

CONFORMAL SEMI-INVARIANT RIEMANNIAN MAPS TO KÄHLER MANIFOLDS

MEHMET AKIF AKYOL AND BAYRAM ŞAHİN

ABSTRACT. As a generalization of CR-submanifolds and semi-invariant Riemannian maps, we introduce conformal semi-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds. We give non-trivial examples, investigate the geometry of foliations, and obtain decomposition theorems by using the existence of conformal Riemannian maps. We also investigate the harmonicity of such maps and find necessary and sufficient conditions for conformal anti-invariant Riemannian maps to be totally geodesic.

1. INTRODUCTION

Let (\bar{M}, g, J) be an almost Hermitian manifold with almost complex structure J . CR-submanifolds of almost Hermitian manifolds were introduced by Bejancu as a generalization of holomorphic submanifolds and totally real submanifolds. Let M be a real submanifold M of an almost Hermitian manifold (\bar{M}, J, \bar{g}) . If there are two complementary orthogonal distributions D and D^\perp on M such that D is invariant (i.e., $JD = D$) and D^\perp is J -anti-invariant (i.e., $JD^\perp \subseteq T(M)^\perp$), then M is called a CR-submanifold, where $T(M)^\perp$ is the normal bundle of M in \bar{M} . Real hypersurfaces of Kähler manifolds are examples of CR-submanifolds; for details, see [4] and [5].

Riemannian maps as a generalization of isometric immersions and Riemannian submersions were defined by Fischer in [6]. Such maps have been studied widely by many authors, see monograph [13]. On the other hand, as a generalization of holomorphic submanifolds and totally real submanifolds, invariant Riemannian maps and anti-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds were introduced in [15]. Semi-invariant Riemannian maps were introduced in [14] and it was shown that such maps include CR-submanifolds (therefore holomorphic immersions and totally real immersions), invariant Riemannian maps, and anti-invariant Riemannian maps. Recently, Riemannian maps have been investigated for various manifolds; see [8, 9, 10, 11, 12, 13].

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As a generalization of Riemannian maps, conformal Riemannian maps have been defined in [16] and it is shown that conformal submersions and conformal immersions are particular conformal Riemannian maps. Recently, conformal anti-invariant Riemannian maps and conformal slant Riemannian maps from Riemannian manifolds to Kähler manifolds have been introduced and studied in [2] and [1], respectively.

In this paper, we introduce and study conformal semi-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds as a generalization of CR-submanifolds of almost Hermitian manifolds and semi-invariant Riemannian maps to almost Hermitian manifolds. We first present the notion of conformal semi-invariant Riemannian maps supported by examples. Then by using the existence of conformal semi-invariant Riemannian maps, we obtain a decomposition theorem. We also observe that conformal semi-invariant maps allow us to obtain new conditions for a map to be harmonic. The total geodesicity of conformal semi-invariant maps is also studied.

2. PRELIMINARIES

In this section, we recall some basic materials from [3, 19]. A $2n$ -dimensional Riemannian manifold (M, g, J_M) is called an almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on M such that $J^2 = -I$ and

$$g(X, Y) = g(JX, JY), \quad \forall X, Y \in \Gamma(TM), \quad (2.1)$$

where I denotes the identity transformation of T_pM . Consider an almost Hermitian manifold (M, g, J) and denote by ∇ the Levi-Civita connection on M with respect to g . Then M is called a Kähler manifold if J is parallel with respect to ∇ , i.e.

$$(\nabla_X J)Y = 0, \quad (2.2)$$

$\forall X, Y \in \Gamma(TM)$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \rightarrow N$ is a smooth map between them. Then the differential φ_* of φ can be viewed a section of the bundle $\text{Hom}(TM, \varphi^{-1}TN) \rightarrow M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$, $p \in M$. $\text{Hom}(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla_X^\varphi\varphi_*(Y) - \varphi_*(\nabla_X^M Y) \quad (2.3)$$

for $X, Y \in \Gamma(TM)$, where ∇^φ is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \rightarrow (N, g_N)$ is said to be harmonic if $\text{trace}(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \text{div } \varphi_* = \sum_{i=1}^m (\nabla\varphi_*)(e_i, e_i), \quad (2.4)$$

where $\{e_1, \dots, e_m\}$ is the orthonormal frame on M . Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$; for details, see [3].

We denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F . Then according to [7], for any vector field X on M_1 and any section V of $(\text{range } F_*)^\perp$, where $(\text{range } F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fiber $(F_*(T_pM))^\perp$ orthogonal complement of $F_*(T_pM)$ for g_2 over p , we have $\nabla_X^{F^\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(T_pM))^\perp$ such that $\nabla^{F^\perp} g_2 = 0$. We now define A_V as

$$\nabla_X^2 V = -A_V F_* X + \nabla_X^{F^\perp} V, \tag{2.5}$$

where $A_V F_* X$ is the tangential component (a vector field along F) of $\nabla_X^2 V$. It is easy to see that $A_V F_* X$ is bilinear in V and F_* and $A_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(A_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)) \tag{2.6}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that A_V is a symmetric linear transformation of $\text{range } F_*$. Here we have the following definition from [17].

Definition 2.1. Let $F : (M^m, g_M) \rightarrow (N^n, g_N)$ be a conformal Riemannian map. Then F is a *horizontally homothetic map* if $\mathcal{H}(\text{grad } \lambda) = 0$.

We now recall the definition of conformal Riemannian maps from [16] as follows. Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds and $F : (M^m, g_M) \rightarrow (N^n, g_N)$ a smooth map between them. Then we say that F is a conformal Riemannian map at $p \in M$ if $0 < \text{rank } F_{*p} \leq \min\{m, n\}$ and F_{*p} maps the horizontal space $\mathcal{H}(p) = (\ker(F_{*p}))^\perp$ conformally onto $\text{range}(F_{*p})$, i.e., there exists a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p} X, F_{*p} Y) = \lambda^2(p) g_M(X, Y)$$

for $X, Y \in \mathcal{H}(p)$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$.

Finally, we recall that, in [16], the second author of the present paper showed that the second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((\ker F_*)^\perp)$, of a conformal Riemannian map is in the form

$$(\nabla F_*)(X, Y)^{\text{range } F_*} = X(\ln \lambda) F_* Y + Y(\ln \lambda) F_* X - g_1(X, Y) F_*(\text{grad } \ln \lambda). \tag{2.7}$$

Thus if we denote the $(\text{range } F_*)^\perp$ component of $(\nabla F_*)(X, Y)$ by $(\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp}$, we can write $(\nabla F_*)(X, Y)$ as

$$(\nabla F_*)(X, Y) = (\nabla F_*)(X, Y)^{\text{range } F_*} + (\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp}, \tag{2.8}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. Hence we have

$$\begin{aligned} (\nabla F_*)(X, Y) &= X(\ln \lambda) F_* Y + Y(\ln \lambda) F_* X - g_1(X, Y) F_*(\text{grad } \ln \lambda) \\ &\quad + (\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp}. \end{aligned} \tag{2.9}$$

3. CONFORMAL SEMI-INVARIANT RIEMANNIAN MAPS

We present the following definition for conformal semi-invariant Riemannian maps as a generalization of CR-submanifolds and semi-invariant Riemannian maps.

Definition 3.1. Let F be a conformal Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J) . Then we say that F is a *conformal semi-invariant Riemannian map* at $p \in M$ if there is a subbundle $D_1 \subseteq (\text{range } F_*)$ such that

$$\text{range } F_* = D_1 \oplus D_2$$

and

$$J(D_1) = D_1, \quad J(D_2) \subseteq (\text{range } F_*)^\perp,$$

where D_2 is orthogonal complementary to D_1 in $\text{range } F_*$. If F is a conformal semi-invariant Riemannian map for any $p \in M$, then F is called a *conformal semi-invariant Riemannian map*.

The following examples are our motivation to introduce and study conformal semi-invariant Riemannian maps.

Example 3.1. Every CR-submanifold of an almost Hermitian manifold is a conformal semi-invariant Riemannian map with $\ker F_* = \{0\}$ and $\lambda = 1$.

The theory of CR-submanifolds has been studied widely in differential geometry, however this subject is an active research area of differential geometry, see for instance [18].

Example 3.2. Every semi-invariant Riemannian map from a Riemannian manifold to an almost Hermitian manifold is a conformal semi-invariant Riemannian map with $\lambda = 1$.

Example 3.3. Every conformal anti-invariant Riemannian map from a Riemannian manifold to an almost Hermitian manifold is a conformal semi-invariant Riemannian map with $D_1 = \{0\}$.

A conformal semi-invariant Riemannian map is said to be proper if it is not an immersion (or submersion) and $\lambda \neq 1$. The following proposition gives a method to obtain examples of conformal semi-invariant Riemannian maps.

Theorem 3.1. Let $F_1 : (M_1, g_1) \rightarrow (M_2, g_2)$ be a conformal submersion from a Riemannian manifold M_1 onto a Riemannian manifold M_2 with the square dilation λ and $F_2 : (M_2, g_2) \rightarrow (M_3, g_3, J)$ a CR-immersion from a Riemannian manifold M_2 to an almost Hermitian manifold M_3 with the complex structure J . Then $F_2 \circ F_1$ is a conformal semi-invariant Riemannian map with the square dilation λ .

This proposition is obvious from [16, Theorem 5.2], and therefore we omit its proof.

As an application of Theorem 3.1, we give the following example of proper conformal semi-invariant Riemannian map.

Example 3.4. Consider the map

$$F : (\mathbb{R}^5, g = e^{x_5}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2)) \rightarrow \mathbb{E}^4$$

$$(x_1, x_2, x_3, x_4, x_5) \mapsto \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}, x_5, 0 \right),$$

which is the composition of the conformal submersion

$$\pi : (\mathbb{R}^5, g = e^{x_5}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2)) \rightarrow \mathbb{E}^3$$

$$(x_1, x_2, x_3, x_4, x_5) \mapsto \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}, x_5 \right)$$

followed by the CR-immersion

$$\phi : \mathbb{E}^3 \rightarrow \mathbb{C}^2$$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0).$$

It is easy to verify that F is a conformal semi-invariant Riemannian map with $\lambda^2 = e^{-x_5}$ with respect to the compatible almost complex structure J on \mathbb{R}^4 .

Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to an almost Hermitian manifold (M_2, g_2, J) . Then for $F_*(X) \in \Gamma(\text{range } F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, we write

$$JF_*(X) = \phi F_*(X) + \omega F_*(X), \tag{3.1}$$

where $\phi F_*(X) \in \Gamma(D_1)$ and $\omega F_*(X) \in \Gamma(JD_2)$. On the other hand, for $V \in \Gamma((\text{range } F_*)^\perp)$, we have

$$JV = \mathcal{B}V + \mathcal{C}V, \tag{3.2}$$

where $\mathcal{B}V \in \Gamma(D_2)$ and $\mathcal{C}V \in \Gamma(\mu)$. Here μ is the complementary orthogonal distribution to $\omega(D_2)$ in $(\text{range } F_*)^\perp$. It is easy to see that μ is invariant with respect to J .

For the geometry of the leaves of D_1 , we have the following.

Theorem 3.2. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then D_1 defines a totally geodesic foliation on M_2 if and only if*

- (i) $g_2(\mathcal{B}V(\ln \lambda)F_*X_1 + F_*(\overset{1}{\nabla}_{X_1}Z), JF_*X_2) = g_2(A_{\mathcal{C}V}F_*X_1, JF_*X_2)$,
- (ii) $\phi A_{JF_*W_1}F_*X_1$ has no components in $\Gamma(D_1)$,

for any $X_1, X_2, Z, W_1 \in \Gamma((\ker \pi_*)^\perp)$ such that $F_*X_1, F_*X_2 \in \Gamma(D_1)$, $F_*W_1 \in \Gamma(D_2)$ and $V \in \Gamma((\text{range } F_*)^\perp)$ such that $F_*Z = \mathcal{B}V$.

Proof. For $F_*X_1, F_*X_2 \in \Gamma(D_1)$, $V \in \Gamma((\text{range } F_*)^\perp)$ and $F_*W_1 \in \Gamma(D_2)$, since F is a conformal Riemannian map, using (2.1), (3.2), (2.8) and (2.5) we have

$$g_2(\overset{2}{\nabla}_{X_1}F_*X_2, V) = -g_2((\nabla F_*)(X_1, Z)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_{X_1}Z), JF_*X_2)$$

$$+ g_2(A_{\mathcal{C}V}F_*X_1, JF_*X_2),$$

where $F_*Z = \mathcal{B}V \in \Gamma(D_2)$ for $Z \in \Gamma((\ker F_*)^\perp)$. Now from (2.9) we get

$$g_2(\overset{2}{\nabla}_{X_1} F_*X_2, V) = -g_2(X_1(\ln \lambda)F_*Z + Z(\ln \lambda)F_*X_1 - g_1(X_1, Z)F_*(\text{grad } \ln \lambda) + F_*(\overset{1}{\nabla}_{X_1} Z), JF_*X_2) + g_2(A_{CV}F_*X_1, JF_*X_2).$$

Hence we have

$$g_2(\overset{2}{\nabla}_{X_1} F_*X_2, V) = -g_2(\mathcal{B}V(\ln \lambda)F_*X_1 + F_*(\overset{1}{\nabla}_{X_1} Z), JF_*X_2) + g_2(A_{CV}F_*X_1, JF_*X_2). \tag{3.3}$$

On the other hand, by using (2.1) we have

$$g_2(\overset{2}{\nabla}_{X_1} F_*X_2, F_*W_1) = g_2(F_*X_2, J\overset{2}{\nabla}_{X_1} JF_*W_1).$$

Then by virtue of (2.5), (3.1) and (3.2), we get

$$g_2(\overset{2}{\nabla}_{X_1} F_*X_2, F_*W_1) = g_2(F_*X_2, \phi A_{JF_*W} F_*X_1). \tag{3.4}$$

Thus the proof follows from (3.3) and (3.4). □

In a similar way, we get the following theorem for D_2 .

Theorem 3.3. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then D_2 defines a totally geodesic foliation on M_2 if and only if*

- (i) $(\nabla F_*)(W_1, Z)^{(\text{range } F_*)} + \nabla_{F_*W_2}^{F_*^\perp} CV$ has no components in $\Gamma(JD_2)$,
- (ii) $\mathcal{B}(\nabla F_*)(W_1, X_3)^{(\text{range } F_*)^\perp}$ has no components in $\Gamma(D_2)$

for any $W_1, W_2, X_3, Z \in \Gamma((\ker \pi_*)^\perp)$ such that $F_*W_1, F_*W_2 \in \Gamma(D_2)$, $F_*X_1 \in \Gamma(D_2)$, $V \in \Gamma((\text{range } F_*)^\perp)$ such that $F_*Z = \mathcal{B}V$.

We now investigate the geometry of the leaves of $(\text{range } F_*)$ and $(\text{range } F_*)^\perp$. First, we give the following result.

Theorem 3.4. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then any two conditions below imply the third:*

- (i) $(\text{range } F_*)$ defines a totally geodesic foliation on M_2 .
- (ii) F is a horizontally homothetic conformal Riemannian map.
- (iii) $g_2(A_{CV}F_*X + F_*(\overset{M_1}{\nabla}_X Z), \phi F_*Y) = g_2(\nabla_X^{F_*^\perp} \omega F_*Y, CW) - g_2(A_{\omega F_*Y} F_*X, \mathcal{B}W)$ for any $X, Y \in \Gamma((\ker \pi_*)^\perp)$ such that $F_*X, F_*Y \in \Gamma(\text{range } F_*)$, $W \in \Gamma((\text{range } F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$ such that $F_*Z = \mathcal{B}V$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $W \in \Gamma((\text{range } F_*)^\perp)$, using (2.1), (3.1) and (3.2) we have

$$g_2(\nabla_X^2 F_*Y, W) = -g_2(\nabla_X^2 F_*Z, \phi F_*Y) + g_2(\nabla_X^2 \omega F_*Y, \mathcal{B}W) - g_2(\nabla_X^2 CW, \phi F_*Y) + g_2(\nabla_X^2 \omega F_*Y, CW),$$

where $F_*Z = \mathcal{B}W$ for $Z \in \Gamma((\ker F_*)^\perp)$. Since F is a conformal Riemannian map, using (2.3), (2.5), (2.8) and (2.9) we obtain

$$\begin{aligned} g_2(\nabla_X^2 F_*Y, W) &= -g_1(X, \mathcal{H} \operatorname{grad} \ln \lambda)g_2(F_*Z, \phi F_*Y) \\ &\quad - g_1(Z, \mathcal{H} \operatorname{grad} \ln \lambda)g_2(F_*X, \phi F_*Y) \\ &\quad - g_1(X, Z)g_2(F_*(\operatorname{grad} \ln \lambda), \phi F_*Y) + g_2(F_*(\nabla_X^1 Z), \phi F_*Y) \\ &\quad - g_2(A_{\omega F_*Y} F_*X, \mathcal{B}W) + g_2(A_{CW} F_*X, \phi F_*Y) \\ &\quad + g_2(\nabla_X^{F_\perp} \omega F_*Y, CW). \end{aligned}$$

Hence we have

$$\begin{aligned} g_2(\nabla_X^2 F_*Y, W) &= g_2(A_{CW} F_*X + F_*(\nabla_X^1 Z) - Z(\ln \lambda)F_*X, \phi F_*Y) \\ &\quad - g_2(A_{\omega F_*Y} F_*X, \mathcal{B}W) + g_2(\nabla_X^{F_\perp} \omega F_*Y, CW). \end{aligned}$$

From the above equation, we can conclude that the two assertions in the theorem imply the third. \square

In a similar way, we obtain the following theorem:

Theorem 3.5. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then $(\operatorname{range} F_*)^\perp$ defines a totally geodesic foliation on M_2 if and only if*

$$g_2(W, [V, F_*X] - \nabla_{F_*X}^{F_\perp} JBV - C\nabla_{F_*X}^{F_\perp} CV) = g_2(CV, (\nabla F_*)(X, Z)^{\operatorname{range} F_*^\perp}) \quad (3.5)$$

for any $V, W \in \Gamma((\operatorname{range} F_*)^\perp)$ and $X, Z, Z' \in \Gamma((\ker F_*)^\perp)$ such that $F_*Z' = \mathcal{B}V$.

From Theorem 3.4 and Theorem 3.5, we have the following theorem:

Theorem 3.6. *Let F be a horizontally homothetic conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then M_2 is a locally product manifold $M_{2(\operatorname{range} F_*)} \times M_{2(\operatorname{range} F_*)^\perp}$ if and only if*

$$g_2(A_{CV} F_*X + F_*(\nabla_X^{M_1} Z), \phi F_*Y) = g_2(\nabla_X^{F_\perp} \omega F_*Y, CW) - g_2(A_{\omega F_*Y} F_*X, \mathcal{B}W)$$

and

$$g_2(W, [V, F_*X] - \nabla_{F_*X}^{F_\perp} JBV - C\nabla_{F_*X}^{F_\perp} CV) = g_2(CV, (\nabla F_*)(X, Z)^{\operatorname{range} F_*^\perp})$$

for any $V, W \in \Gamma((\operatorname{range} F_*)^\perp)$ and $X, Y, Z \in \Gamma((\ker F_*)^\perp)$ such that $F_*Z = \mathcal{B}V$.

Now, we give necessary and sufficient conditions for a conformal semi-invariant Riemannian map to be totally geodesic. We recall that a differentiable map F between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$.

Theorem 3.7. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then F is totally geodesic if and only if*

- (a) *The vertical distribution $(\ker F_*)$ defines a totally geodesic foliation on M_1 .*

(b) *The horizontal distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M_1 .*

$$\begin{aligned}
 \text{(c)} \quad & \phi((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) \\
 &= -\mathcal{B}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} + \nabla_X^{F_\perp} \omega F_* Z) - F_*(\overset{1}{\nabla}_X Z), \\
 & \omega((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) \\
 &= -\mathcal{C}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} + \nabla_X^{F_\perp} \omega F_* Z), \\
 & \text{for any } X, Y, Z \in \Gamma(\ker F_*)^\perp \text{ such that } F_* Y = \phi F_* Z.
 \end{aligned}$$

Proof. (a) and (b) are clear from the second fundamental form. For $X, Z \in \Gamma(\ker F_*)^\perp$ and $V \in \Gamma(\text{range } F_*)$, using (2.1), (2.3), (3.2) we have

$$(\nabla F_*)(X, Z) = -J(\overset{2}{\nabla}_X(\phi F_* Z + \omega F_* Z)) - F_*(\overset{1}{\nabla}_X Z).$$

Then (2.8) and (2.5) imply that

$$\begin{aligned}
 (\nabla F_*)(X, Z) &= -J((\nabla F_*)(X, Y)^{(\text{range } F_*)} + (\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} + F_*(\overset{1}{\nabla}_X Y) \\
 &\quad - A_{\omega F_* Z} F_* X + \nabla_X^{F_\perp} \omega F_* Z - F_*(\overset{1}{\nabla}_X Z)),
 \end{aligned}$$

where $F_* Y = \phi F_* Z$ for $Y \in \Gamma((\ker F_*)^\perp)$. Since F is a conformal Riemannian map, from (3.1) and (3.2) and taking the $(\text{range } F_*)$ and $(\text{range } F_*)^\perp$ components, we get

$$\begin{aligned}
 (\nabla F_*)(X, Z)^{(\text{range } F_*)} &= -\phi((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) \\
 &\quad - \mathcal{B}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} + \nabla_X^{F_\perp} \omega F_* Z) - F_*(\overset{1}{\nabla}_X Z)
 \end{aligned}$$

and

$$\begin{aligned}
 (\nabla F_*)(X, Z)^{(\text{range } F_*)^\perp} &= -\omega((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) \\
 &\quad - \mathcal{C}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} + \nabla_X^{F_\perp} \omega F_* Z).
 \end{aligned}$$

Thus $(\nabla F_*)(X, Z) = 0$ if and only if

$$\begin{aligned}
 \phi((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) &= -\mathcal{B}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} \\
 &\quad + \nabla_X^{F_\perp} \omega F_* Z) - F_*(\overset{1}{\nabla}_X Z)
 \end{aligned}$$

and

$$\begin{aligned}
 \omega((\nabla F_*)(X, Y)^{(\text{range } F_*)} + F_*(\overset{1}{\nabla}_X Y) - A_{\omega F_* Z} F_* X) &= -\mathcal{C}((\nabla F_*)(X, Y)^{(\text{range } F_*)^\perp} \\
 &\quad + \nabla_X^{F_\perp} \omega F_* Z)
 \end{aligned}$$

are satisfied. This completes the proof. □

In the sequel we are going to investigate the harmonicity of conformal semi-invariant Riemannian maps. We first have the following general result.

Theorem 3.8. *Let F be a conformal semi-invariant Riemannian map from a Riemannian manifold (M_1, g_1) to a Kähler manifold (M_2, g_2, J) . Then F is harmonic if and only if the following conditions are satisfied:*

- (a) *the fibres are minimal,*
- (b) $\text{trace } \phi A_{\omega F_*(\cdot)}(\cdot) - \mathcal{B}\nabla_{(\cdot)}^{F\perp} \omega F_*(\cdot) - F_*(\nabla_{(\cdot)}^1(\cdot)) - (\nabla_{(\cdot)}^2 J\phi F_*(\cdot))^{(\text{range } F_*)} = 0,$
- (c) $\text{trace } \omega A_{\omega F_*(\cdot)}(\cdot) - \mathcal{C}\nabla_{(\cdot)}^{F\perp} \omega F_*(\cdot) - (\nabla_{(\cdot)}^2 J\phi F_*(\cdot))^{(\text{range } F_*)^\perp} = 0.$

Proof. For $U \in \Gamma(\ker F_*)$, using (2.3) we have

$$(\nabla F_*)(U, U) = -F_*(\nabla_U^1 U). \tag{3.6}$$

For $X \in \Gamma(\ker F_*)^\perp$, using (2.1), (2.3), (3.2), (2.8) and (2.5) we have

$$(\nabla F_*)(X, X) = -\nabla_X^2 J\phi F_* X - J(-A_{\omega F_* X} F_* X + \nabla_X^{F\perp} \omega F_* X) - F_*(\nabla_X^1 X).$$

Since F is a conformal Riemannian map, from (3.1) and (3.2), and taking the $(\text{range } F_*)$ and $(\text{range } F_*)^\perp$ components we obtain

$$\begin{aligned} (\nabla F_*)(X, X)^{(\text{range } F_*)} &= \phi A_{\omega F_* X} F_* X - \mathcal{B}\nabla_X^{F\perp} \omega F_* X - F_*(\nabla_X^1 X) \\ &\quad - (\nabla_X^2 J\phi F_* X)^{(\text{range } F_*)} \end{aligned} \tag{3.7}$$

and

$$(\nabla F_*)(X, X)^{(\text{range } F_*)^\perp} = \omega A_{\omega F_* X} F_* X - \mathcal{C}\nabla_X^{F\perp} \omega F_* X - (\nabla_X^2 J\phi F_* X)^{(\text{range } F_*)^\perp}. \tag{3.8}$$

Then the proof follows from (3.6), (3.7), and (3.8). □

REFERENCES

- [1] Akyol, M. A., Şahin, B., Conformal slant Riemannian maps to Kähler manifolds, *Tokyo J. Math.* **42** (2019), no. 1, 225–237. MR 3982056.
- [2] Akyol, M. A., Şahin, B., Conformal anti-invariant Riemannian maps to Kähler manifolds, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **80** (2018), no. 4, 187–198. MR 3887302.
- [3] Baird, P., Wood, J. C., *Harmonic morphisms between Riemannian manifolds*, Clarendon Press, Oxford, 2003. MR 2044031.
- [4] Bejancu, A., *Geometry of CR-submanifolds*, D. Reidel, Dordrecht, 1986. MR 0861408.
- [5] Chen, B.-Y., Riemannian submanifolds, in *Handbook of Differential Geometry*, Vol. I, 187–418, North-Holland, Amsterdam, 2000. MR 1736854.
- [6] Fischer, A. E., Riemannian maps between Riemannian manifolds, in *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, 331–366, Amer. Math. Soc., Providence, RI, 1992. MR 1188447.
- [7] Nore, T., Second fundamental form of a map, *Ann. Mat. Pura Appl.* **146** (1987), 281–310. MR 0916696.
- [8] Panday, B., Jaiswal, J. P., Ojha, R. H., Necessary and sufficient conditions for the Riemannian map to be a harmonic map on cosymplectic manifolds, *Proc. Nat. Acad. Sci. India Sect. A* **85** (2015), no. 2, 265–268. MR 3351900.
- [9] Park, K.-S., Almost h-semi-slant Riemannian maps to almost quaternionic Hermitian manifolds. *Commun. Contemp. Math.* **17** (2015), no. 6, 1550008, 23 pp. MR 3485873.

- [10] Park, K.-S., Şahin, B. *Semi-slant Riemannian maps into almost Hermitian manifolds*, *Czechoslovak Math. J.* **64** (2014), no. 4, 1045–1061. MR 3304797.
- [11] Prasad, R., Kumar, S., *Slant Riemannian maps from Kenmotsu manifolds into Riemannian manifolds*, *Global J. Pure Appl. Math.* **13** (2017), no. 4, 1143–1155.
- [12] Prasad, R., Pandey, S., *Slant Riemannian maps from an almost contact manifold*, *Filomat* **31** (2017), no. 13, 3999–4007. MR 3721333.
- [13] Şahin, B., *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Elsevier/Academic Press, London, 2017. MR 3644540.
- [14] Şahin, B., *Semi-invariant Riemannian maps to Kähler manifolds*. *Int. J. Geom. Methods Mod. Phys.* **8** (2011), no. 7, 1439–1454. MR 2873816.
- [15] Şahin, B., *Invariant and anti-invariant Riemannian maps to Kähler manifolds*, *Int. J. Geom. Methods Mod. Phys.* **7** (2010), no. 3, 337–355. MR 2646767.
- [16] Şahin, B., *Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems*, *Acta Appl. Math.* **109** (2010), no. 3, 829–847. MR 2596178.
- [17] Şahin, B., Yanan, Ş., *Conformal Riemannian maps from almost Hermitian manifolds*, *Turkish J. Math.* **42** (2018), no. 5, 2436–2451. MR 3866163.
- [18] Vilcu, G. E., *Ruled CR-submanifolds of locally conformal Kähler manifolds*, *J. Geom. Phys.* **62** (2012), no. 6, 1366–1372. MR 2911212.
- [19] Yano, K., Kon, M., *Structures on Manifolds*, World Scientific, Singapore, 1984. MR 0794310.

M. A. Akyol

Department of Mathematics, Faculty of Art and Science, Bingol University, 12000, Bingol, Turkey
mehmetakifakyol@bingol.edu.tr

B. Şahin 

Department of Mathematics, Ege University, 35100, Bornova, Izmir, Turkey
bayram.sahin@ymail.com

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