REAL HYPERSURFACES IN COMPLEX GRASSMANNIANS OF RANK TWO WITH SEMI-PARALLEL STRUCTURE JACOBI OPERATOR

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ABSTRACT. We prove that there does not exist any real hypersurface in complex Grassmannians of rank two with semi-parallel structure Jacobi operator. With this result, the non-existence of real hypersurfaces in complex Grassmannians of rank two with recurrent structure Jacobi operator is proved.

1. INTRODUCTION

Let $\hat{M}^m(c)$ be the compact complex Grassmannian $SU_{m+2}/S(U_2U_m)$ of rank two (resp. noncompact complex Grassmannian $SU_{2,m}/S(U_2U_m)$ of rank two) for $c > 0$ (resp. $c < 0$), where $c = \max \|K\|/8$ is a scaling factor for the Riemannian metric $g$ and $K$ is the sectional curvature for $\hat{M}^m(c)$. It is an irreducible Riemannian symmetric space equipped with a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ not containing $J$.

Let $M$ be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 2$, and $N$ be a unit normal vector field on $M$. Denote by the same $g$ the Riemannian metric on $M$. The Reeb vector field $\xi$ is defined by $\xi = -JN$, and we define $\xi_a = -J_aN$, $a \in \{1,2,3\}$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$. Denote by $D^\perp$ (resp. $\mathcal{D}^\perp$) the distribution on $M$ spanned by $\xi$ (resp. $\{\xi_1, \xi_2, \xi_3\}$). A real hypersurface $M$ in a Kähler manifold is said to be Hopf if the Reeb vector field is principal, that is, $A\xi = \alpha\xi$.

The study of real hypersurfaces in $\hat{M}^m(c)$ was initiated by Berndt and Suh in [1, 2]. They considered the invariance of $\mathcal{D}^\perp$ under the shape operator $A$ of Hopf hypersurfaces $M$ in $\hat{M}^m(c)$ and proved a classification of such Hopf hypersurfaces in $\hat{M}^m(c)$.

The structures $J$ and $\mathfrak{J}$ of the ambient space impose several restrictions on the geometry of its real hypersurfaces; for example, there does not exist any semi-parallel real hypersurface in $SU_{m+2}/S(U_2U_m)$ [13], while the non-existence problem of Hopf hypersurfaces in $\hat{M}^m(c)$ with parallel Ricci tensor was studied in [15, 16].

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Besides the shape operator and the Ricci tensor, there are particularly two operators on a real hypersurface $M$ which draw much attention, namely the normal Jacobi operator $R_X$ and the structure Jacobi operator $R_\xi$. Denote by $\hat{R}$ and $R$ the curvature tensor on $\mathbb{M}^m(c)$ and that induced on $M$, respectively. We define $R_X(X) = \hat{R}(X, N)N$ and $R_\xi(X) = R(X, \xi)\xi$ for any vector field $X$ tangent to $M$.

A $(1, s)$-tensor field $P$ is said to be semi-parallel if $R \cdot P = 0$, where the curvature tensor $R$ acts on $P$ as a derivation. More precisely,

$$(R(X, Y) \cdot P)(X_1, \ldots, X_s) = R(X, Y)P(X_1, \ldots, X_s) - \sum_{j=1}^{s} P(X_1, \ldots, R(X, Y)X_j, \ldots, X_s).$$

The tensor field $P$ is said to be recurrent if there exists a 1-form $\omega$ on $M$ such that

$$(\nabla_X P)(X_1, \ldots, X_s) = \omega(X)P(X_1, \ldots, X_s).$$

Clearly, a vanishing $\omega$ leads to parallelism of $P$.

Recently, we proved the non-existence of real hypersurfaces in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with pseudo-parallel normal Jacobi operator $[5]$. On the other hand, related to the structure Jacobi operator $R_\xi$, Jeong et al. proved that there does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with parallel structure Jacobi operator $[10]$. Also, the non-existence of Hopf hypersurfaces with $D^\perp$-parallel structure Jacobi operator is obtained under certain conditions $[9]$. Jeong et al. considered Reeb-parallel structure Jacobi operator and proved the following:

Theorem 1.1 ($[8]$). Let $M$ be a Hopf hypersurface in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field $\xi$ on $M$ is non-vanishing and constant along the direction of the Reeb vector field $\xi$, then $M$ is an open part of a tube around a totally geodesic $SU_{m+1}/S(U_2 U_{m-1})$ in $SU_{m+2}/S(U_2 U_m)$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup \left(\frac{\pi}{4\sqrt{2}} + \frac{\pi}{\sqrt{8}}\right)$.

We say that a real hypersurface $M$ has commuting structure Jacobi operator if it commutes with any other Jacobi operator defined on $M$, that is, $R_\xi \cdot R_X = R_X \cdot R_\xi$ for any $X$ tangent to $M$. Machado et al. proved the non-existence of Hopf real hypersurfaces in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with commuting structure Jacobi operator under certain conditions $[14]$. They also classified real hypersurfaces in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with $\hat{R}_N \cdot R_\xi = R_\xi \cdot \hat{R}_N$. In $[11]$, Jeong et al. proved the following:

Theorem 1.2 ($[11]$). There does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with recurrent structure Jacobi operator if the distribution $\mathcal{D}$ or $\mathcal{D}^\perp$-component of the Reeb vector field is invariant under the shape operator.

On the other hand, under certain restrictions, Hwang et al. obtained the following non-existence result.

Theorem 1.3 ($[6]$). There does not exist any Hopf hypersurface in $SU_{m+2}/S(U_2 U_m)$, $m \geq 3$, with semi-parallel structure Jacobi operator if the smooth function $\alpha = g(A_\xi, \xi)$ is constant along the direction of $\xi$. 

Motivated from the above studies, a natural question arises:

**Problem 1.1.** Does there exist a real hypersurface in $\hat{M}^m(c)$ with parallel, recurrent or semi-parallel structure Jacobi operator?

In the present paper we first prove the following:

**Theorem 1.4.** There does not exist any connected real hypersurface in $\hat{M}^m(c)$, $m \geq 3$, with semi-parallel structure Jacobi operator.

The non-existence of real hypersurfaces with semi-parallel structure Jacobi operator in a non flat complex space form has been proved in [3, 7]. We also remark that Theorem 1.4 holds for non-Hopf real hypersurfaces as well and no further conditions are imposed. By a result in [4], we learn that if a tensor field is recurrent, it is always semi-parallel. Hence, as a corollary we obtain the following:

**Corollary 1.1.** There does not exist any real hypersurface in $\hat{M}^m(c)$, $m \geq 3$, with parallel or recurrent structure Jacobi operator.

2. Preliminaries

In this section, we recall some fundamental identities for real hypersurfaces in complex Grassmannians of rank two, which have been proven in [1, 2, 12, 13].

Let $M$ be a connected, oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$, $m \geq 3$. The almost contact metric 3-structure $(\phi, \xi, \eta, g)$ on $M$ is given by

$$J_aX = \phi_aX + \eta_a(X)N, \quad J_aN = -\xi_a, \quad \eta_a(X) = g(X, \xi_a),$$

for any $X \in TM$, where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$ on $\hat{M}^m(c)$. It follows that

$$\phi_a\phi_{a+1} - \xi_a \otimes \eta_{a+1} = \phi_{a+2}$$
$$\phi_a\xi_{a+1} = \xi_{a+2} = -\phi_{a+1}\xi_a$$

for $a \in \{1, 2, 3\}$. The indices in the preceding equations are taken modulo three.

The Kähler structure $J$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$, namely,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi).$$

Let $\mathfrak{D}^\perp = \mathfrak{J}TM^\perp$, and $\mathfrak{D}$ its orthogonal complement in $TM$. We define a local $(1,1)$-tensor field $\theta_a$ on $M$ by

$$\theta_a := \phi_a\phi - \xi_a \otimes \eta.$$

Denote by $\nabla$ the Levi-Civita connection on $M$. Then there exist local 1-forms $q_a$, $a \in \{1, 2, 3\}$, such that

$$\begin{align*}
\nabla_X\xi &= \phi AX \\
\nabla_X\xi_a &= \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} \\
\nabla_X\phi\xi_a &= \theta_a AX + \eta_a(\xi)AX + q_{a+2}(X)\phi\xi_{a+1} - q_{a+1}(X)\phi\xi_{a+2}.
\end{align*}$$

The following identities are known.
Lemma 2.1 [12].

(a) $\theta_a$ is symmetric,
(b) $\phi \xi_a = \phi_0 \xi_a$, 
(c) $\theta_a \xi = -\xi$, $\theta_0 \xi = -\xi$, $\theta_a \phi \xi_a = \eta(\xi_a) \phi \xi_a$,
(d) $\theta_a \xi_a + 1 = \phi \xi_a + 2 = -\theta_a + 1 \xi_a$, 
(e) $-\theta_a \phi \xi_a + 1 + \eta(\xi_a + 1) \phi \xi_a = \xi_a + 2 = \theta_a + 1 \phi \xi_a - \eta(\xi_a) \phi \xi_a + 1$.

Lemma 2.2 [12]. If $\xi \in \mathcal{O}$ everywhere, then $A \phi \xi_a = 0$ for $a \in \{1, 2, 3\}$.

For each $x \in M$, we define a subspace $\mathcal{H}^\perp$ of $T_x M$ by

$$\mathcal{H}^\perp := \text{span}\{\xi, \xi_1, \xi_2, \xi_3, \phi_1, \phi_2, \phi_3\}.$$ 

Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{H}^\perp$ in $T_x M$. Then dim $\mathcal{H} = 4m - 4$ (resp. dim $\mathcal{H} = 4m - 8$) when $\xi \in \mathcal{O}^\perp$ (resp. $\xi \notin \mathcal{O}^\perp$). Moreover, $\theta_{a|\mathcal{H}}$ has two eigenvalues: 1 and $-1$. Denote by $\mathcal{H}_a(\varepsilon)$ the eigenspace corresponding to the eigenvalue $\varepsilon$ of $\theta_{a|\mathcal{H}}$. Then dim $\mathcal{H}_a(1) = \dim \mathcal{H}_a(-1)$ is even, and

$$\phi \mathcal{H}_a(\varepsilon) = \phi_a \mathcal{H}_a(\varepsilon) = \theta_a \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon)$$
$$\phi_0 \mathcal{H}_a(\varepsilon) = \theta_0 \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(-\varepsilon), \quad (a \neq b).$$

We define the tensor fields $\theta, \phi^\perp, \xi^\perp$, and $\eta^\perp$ on $M$ as follows:

$$\theta := \sum_{a=1}^{3} \eta_a(\xi) \theta_a, \quad \phi^\perp := \sum_{a=1}^{3} \eta_a(\xi) \phi_a, \quad \xi^\perp := \sum_{a=1}^{3} \eta_a(\xi) \xi_a, \quad \eta^\perp := \sum_{a=1}^{3} \eta_a(\xi) \eta_a.$$ 

Then for each $x \in M$ with $\xi^\perp \neq 0$, $\theta_{a|\mathcal{H}}$ has two eigenvalues $\varepsilon \|\xi^\perp\|, \varepsilon \in \{1, -1\}$. Let $\mathcal{H}(\varepsilon)$ be the eigenspace of $\theta_{a|\mathcal{H}}$ corresponding to $\varepsilon \|\xi^\perp\|$. Then

(a) $\phi \mathcal{H}(\varepsilon) = \phi^\perp \mathcal{H}(\varepsilon) = \mathcal{H}(\varepsilon),$ 
(b) $\dim \mathcal{H}(1) = \dim \mathcal{H}(-1)$ is even.

Moreover, we can take a canonical local basis of $\mathcal{J}$ on a neighborhood $G \subset M$ of such a point $x$ such that

$$\xi_1 = \frac{\xi^\perp}{\|\xi^\perp\|}, \quad 0 < \eta_1(\xi) = \|\xi^\perp\| \leq 1, \quad \eta_2(\xi) = \eta_3(\xi) = 0$$

$$\mathcal{H}(\varepsilon) = \mathcal{H}_1(\varepsilon), \quad \theta = \eta_1(\xi) \theta_1, \quad \phi^\perp = \eta_1(\xi) \phi_1, \quad \eta^\perp = \eta_1(\xi) \eta_1.$$ 

In particular, if $\|\xi^\perp\| = 1$ at $x$, then

$$\xi_1 = \xi = \xi^\perp, \quad \xi_2 = \theta \xi_2 = \phi \xi_3, \quad \xi_3 = \theta \xi_3 = -\phi \xi_2.$$ 

Throughout this paper, we always consider such a local orthonormal frame $\{\xi_1, \xi_2, \xi_3\}$ on $\mathcal{O}^\perp$ under these situations.

A straightforward calculation gives

$$(\nabla_X \theta) Y = \eta^\perp(\phi Y) AX - g(AX, Y) \phi \xi^\perp + 2 \sum_{a=1}^{3} \eta_a(\phi AX) \theta_a Y. \quad (2.2)$$
The equations of Gauss and Codazzi are respectively given by
\[
\begin{align*}
R(X,Y)Z &= g(AY,Z)AX - g(AX,Z)AY + c\{g(Y,Z)X - g(X,Z)Y \\
& \quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\
& \quad + c \sum_{a=1}^{3}\{g(\phi_a Y, Z)\phi_a X - g(\phi_a X, Z)\phi_a Y - 2g(\phi_a X, Y)\phi_a Z \\
& \quad + g(\theta_a Y, Z)\theta_a X - g(\theta_a X, Z)\theta_a Y\};
\end{align*}
\]
\[
(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(Y)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}
\]
\[
+ c \sum_{a=1}^{3}\{\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X, Y)\xi_a \\
+ \eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X\}.
\]
As \(M\) is a real hypersurface in \(\hat{M}^m(c)\), by the Gauss equation we have
\[
R_\xi X = \alpha AY - \eta(AY)A\xi + c\{X - \eta(X)\xi - \theta X\}
\]
\[
- c \sum_{a=1}^{3}\{\eta_a(X)\xi_a + 3\eta_a(\phi X)\phi\xi_a\}. \tag{2.3}
\]
We end this section with the following general result.

**Theorem 2.1.** Let \(M\) be an almost contact metric manifold. The structure Jacobi operator \(R_\xi\) is semi-parallel if and only if \(R_\xi = 0\).

**Proof.** Suppose the structure Jacobi operator is semi-parallel. Then
\[
R(X,Y)R_\xi Z - R_\xi R(X,Y)Z = (R(X,Y) \cdot R_\xi)Z = 0
\]
for any \(X,Y,Z \in TM\). In particular, for \(Y = Z = \xi\), we obtain \(R_\xi^2 X = 0\). Since \(R_\xi\) is self-adjoint, \(R_\xi = 0\). The converse is trivial. \(\square\)

3. **Proof of Theorem 1.4**

By virtue of Theorem 2.1, it suffices to show that the structure Jacobi operator cannot be identically zero. Suppose to the contrary that \(R_\xi = 0\). Then by (2.3), we have
\[
\alpha AY - \eta(AY)A\xi + c\{Y - \eta(Y)\xi - \theta Y\}
\]
\[
- c \sum_{a=1}^{3}\{\eta_a(Y)\xi_a + 3\eta_a(\phi Y)\phi\xi_a\} = 0. \tag{3.1}
\]

**Claim 3.1.** \(\xi \notin \mathfrak{D}\) on an open dense subset of \(M\).

**Proof.** Suppose \(\xi \in \mathfrak{D}\) on an open subset \(G\) of \(M\). For each \(x \in G\), we have \(\theta = 0\). It follows from Lemma 2.2 that \(\phi\xi_1 = 0\) after putting \(Y = \phi\xi_1\) in (3.1), a contradiction. Hence we obtain the claim. \(\square\)
Consider a point \( x \in M \) on which \( \xi \notin \mathcal{D} \) on a neighborhood \( G \) of \( x \) in \( M \). We define subspaces \( \mathcal{F}, \mathcal{F}(1) \) and \( \mathcal{F}(-1) \) of \( T_x M \) by

\[
\mathcal{F} = \{ X \in \mathcal{H} : \eta(AX) = 0 \}, \quad \mathcal{F}(1) = \mathcal{F} \cap \mathcal{H}(1), \quad \mathcal{F}(-1) = \mathcal{F} \cap \mathcal{H}(-1).
\]

It is clear that

\[
AY = \lambda_c Y, \quad \alpha \lambda_c + c(1 - \varepsilon \|\xi^\perp\|) = 0
\]

for any \( Y \in \mathcal{F}(\varepsilon) \) and \( \varepsilon \in \{1, -1\} \). By (3.1), we have

\[
(X\alpha)AY + \alpha(\nabla_X A)Y - \eta(AY)\{(\nabla_X A)\xi + A\nabla_X \xi\}
- \{g(\nabla_X A)Y, \xi) + g(AY, \nabla_X \xi)\}A\xi + c\{-g(\nabla_X \xi, Y) - \eta(Y)\nabla_X \xi - (\nabla_X \theta)Y\}
- c\sum_{a=1}^3\{g(\nabla_X \xi_a, Y)\xi_a + \eta_a(Y)\nabla_X \xi_a - 3g(\nabla_X \phi\xi_a, Y)\phi\xi_a + 3\eta_a(\phi Y)\nabla_X \phi\xi_a\} = 0
\]

for any \( X, Y \in T_x M \). By using (2.1) and (2.2), the preceding equation becomes

\[
(X\alpha)AY + \alpha(\nabla_X A)Y - \eta(AY)\{(\nabla_X A)\xi + A\phi AX\}
- \{g(\nabla_X A)Y, \xi\} + g(A\phi AX, Y)\}A\xi
+ c\{-g(\phi AX, Y) - \eta(Y)\phi AX + 4g(AX, Y)\phi\xi^\perp - 4\eta^\perp(\phi Y)AX\}
+ c\sum_{a=1}^3\{-g(\phi_a AX, Y)\xi_a - \eta_a(Y)\phi_a AX

+ 3g(\theta_a AX, Y)\phi\xi_a - 3\eta_a(\phi Y)\theta_a AX - 2\eta_a(\phi AX)\theta_a Y\} = 0.
\]

By the preceding equation and the Codazzi equation, we have

\[
(X\alpha)AY - (Y\alpha)AX - \eta(AY)\{(\nabla_X A)\xi + A\phi AX\} + \eta(AX)\{(\nabla_Y A)\xi + A\phi AY\}
+ c\{\eta(X)(\phi AY + \alpha\phi Y) - \eta(Y)(\phi AX + \alpha\phi X) - 4\eta^\perp(\phi Y)AX + 4\eta^\perp(\phi X)AY\}
+ 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)\}A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi
+ c\sum_{a=1}^3\{\eta_a(X)(\phi_a AY + \alpha\phi_a Y) - \eta_a(Y)(\phi_a AX + \alpha\phi_a X)
- \alpha\eta_a(\phi Y)\theta_a X + \alpha\eta_a(\phi X)\theta_a Y - 3\eta_a(\phi Y)\theta_a AX + 3\eta_a(\phi X)\theta_a AY

+ 2\eta_a(\phi AX)\theta_a Y - 2\eta_a(X)\eta_a(\phi Y)A\xi + 2\eta_a(Y)\eta_a(\phi X)A\xi

- g(2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a + 3g((\theta_a A - A\theta_a)X, Y)\phi\xi_a\} = 0
\]

for any \( X, Y \in T_x M \). By putting \( X, Y \in \mathcal{F} \) in (3.2), we have

\[
(X\alpha)AY - (Y\alpha)AX
+ 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi
+ c\sum_{a=1}^3\{-g(2\alpha\phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a + 3g((\theta_a A - A\theta_a)X, Y)\phi\xi_a\} = 0.
\]

Since the first two terms are in $\mathcal{H}$ and the remaining are in $\mathcal{H}^\perp$, we have
\[ 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha\phi X + (\phi A + A\phi)X, Y)\xi \]
\[ + c \sum_{a=1}^{3} \{-g(2\alpha_\phi X + (\phi_a A + A\phi_a)X, Y)\xi_a \} \]
\[ + 3g((\theta_a A - A\theta_a)X, Y)\phi_\xi a \} = 0 \tag{3.3} \]
for any $X, Y \in \mathcal{F}$. For any $\varepsilon \in \{1, -1\}$, we can further deduce from (3.3) that
\[ 0 = 2g(\phi X, Y)c(1 - \varepsilon \|\xi^\perp\|) - \lambda^2_\varepsilon) A\xi + g(\phi X, Y)c(2\alpha + 2\lambda_\varepsilon) \left\{-\varepsilon + \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp \right\} \]
\[ = g(\phi X, Y)(\alpha + \lambda_\varepsilon) \left\{-\lambda_\varepsilon A\xi - c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp \right) \right\} \tag{3.4} \]
for any $X, Y \in \mathcal{F}(\varepsilon)$; and
\[ 0 = \sum_{a=1}^{3} \{-g(2\alpha + \lambda_\varepsilon + \lambda_{-\varepsilon})\phi(\phi_a AX, Y)\xi_a + 3(\lambda_{1\varepsilon} - \lambda_{-\varepsilon})g(\theta_a X, Y)\phi_\xi a \} \tag{3.5} \]
for any $X \in \mathcal{F}(\varepsilon)$ and $Y \in \mathcal{F}(-\varepsilon)$.

**Claim 3.2.** $\dim \mathcal{H} \geq 8$.

*Proof.* Suppose $\dim \mathcal{H} = 4$. Since $\dim M = 4m - 1 \geq 11$, we have $\xi \notin \mathcal{D}^\perp$ or $0 < \|\xi^\perp\| < 1$. Take a unit vector $V \in \mathcal{F}(1)$ such that $\mathcal{H}(1) = \mathbb{R}V \oplus \mathbb{R}\phi_1 V$ and $\mathcal{H}(-1) = \mathbb{R}\phi_2 V \oplus \mathbb{R}\phi_3 V$.

Substituting $X = V$ and $Y = \phi_2 V$ in (3.5), we obtain
\[ 0 = -(2\alpha + \lambda_1 + \lambda_{-1})\xi_2 - 3(\lambda_1 - \lambda_{-1})\phi_3 \]
for any $Y \in \mathcal{F}(1)$. Since $\{\xi_a, \phi_\xi a \}_{a \in \{1,2,3\}}$ is linearly independent, $-2\alpha^{-1}\|\xi^\perp\| = \lambda_1 - \lambda_{-1} = 0$, a contradiction. Hence, the claim is obtained. \(\square\)

According to Claim 3.2 there exists $X \in \mathcal{F}(\varepsilon)$ such that $X \perp \phi A\xi$. Taking such a vector $X$ and $Y = \phi X$ in (3.4), we obtain
\[ (\alpha + \lambda_\varepsilon) \left\{\lambda_\varepsilon A\xi + c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp \right) \right\} = 0 \tag{3.6} \]
for any $\varepsilon \in \{1, -1\}$.

**Claim 3.3.** $\|\xi^\perp\| = 1$ on $M$.

*Proof.* Suppose $0 < \|\xi^\perp\| < 1$ on the open subset $G$ of $M$. It is clear that $\alpha + \lambda_1$ and $\alpha + \lambda_{-1}$ cannot be both zero as $\lambda_1 - \lambda_{-1} = -2\alpha^{-1}\|\xi^\perp\| \neq 0$. Fix $\varepsilon \in \{1, -1\}$ such that $\alpha + \lambda_\varepsilon \neq 0$. It follows from (3.6) that
\[ \lambda_\varepsilon A\xi + c\left(\xi - \frac{\varepsilon}{\|\xi^\perp\|}\xi^\perp \right) = 0. \]
This implies that $A\xi \perp \mathcal{H}$, so $\mathcal{F}(1) = \mathcal{H}(1)$ and $\mathcal{F}(-1) = \mathcal{H}(-1)$. Taking $X \in \mathcal{H}(1)$ and $Y = \phi_2 X$ in (3.5), we can obtain a contradiction by using a similar method as
in the proof of Claim 3.2. Hence, \(\|\xi^\perp\| = 1\) at the point \(x\). By the connectedness of \(M\) and the continuity of \(\|\xi^\perp\|\), we conclude that \(\|\xi^\perp\| = 1\) on \(M\).

Since \(\|\xi^\perp\| = 1\) on \(M\) or \(\xi \in \mathcal{D}^\perp\) everywhere, we have \(\lambda_1 = 0\) and \(\lambda_{-1} = -2c/\alpha\) (= \(\lambda\), for simplicity). Moreover, we have

\[
-\sum_{a=1}^{3} \eta_a(\phi Y)\phi \xi_1 = \sum_{a=1}^{3} \eta_a(Y)\xi_a - \eta(Y)\xi. \tag{3.7}
\]

It follows from (3.1) and (3.7) that

\[
\alpha Y - \eta(AY)A\xi + c\{Y - 4\eta(Y)\xi - \theta Y\} + 2c \sum_{a=1}^{3} \eta_a(Y)\xi_a = 0. \tag{3.8}
\]

On the other hand, we have

\[
\sum_{a=1}^{3} \{g(Y, \nabla_x \phi \xi_a)\phi \xi_a - \eta_a(\phi Y)\nabla_x \phi \xi_a\} = \sum_{a=1}^{3} \{g(Y, \nabla_x \xi_a)\xi_a + \eta_a(Y)\nabla_x \xi_a\} - g(Y, \nabla_x \xi) - \eta(Y)\nabla_x \xi.
\]

By using (2.1), we have

\[
\sum_{a=1}^{3} \{g(Y, \theta_a AX)\phi \xi_a - \eta_a(\phi Y)\theta_a AX\} = \sum_{a=1}^{3} \{g(Y, \phi_a AX)\xi_a + \eta_a(Y)\phi_a AX\} - g(Y, \phi AX)\xi - \eta(Y)\phi AX. \tag{3.9}
\]

**Claim 3.4.** \(A\xi + 2c\xi \neq 0\).

**Proof.** Suppose \(A\xi + 2c\xi = 0\). Then \(A\xi = \alpha\xi\) and \(\alpha\lambda + 2c = 0\). It follows from (3.8) that

\[
AY = -\frac{c}{\alpha}(Y - \theta Y) + \frac{\alpha^2 + 4c}{\alpha} \eta(Y)\xi - \frac{2c}{\alpha} \sum_{a=1}^{3} \eta_a(Y)\xi_a.
\]

By [12] Theorem 6.1, we obtain

\[
\alpha^2 + 4c = 0 \tag{3.10}
\]

and so \(c < 0\). Furthermore, we have either

\[
\frac{-c}{\alpha} = \sqrt{-2c} \tanh\left(\sqrt{-2cr}\right), \quad \frac{-2c}{\alpha} = -\sqrt{-2c} \coth\left(\sqrt{-2cr}\right), \quad r > 0
\]

or

\[
\frac{-c}{\alpha} = \sqrt{\frac{-2c}{2}}, \quad \frac{-2c}{\alpha} = \sqrt{-2c}.
\]

However, both cases contradict (3.10). Accordingly, we obtain the claim. \(\square\)
By using Claim 3.4, (3.6) and (3.8), there exists a unit vector field $U$ tangent to $H(-1) \oplus (D_\perp \ominus \mathbb{R}\xi)$ and functions $\tau, \beta (\beta \neq 0)$ on $M$ such that

$$
\begin{align*}
A\xi &= \alpha \xi + \beta U \\
AU &= \beta \xi + \tau U \\
AY &= \lambda Y \quad (Y \in \mathcal{H}(-1)) \\
AX &= 0 \quad (X \in \mathcal{H}(1)) \\
\alpha \tau + \alpha^2 &= \beta^2 \\
\lambda + \alpha &= 0, \quad \alpha^2 = 2c.
\end{align*}
$$

By using (3.9) and (3.11), (3.2) is simplified as

$$
\begin{align*}
- \eta(AY)\{\alpha \phi AX + (X \beta)U + \beta \nabla_X U\} + \eta(AX)\{\alpha \phi AY + (Y \beta)U + \beta \nabla_Y U\} \\
+ c\{\eta(X)(4\phi AY + \alpha \phi Y) - \eta(Y)(4\phi AX + \alpha \phi X)\} \\
+ 2g(c(\phi + \phi^\perp)X - A\phi AX, Y)A\xi - cg(2\alpha \phi X + 4(\phi A + A\phi)X, Y)\xi \\
+ c \sum_{a=1}^3 \{\eta_a(X)(-2\phi_a AY + \alpha \phi_a Y) - \eta_a(Y)(-2\phi_a AX + \alpha \phi_a X) \\
- \alpha \eta_a(\phi Y)\theta_a X + \alpha \eta_a(\phi X)\theta_a Y + 2\eta_a(\phi AY)\theta_a X - 2\eta_a(\phi AX)\theta_a Y \\
- 2\eta_a(X)\eta_a(\phi Y)A\xi + 2\eta_a(Y)\eta_a(\phi X)A\xi \\
+ g(-2\alpha \phi_a X + (\phi_a A + A\phi_a)X, Y)\xi_a\} = 0.
\end{align*}
$$

On the other hand, by the Codazzi equation, we obtain

$$
\begin{align*}
ct(\phi + \phi^\perp)Y, Z - 2c \sum_{a=1}^3 \eta_a(Y)\eta_a(\phi Z) = g((\nabla_\xi A)Y - (\nabla_Y A)\xi, Z).
\end{align*}
$$

Substituting $Z = \xi$ in (3.13) gives

$$
(\xi \beta)g(U, Y) + \beta g(\nabla_\xi U, Y) + 4\beta \alpha g(\phi U, Y).
$$

Letting $Y = U$ in the preceding equation gives

$$
\xi \beta = 0 \quad (3.14)
$$

$$
\nabla_\xi U + 4\alpha \phi U = 0 \quad (3.15)
$$

Next, with the help of (3.11), after putting $Y \in \mathcal{H}(1)$ and $Z = U$ in (3.13) gives

$$
Y \beta = 0 \quad (3.16)
$$

for any $Y \in \mathcal{H}(1)$. By putting $X = \xi$ and $Y \perp \xi$ in (3.12), we have

$$
Y \beta + (\alpha^2 + 2\beta^2)g(Y, \phi U) + 3cg(Y, \phi U - \phi^\perp U) + \sum_{a=1}^3 \{-2\alpha \eta_a(\phi AY)\eta_a(U) \\
- \alpha \beta g(\phi_a Y, U)\eta_a(U) - \alpha \beta g(\theta_a Y, U)\eta_a(\phi U) - 2c\alpha \eta_a(U)\eta_a(\phi U)\} = 0
$$

(3.17)
for any $Y \perp \xi$. In particular, for $Y \in \mathcal{H}(1)$, with the help of (3.16) we obtain
\[
0 = -3 \sum_{a=1}^{3} \{g(\phi_a Y, U)\eta_a(U) - g(\theta_a Y, U)\eta_a(\phi U)\} = 2 \sum_{a=1}^{3} \eta_a(U)g(\phi_a U, Y)
\]
for any $Y \in \mathcal{H}(1)$.

Denote by $U^-$ the $\mathcal{H}(-1)$-component of $U$. If $U$ is tangent to $\mathcal{D}^\perp$ on an open subset $G$ of $M$. Then for each $x \in G$, $\mathcal{F}(-1) = \mathcal{H}(-1)$ and so $A_\mathcal{D}^\perp \subset \mathcal{D}^\perp$. By virtue of [12, Theorem 3.6], $\xi$ is principal on $G$. This contradicts Claim 3.4. Hence, we assume that $U^- \neq 0$. By putting $Y = \phi_b U^-$, $b \in \{2, 3\}$ in the preceding equation, we obtain $\eta_2(U) = \eta_3(U) = 0$. Consequently, $U = U^- \in \mathcal{H}(-1)$ and $\phi U = \phi^\perp U$. These, together with (3.14) and (3.17), give
\[
Y\beta = - (\alpha^2 + 2\beta^2)g(Y, \phi U)
\]
for any vector field $Y$ tangent to $M$. It follows that
\[
(XY - \nabla_X Y)\beta = (\alpha^2 + 2\beta^2)\{4\beta g(X, \phi U)g(Y, \phi U) - g(Y, \nabla_X \phi U)\}.
\]
Hence
\[
g(Y, \nabla_X \phi U) - g(X, \nabla_Y U) = 0.
\]
By virtue of (3.11) and (3.15), after substituting $X = \xi$ and $Y = U$ in the preceding equation we get $4\alpha + \tau = 0$. But then $\beta^2 = \alpha\tau + \alpha^2 = -3\alpha^2$, a contradiction. This completes the proof.

References


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