Gamma(\upsilon),$$
$$\Psi : \Gamma(\upsilon) \longrightarrow \mathfrak{X}(M), \quad G : \Gamma(\upsilon) \longrightarrow \Gamma(\upsilon),$$

that is, φX, ΨN are the tangential components of JX and JN respectively and FX, GN are the normal components of JX and JN respectively.

Define a symmetric tensor C of type $(1,1)$ by $C(X) = A_{\overline{N}} X$, $X \in \mathfrak{X}(M)$, and a smooth function $E : M \to \mathbb{R}$ on the real submanifold M by $E = \langle H, \overline{N} \rangle$. Then we have

$$\mathrm{tr}\, C = nE.$$

Lemma 2.1. *Let M be an n-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$. Then*

$$\nabla_X v = \varphi X + C(X) \quad \textit{and} \quad \nabla_X^{\perp} \overline{N} = FX - h(X, v).$$

Proof. As J is a complex structure, we have

$$\overline{\nabla}_X J\psi = J\overline{\nabla}_X \psi,$$

which in view of equation (2.1) gives

$$\nabla_X v + h(X, v) + \nabla_X^{\perp} \overline{N} - C(X) = \varphi X + FX, \quad X \in \mathfrak{X}(M).$$

Equating the tangential and the normal components we get the result. □

Lemma 2.2. *Let M be an n-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$. Then for $X, Y \in \mathfrak{X}(M)$ and $N \in \Gamma(\upsilon)$, we have*

$$(\nabla\varphi)(X, Y) = A_{F(Y)} X + \Psi(h(X, Y)), \ \textit{where}\ (\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi \nabla_X Y$$
$$(D_X F) Y = G(h(X, Y)) - h(X, \varphi Y), \ \textit{where}\ (D_X F) Y = \nabla_X^{\perp} FY - F(\nabla_X Y)$$
$$(D_X \Psi) N = A_{G(N)} X - \varphi A_N X, \ \textit{where}\ (D_X \Psi) N = \nabla_X \Psi(N) - \Psi\left(\nabla_X^{\perp} N\right)$$
$$\left(\nabla_X^{\perp} G\right) N = F(A_N X) - h(X, \Psi(N)), \ \textit{where}\ \left(\nabla_X^{\perp} G\right) N = \nabla_X^{\perp} GN - G\left(\nabla_X^{\perp} N\right).$$

Proof. As J is parallel, we have

$$\overline{\nabla}_X (\varphi Y + F(Y)) = J(\nabla_X Y + h(X,Y)),$$

which in view of equation (2.1) takes the form

$$(\nabla\varphi)(X,Y) + (D_X F)Y = A_{F(Y)}X + \Psi(h(X,Y)) + G(h(X,Y)) - h(X,\varphi Y),$$

which on equating the tangential and the normal components gives the first two relations. Similarly, on using $\left(\overline{\nabla}_X J\right)N = 0$, we get the remaining two. □

Using Lemma 2.1, we find the divergence of the vector field v as $\operatorname{div} v = nE$ and consequently, we have the following:

Lemma 2.3. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Then*

$$\int_M E\, dV = 0.$$

The following lemma is an immediate consequence of Lemma 2.1.

Lemma 2.4. *Let M be an n-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Then the tensor C satisfies*

(i) $(\nabla C)(X,Y) - (\nabla C)(Y,X) = R(X,Y)v + (\nabla\varphi)(Y,X) - (\nabla\varphi)(X,Y)$,

(ii) $\sum_{i=1}^n (\nabla C)(e_i,e_i) = n\nabla E + Q(v) + \sum_{i=1}^n (\nabla\varphi)(e_i,e_i)$,

where $(\nabla C)(X,Y) = \nabla_X C(Y) - C(\nabla_X Y)$, $X,Y \in \mathfrak{X}(M)$, and $\{e_1,\dots,e_n\}$ is a local orthonormal frame of M.

Lemma 2.5. *Let M be an n-dimensional real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Then the skew symmetric tensor φ satisfies*

(i) $(\nabla\varphi)(X,Y) - (\nabla\varphi)(Y,X) = A_{FY}X - A_{FX}Y$,

(ii) $\sum_{i=1}^n (\nabla\varphi)(e_i,e_i) = n\Psi(H) + \sum_{i=1}^n A_{Fe_i}e_i$,

where $X,Y \in \mathfrak{X}(M)$ and $\{e_1,\dots,e_n\}$ is a local orthonormal frame of M.

Proof. (i) Using Lemma 2.2, we get

$$\begin{aligned}(\nabla\varphi)(X,Y) - (\nabla\varphi)(Y,X) &= A_{FY}X + \Psi(h(X,Y)) - A_{FX}Y - \Psi(h(Y,X))\\ &= A_{FY}X - A_{FX}Y, \quad X,Y \in \mathfrak{X}(M).\end{aligned}$$

(ii) As $\operatorname{tr}\varphi = 0$, we have

$$\sum_{i=1}^n g((\nabla\varphi)(X,e_i), e_i) = 0,$$

which gives

$$\sum_{i=1}^n \{g((\nabla\varphi)(e_i,X), e_i) + g(A_{Fe_i}X, e_i) - g(A_{FX}e_i, e_i)\} = 0,$$

that is,

$$\sum_{i=1}^n \{g(-(\nabla\varphi)(e_i,e_i) + A_{Fe_i}e_i, X) + g(n\Psi(H), X)\} = 0.$$

Hence,

$$\sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i) = n\Psi(H) + \sum_{i=1}^{n} A_{Fe_i} e_i. \qquad \square$$

Lemma 2.6. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Then*

$$\int_M \left(\operatorname{Ric}(v,v) + \|C\|^2 - \|\varphi\|^2 - n^2E^2\right) dV = 0.$$

Proof. Using Lemmas 2.4 and 2.5, we get

$$\operatorname{div}\varphi v = -\sum_{i=1}^{n} g\left(A_{F(e_i)}e_i, v\right) - ng\left(\Psi(H), v\right) - \|\varphi\|^2, \tag{2.2}$$

$$\operatorname{div} Cv = \operatorname{Ric}(v,v) + nv(E) + ng\left(\Psi(H), v\right) + \|C\|^2 + \sum_{i=1}^{n} g\left(A_{Fe_i}e_i, v\right),$$

and

$$\operatorname{div} Ev = v(E) + nE^2. \tag{2.3}$$

Using these equations, we conclude that

$$\operatorname{div} Cv = \operatorname{Ric}(v,v) + n\operatorname{div} Ev - n^2E^2 - \operatorname{div}\varphi v - \|\varphi\|^2 + \|C\|^2,$$

which on integration gives the result. $\square$

Lemma 2.7. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. If v satisfies $\triangle v = -\lambda v$ for a constant $\lambda > 0$, where $\triangle$ is the Laplace operator acting on smooth vector fields on M, then*

$$\int_M \left\{\operatorname{Ric}(v,v) + \lambda\|v\|^2 - 2\|\varphi\|^2 - n^2E^2\right\} dV = 0.$$

Proof. Using the definition of the operator C and Lemma 2.1, we have

$$\begin{aligned}(\nabla C)(X,Y) &= \nabla_X CY - C\nabla_X Y \\ &= \nabla_X\left(\nabla_Y v - \varphi Y\right) - \nabla_{\nabla_X Y} v + \varphi\nabla_X Y \\ &= \nabla_X\nabla_Y v - \nabla_{\nabla_X Y} v - (\nabla\varphi)(X,Y), \quad X, Y \in \mathfrak{X}(M).\end{aligned}$$

Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$, the above equation leads to

$$\begin{aligned}\sum_{i=1}^{n} (\nabla C)(e_i, e_i) &= \sum_{i=1}^{n}\left(\nabla_{e_i}\nabla_{e_i} v - \nabla_{\nabla_{e_i}e_i} v\right) - \sum_{i=1}^{n}(\nabla\varphi)(e_i, e_i) \\ &= \triangle v - \sum_{i=1}^{n}(\nabla\varphi)(e_i, e_i) \\ &= -\lambda v - \sum_{i=1}^{n}(\nabla\varphi)(e_i, e_i),\end{aligned}$$

where we used the definition of the Laplace operator acting on smooth vector fields.

Now, using Lemma 2.4 (ii) and Lemma 2.5, we conclude

$$-\lambda \|v\|^2 = \operatorname{Ric}(v,v) + nv(E) + 2g\left(\sum_{i=1}^{n} A_{F(e_i)}e_i, v\right) + 2ng(\Psi(H), v),$$

and this equation together with equations (2.2) and (2.3) by integration gives

$$\int_M \left\{\operatorname{Ric}(v,v) + \lambda \|v\|^2 - 2\|\varphi\|^2 - n^2E^2\right\} dV = 0. \qquad \square$$

3. Submanifolds with v as a conformal vector field

Recall that a smooth vector field ξ on a Riemannian manifold (M,g) is said to be a conformal vector field if the flow of ξ consists of conformal transformations of the Riemannian manifold (M,g). Equivalently, a smooth vector field ξ on a Riemannian manifold (M,g) is a conformal vector field if there exists a smooth function ρ on M that satisfies $\pounds_\xi g = 2\rho g$, where $\pounds_\xi g$ is the Lie derivative of g with respect to ξ. The smooth function ρ is called the potential function of the conformal vector field ξ. A conformal vector field ξ is said to be a non trivial conformal vector field if the potential function ρ is not a constant. In this section, we find conditions under which the vector field v on the real submanifold M of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$ is a conformal vector field.

Theorem 3.1. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. If the Ricci curvature* $\operatorname{Ric}(v,v)$ *of M satisfies*

$$\operatorname{Ric}(v,v) \geq n(n-1)E^2 + \|\varphi\|^2,$$

then v is a conformal vector field on M.

Proof. Using Lemma 2.6, we have

$$\int_M \left(\operatorname{Ric}(v,v) - n(n-1)E^2 - \|\varphi\|^2 + \|C\|^2 - nE^2\right) dV = 0,$$

which together with the condition in the hypothesis and Schwarz's inequality $\|C\|^2 \geq nE^2$ gives

$$\operatorname{Ric}(v,v) = n(n-1)E^2 + \|\varphi\|^2 \quad \text{and} \quad \|C\|^2 = nE^2.$$

The second equality holds if and only if $C = EI$, and consequently, the first equation in Lemma 2.1 reads

$$\nabla_X v = EX + \varphi X, \quad X \in \mathfrak{X}(M).$$

This equation proves that

$$(\pounds_v g)(X,Y) = 2Eg(X,Y), \quad X,Y \in \mathfrak{X}(M),$$

that is, v is a conformal vector field with potential function E. $\square$

Theorem 3.2. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. If the vector field v is an eigenvector of the Laplace operator, $\Delta v = -\lambda v$, and the Ricci curvature* $\mathrm{Ric}(v,v)$ *satisfies*

$$\mathrm{Ric}\,(v,v) \geq n\,(n-2)\,E^2 + \lambda\,\|v\|^2\,,$$

then v is a conformal vector field.

Proof. Lemma 2.6 implies

$$-\int\limits_M \|\varphi\|^2\,dv = \int\limits_M \Big(-\,\mathrm{Ric}\,(v,v) - \|C\|^2 + n^2E^2\Big)\,dV,$$

which in view of Lemma 2.7, gives

$$\int\limits_M \Big(\mathrm{Ric}\,(v,v) - \lambda\,\|v\|^2 + 2\,\|C\|^2 - n^2E^2\Big)\,dV = 0,$$

that is,

$$\int\limits_M \Big(\mathrm{Ric}\,(v,v) - \lambda\,\|v\|^2 - n\,(n-2)\,E^2 + 2(\|C\|^2 - nE^2)\Big)\,dV = 0.$$

Thus, using the hypothesis and Schwarz's inequality $\|C\|^2 \geq nE^2$, we get

$$\mathrm{Ric}\,(v,v) = n\,(n-2)\,E^2 + \lambda\,\|v\|^2 \quad \text{and} \quad \|C\|^2 = nE^2,$$

that is, $C = EI$. Hence, by Lemma 2.1, we get that v is a conformal vector field. ☐

4. Submanifolds with v as a Killing vector field

Recall that a smooth vector field ξ on a Riemannian manifold (M,g) is said to be a Killing vector field if the flow of ξ consists of isometries of the Riemannian manifold (M,g). Equivalently, a smooth vector field ξ on a Riemannian manifold (M,g) is a Killing vector field if $\pounds_\xi g = 0$. In this section, we find conditions under which the vector field v on the real submanifold M of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$ is a Killing vector field.

Theorem 4.1. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Suppose that v satisfies*

(i) *v is an eigenvector of the Laplace operator with eigenvalue $-\lambda$,*
(ii) $\mathrm{Ric}\,(v,v) \geq n(n-1)E^2 + \|\varphi\|^2$,
(iii) $\|\varphi\|^2 \geq \lambda\,\|v\|^2$.

Then v is a Killing vector field.

Proof. The condition (ii), in view of Theorem 3.1, implies that v is a conformal vector field with $C = EI$ and

$$\mathrm{Ric}(v,v) = n(n-1)E^2 + \|\varphi\|^2\,. \tag{4.1}$$

Now, the condition (i), $\Delta v = -\lambda v$, combined with Lemma 2.7 and the above conclusion, gives

$$\int_M \left(n(n-1)E^2 + \|\varphi\|^2 + \lambda \|v\|^2 - 2\|\varphi\|^2 - n^2E^2\right) dV = 0,$$

that is,

$$\int_M \left((\|\varphi\|^2 - \lambda \|v\|^2) + nE^2\right) dV = 0. \tag{4.2}$$

Using condition (iii), we conclude that $E = 0$ and consequently $C = 0$. Thus, Lemma 2.1 gives

$$\nabla_X v = \varphi X, \quad X \in \mathfrak{X}(M),$$

that is,

$$(\pounds_v g)(X, Y) = 0, \quad X, Y \in \mathfrak{X}(M).$$

Hence, v is a Killing vector field. □

Corollary 4.2. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$, with positive sectional curvature. Suppose that v satisfies*

(i) v is an eigenvector of the Laplace operator with eigenvalue $-\lambda$, that is, $\Delta v = -\lambda v$,
(ii) $\operatorname{Ric}(v, v) \geq n(n-1)E^2 + \|\varphi\|^2$,
(iii) $\|\varphi\|^2 \geq \lambda \|v\|^2$.

Then either n is odd or $m \geq n$.

Proof. Notice that $n < 2m$. Suppose the conditions (i)–(iii) hold. Then equation (4.2) implies $E = 0$, $\lambda \|v\|^2 = \|\varphi\|^2$, and combining these with equation (4.1), we get

$$\operatorname{Ric}(v, v) = \lambda \|v\|^2 = \|\varphi\|^2. \tag{4.3}$$

Now, consider the smooth function $f = \frac{1}{2}\|v\|^2$, which by Lemma 2.1 and $E = 0$, gives the gradient $\nabla f = -\varphi v$, and we compute

$$\Delta f = -\sum_{i=1}^{n} g(\nabla_{e_i} \varphi v, e_i) = -\sum_{i=1}^{n} g(\nabla_{e_i} \nabla_v v, e_i). \tag{4.4}$$

Note that $E = 0$, as in the proof of Theorem 4.1, we get $C = 0$ and thus, an easy computation on using Lemma 2.1 with $E = 0$ gives

$$R(X, v)\, v = \nabla_X \nabla_v v - \varphi^2 X,$$

that is,

$$R(X, v, v, X) = g(\nabla_X \nabla_v v, X) + \|\varphi X\|^2.$$

This equation in view of equation (4.4) implies

$$\operatorname{Ric}(v, v) = -\Delta f + \|\varphi\|^2,$$

which together with equation (4.3) gives $\Delta f = 0$. Hence, f is a constant, that is, v has constant length and consequently, $\varphi v = 0$.

If $v = 0$, then Lemma 2.1 implies $\varphi = 0$, that is, $J\psi = \overline{N}$, which on taking covariant derivative and using Lemma 2.1 gives $JX = FX$, $X \in \mathfrak{X}(M)$, and we get that M is a totally real real submanifold of $\mathbb{C}^m$. Hence, in this case we have $2n \leq 2m$.

If $v \neq 0$, as v is a Killing vector field of constant length $v\,(p) \neq 0$ for each $p \in M$, and as M is compact connected with positive sectional curvature, then M is odd-dimensional (for on an even-dimensional compact connected manifold of positive sectional curvature a Killing vector field has a zero). □

Theorem 4.3. *Let M be an n-dimensional compact real submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\,,\rangle)$. Suppose that $v \neq 0$ is not closed and satisfies $\varphi v = 0$, with Ricci curvature*

$$\operatorname{Ric}(v, v) \geq n\,(n-1)\,E^2 + \|\varphi\|^2 .$$

Then v is a Killing vector field of constant length.

Proof. As in Theorem 3.1, the condition $\operatorname{Ric}(v, v) \geq n\,(n-1)\,E^2 + \|\varphi\|^2$ implies that v is a conformal vector field and the following hold:

$$\nabla_X v = \varphi X + EX, X \in \mathfrak{X}\,(M) \quad \text{and} \quad \operatorname{Ric}(v, v) = n\,(n-1)\,E^2 + \|\varphi\|^2 . \tag{4.5}$$

Using the first equation in (4.5), we get

$$R(X, Y)v = X(E)Y - Y(E)X + (\nabla\varphi)\,(X, Y) - (\nabla\varphi)\,(Y, X),$$

which gives

$$\operatorname{Ric}(Y, v) = -(n-1)Y(E) - g\left(Y, \sum_{i=1}^{n} (\nabla\varphi)\,(e_i, e_i)\right),$$

that is,

$$\operatorname{Ric}(v, v) = -(n-1)v(E) - g\left(v, \sum_{i=1}^{n} (\nabla\varphi)\,(e_i, e_i)\right). \tag{4.6}$$

Now, taking divergence on both sides of the equation $\varphi v = 0$, in view of equation (4.5), we have

$$-\|\varphi\|^2 - g\left(v, \sum_{i=1}^{n} (\nabla\varphi)\,(e_i, e_i)\right) = 0, \tag{4.7}$$

and inserting this equation in (4.6) leads to

$$\operatorname{Ric}(v, v) = -(n-1)v(E) + \|\varphi\|^2 ,$$

which on comparing with the second equation in (4.5) implies

$$v(E) = -nE^2. \tag{4.8}$$

Also, using $\varphi v = 0$ in the first equation in (4.5) gives

$$\nabla_v v = Ev, \tag{4.9}$$

which in view of equations (4.5) and (4.8) leads to

$$R\,(X, v)\,v = X\,(E)\,v + nE^2 X - (\nabla\varphi)\,(v, X) - E\varphi X - \varphi^2 X,$$

which on taking the inner product with v and using $\varphi(\nabla_v v) = 0$ (outcome of equation (4.9)), gives $X(E)\|v\|^2 + nE^2 g(X,v) = 0$, that is,

$$\|v\|^2 \nabla E = -nE^2 v. \tag{4.10}$$

Hence, as $v \neq 0$, we get $\varphi(\nabla E) = 0$, and taking divergence on both sides of this equation leads to $\operatorname{div}(\varphi(\nabla E)) = 0$, that is,

$$g\left(\nabla E, \sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i)\right) = 0,$$

which in view of equation (4.10) implies

$$-nE^2 g\left(v, \sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i)\right) = 0.$$

Using (4.7) in the above equation, we get

$$nE^2 \|\varphi\|^2 = 0,$$

and as v is not closed, from above equation, we conclude that $E = 0$, and thus equation (4.5) reads, $\nabla_X v = \varphi X$, $X \in \mathfrak{X}(M)$, which proves that v is a Killing vector field.

Moreover, if $f = \frac{1}{2}\|v\|^2$, then we have

$$X(f) = g(\varphi X, v) = 0, \quad X \in \mathfrak{X}(M),$$

that is, v has constant length. □

5. A characterization of spheres

In this section we consider an n-dimensional compact real submanifold M of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$, and prove the following characterization for the spheres.

Theorem 5.1. *Let M be an n-dimensional compact Einstein submanifold of the canonical complex space form $(\mathbb{C}^m, J, \langle\, ,\rangle)$, $n > 2$. Suppose that v satisfies*

(i) v is an eigenvector of the Laplace operator with eigenvalue $-\lambda < \frac{S}{n}$,

(ii) $\operatorname{Ric}(v,v) \geq n(n-1)E^2 + \|\varphi\|^2$, where S is the constant scalar curvature.

Then M is isometric to the sphere $S^n(c)$, for a constant $c > 0$.

Proof. Using Theorem 3.1, we get that v is a conformal vector field on M and equation (4.5) holds. Thus, using the first equation in (4.5), we conclude

$$(\nabla\varphi)(X,Y) = \nabla_X\nabla_Y v - \nabla_{\nabla_X Y} v - X(E)Y, \quad X, Y \in \mathfrak{X}(M), \tag{5.1}$$

where $(\nabla\varphi)(X,Y) = \nabla_X \varphi y - \varphi\nabla_X Y$. Taking sum in the above equation over a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M and using $\Delta v = -\lambda v$, we get

$$\sum_{i=1}^{n} (\nabla\varphi)(e_i, e_i) = \Delta v - \nabla E = -\lambda v - \nabla E. \tag{5.2}$$

Also, using equation (5.1), we find

$$(\nabla\varphi)(X,Y)-(\nabla\varphi)(Y,X)=R(X,Y)v+Y(E)X-X(E)Y,$$

which on choosing $X=e_i$ and taking the inner product with e_i and adding these n equations corresponding to a local orthonormal frame $\{e_1,\dots,e_n\}$ on M, we get

$$-g\left(\sum_{i=1}^{n}(\nabla\varphi)(e_i,e_i),Y\right)=\operatorname{Ric}(Y,v)+(n-1)Y(E), \tag{5.3}$$

where we used the fact that φ is skew-symmetric and consequently $\sum g(\varphi e_i,e_i)=0$, and that $g((\nabla\varphi)(X,Y),Z)=-g((\nabla\varphi)(X,Z),Y)$. Combining equations (5.2) and (5.3), we arrive at

$$Q(v)=\lambda v-(n-2)\nabla E. \tag{5.4}$$

Moreover, M being an Einstein manifold, $Q(v)=\frac{S}{n}v$, and thus using equation (5.4) we get

$$\nabla E=-\frac{S-n\lambda}{n(n-2)}v,$$

and as S is a constant, we have $\nabla E=-cv$ for a constant c. This leads to

$$\nabla_X(\nabla E)=-c\nabla_X v=-c(EX+\varphi X), \tag{5.5}$$

that is, the Hessian of the smooth function E is given by

$$H_E(X,Y)=-cEg(X,Y)-cg(\varphi X,Y)a, \qquad X,Y\in\mathfrak{X}(M),$$
$$H_E(X,Y)-H_E(Y,X)=2cg(\varphi Y,X).$$

Since the Hessian is symmetric, we get $cg(\varphi Y,X)=0$, $X,Y\in\mathfrak{X}(M)$. However, condition (i) in the hypothesis does not allow $c=0$ (as $c=0$ implies $S=n\lambda$); consequently we get $\varphi=0$, which changes equation (5.5) to

$$\nabla_X(\nabla E)=-cEX, \quad X\in\mathfrak{X}(M),$$

where c is a positive constant by condition (i). Hence, by Obata's Theorem (cf. [11]), we get that M is isometric to $S^n(c)$. □

6. Examples

In this section, we give two examples of real submanifolds of a canonical complex space form $(\mathbb{C}^m,J,\langle\,,\rangle)$, one admitting a conformal vector field that is not Killing and other admitting a Killing vector field that is not parallel.

(i) Consider

$$S^{2n}(c)=\left\{x=(x_1,\dots,x_{2n+1})\in R^{2n+1}:\|x\|=\frac{1}{\sqrt{c}},\ c>1\right\}$$

and an immersion $\psi:S^{2n}(c)\to C^{n+1}$ defined by

$$\psi(x)=\left(x_1,\dots,x_{2n+1},\sqrt{1-\frac{1}{c}}\right),$$

which is clearly a smooth immersion. Observe that

$$T_p(S^{2n}(c)) = \left\{X \in R^{2n+1} : \langle X, p\rangle = 0\right\}.$$

The two orthogonal unit normals N_1, N_2 for the real submanifold $S^{2n}(c)$ in C^{n+1} are given by

$$N_1 = \left(-\sqrt{c-1}x_1, \dots, -\sqrt{c-1}x_{2n+1}, \frac{1}{\sqrt{c}}\right)$$

and

$$N_2 = \left(x_1, \dots, x_{2n+1}, \sqrt{1-\frac{1}{c}}\right).$$

Also, the standard complex structure J on C^{n+1} gives

$$J\psi = \left(-x_{n+2}, \dots, -x_{2n+1}, -\sqrt{1-\frac{1}{c}}, x_1, \dots, x_{n+1}\right) \tag{6.1}$$

and it is easy to check that

$$\langle J\psi, N_1\rangle = \sqrt{c}x_{n+1} \quad \text{and} \quad \langle J\psi, N_2\rangle = 0.$$

Expressing $J\psi = v + \overline{N}$, where $v \in \mathfrak{X}(S^{2n}(c))$, we get

$$v = J\psi - \sqrt{c}x_{n+1}\left(-\sqrt{c-1}x_1, \dots, -\sqrt{c-1}x_{2n+1}, \frac{1}{\sqrt{c}}\right), \tag{6.2}$$

that is,

$$\begin{aligned} v &= \left(-x_{n+2}, \dots, -x_{2n+1}, -\sqrt{1-\frac{1}{c}}, x_1, \dots, x_{n+1}\right) \\ &\quad + \left(\sqrt{c^2-c}x_1x_{n+1}, \dots, \sqrt{c^2-c}x_{n+1}x_{2n+1}, -x_{n+1}\right) \\ &= \Bigg(\sqrt{c^2-c}x_1x_{n+1} - x_{n+2}, \dots, \sqrt{c^2-c}x_{n+1}^2 - \sqrt{1-\frac{1}{c}}, \\ &\qquad \sqrt{c^2-c}x_{n+1}x_{n+2} + x_1, \dots, \sqrt{c^2-c}x_{n+1}x_{2n+1} + x_n, 0\Bigg). \end{aligned} \tag{6.3}$$

Now, using expressions of N_1 and N_2 it is straightforward to show that

$$A_{N_1} = \sqrt{c-1}I \quad \text{and} \quad A_{N_2} = -I,$$

and consequently that

$$A_{\overline{N}} = \sqrt{c^2-c}x_{n+1}I.$$

This proves that the vector field v given by equation (6.3) satisfies

$$\pounds_v g = 2\sqrt{c^2-c}x_{n+1}g,$$

that is, v is a conformal vector field. Note that this vector field is not a Killing vector field on $S^{2n}(c)$. To verify the last assertion, we see from the last equation that if v is Killing, $x_{n+1} = 0$, and consequently equation (6.2) gives that $v = J\psi$. Moreover, $S^{2n}(c)$ being an even-dimensional compact and connected manifold of positive sectional curvature, there would exist a point p where $(J\psi)(p) = 0$; using this in equation (6.1) we get $c = 1$, a contradiction.

(ii) Consider the unit sphere S^{2n-1} in R^{2n} and an immersion $\psi : S^{2n-1} \to \mathbb{C}^m$, $m > n$, defined by

$$\psi(x_1, \dots, x_n, \dots, x_{2n}) = (x_1, \dots, x_{2n}, c_1, \dots, c_{2m-2n}),$$

where c_i, $1 \le i \le 2m - 2n$, are constants and $\mathbb{C}^m$ is identified with R^{2m}. A local frame of orthonormal normal vector fields for this immersion is given by $\{N_1, N_2, \dots, N_{2m-2n+1}\}$, where

$$N_1 = (x_1, \dots, x_{2n}, 0, \dots, 0)$$

and

$$N_\alpha = (0, \dots, 0, 1, 0, \dots, 0),\ 1 \text{ at the } (2n+\alpha)^{\text{th}} \text{ place},\ 2 \le \alpha \le 2m - 2n + 1.$$

Consider a complex structure J on $\mathbb{C}^m$ defined by

$$JE = (-E(x_2), E(x_1), -E(x_4), E(x_3), \dots, -E(x_{2m}), E(x_{2m-1})), \quad E \in \mathfrak{X}(\mathbb{C}^m),$$

which makes $(\mathbb{C}^m, J, \langle , \rangle)$ a Kaehler manifold. Now set $J\psi = v + \overline{N}$, where $v \in \mathfrak{X}(S^{2n-1})$ is the tangential component and $\overline{N}$ is the normal component of $J\psi$. We get

$$J\psi = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1}, -c_2, c_1, \dots, -c_{2m-2n}, c_{2m-2n-1}), \tag{6.4}$$

$$\langle J\psi, N_1\rangle = 0, \quad \langle J\psi, N_\alpha\rangle = -(-1)^\alpha c_\alpha, \quad 2 \le \alpha \le 2m - 2n + 1,$$

and consequently,

$$\overline{N} = \sum_{\alpha=1}^{2m-2n+1} \left\langle \overline{N}, N_\alpha \right\rangle N_\alpha = (0, \dots, 0, -c_2, c_1, \dots, -c_{2m-2n}, c_{2m-2n-1}). \tag{6.5}$$

Thus, equations (6.4) and (6.5) imply

$$v = J\psi - \overline{N} = (-x_2, x_1, , \dots, -x_{2n}, x_{2n-1}, 0, \dots, 0). \tag{6.6}$$

Let $\overline{\nabla}$ and ∇ be the Euclidean connection on $\mathbb{C}^m$ and the Riemannian connection on the real submanifold (S^{2n-1}, g) with respect to the induced metric g. Then using equation (6.6) we get

$$\begin{aligned}\nabla_X v &= \overline{\nabla}_X v - h(X, v)\\ &= (-X(x_2), X(x_1), \dots, -X(x_{2n}), X(x_{2n-1}), 0, \dots, 0) - h(X, v),\end{aligned}$$

$X \in \mathfrak{X}(S^{2n-1})$, where h is the second fundamental form. Taking the inner product with $Y \in \mathfrak{X}(S^{2n-1})$ in the above equation we arrive at

$$g(\nabla_X v, Y) = -X(x_2)Y(x_1) + \cdots - X(x_{2n})Y(x_{2n-1}) + X(x_{2n-1})Y(x_{2n}), \tag{6.7}$$

which leads to

$$g(\nabla_X v, Y) + g(\nabla_Y v, X) = 0, \quad X, Y \in \mathfrak{X}(S^{2n-1}).$$

Thus, the vector field v satisfies

$$\pounds_v g = 0,$$

that is, v is a Killing vector field on S^{2n-1}. That the Killing vector field v is not parallel follows from equation (6.7), that is, v is a nontrivial Killing vector field.

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