

MATRIX MOMENT PERTURBATIONS AND THE INVERSE SZEGŐ MATRIX TRANSFORMATION

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ABSTRACT. Given a perturbation of a matrix measure supported on the unit circle, we analyze the perturbation obtained on the corresponding matrix measure on the real line, when both measures are related through the Szegő matrix transformation. Moreover, using the connection formulas for the corresponding sequences of matrix orthogonal polynomials, we deduce several properties such as relations between the corresponding *norms*. We illustrate the obtained results with an example.

1. INTRODUCTION

The term *moment problem* was used for the first time in T. J. Stieltjes' classic memoir [32] (published posthumously between 1894 and 1895) dedicated to the study of continued fractions. The moment problem is a question in classical analysis that has produced a rich theory in applied and pure mathematics. This problem is beautifully connected to the theory of orthogonal polynomials, spectral representation of operators, matrix factorization problems, probability, statistics, prediction of stochastic processes, polynomial optimization, inverse problems in financial mathematics and function theory, among many other areas. In the matrix case, M. Krein was the first to consider this problem in [21], and later on some density questions related to the matrix moment problem were addressed in [14, 15, 24, 25]. Recently, the theory of the matrix moment problem is used in [10] for the analysis of random matrix-valued measures. Since the matrix moment problem is closely related to the theory of matrix orthogonal polynomials, M. Krein was the first to consider these polynomials in [22]. Later, several researchers have made contributions to this theory until today. In the last 30 years, several known properties of orthogonal polynomials in the scalar case have been extended to the matrix case, such as algebraic aspects related to their zeros, recurrence relations, Favard type theorems, and Christoffel–Darboux formulas, among many others.

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A nice summary, as well as many references, can be found in [8]. As in the scalar case, matrix orthogonal polynomials have proved to be a useful tool in the analysis of many problems of mathematics such as differential equations [13, 28], rational approximation theory [20], spectral theory of Jacobi matrices [1, 29], analysis of polynomial sequences satisfying higher order recurrence relations [11, 17], quantum random walks [3], and Gaussian quadrature formulas [2, 12, 16, 31], among many others. In this contribution, we are interested in the study of some properties related with a perturbation of a sequence of matrix moments, within the framework of the theory of matrix orthogonal polynomials both on the real line and on the unit circle.

The structure of the manuscript is as follows. In Section 2, we state some properties of matrix orthogonal polynomials on the real line and on the unit circle that will be used in the sequel. Section 3 deals with the so-called Szegő matrix transformation and some of its properties. This has been previously studied in [8, 9, 33]. Then, we introduce a perturbation on a matrix measure supported on the unit circle that results in a perturbation of the corresponding sequence of matrix moments, and analyze the resulting perturbation on the associated matrix measure on the real line, when both measures are related through the Szegő transformation. This is the main contribution of our manuscript. Sections 4 and 5 deal with connection formulas and some other properties associated with the corresponding matrix orthogonal polynomials sequences on the real line and on the unit circle, respectively. Connection formulas for perturbations of matrix moments have been studied previously in [7, 6]. Finally, we illustrate the obtained results with an example in Section 6.

2. MATRIX ORTHOGONAL POLYNOMIALS

We will use the following notation. Let $\mathcal{M}_l = \mathcal{M}_l(\mathbb{C})$ be the ring of $l \times l$ matrices with complex entries. The matrices \mathbf{I} and $\mathbf{0}$ will denote the $l \times l$ identity matrix and zero matrix, respectively. An Hermitian matrix $A \in \mathcal{M}_l$ satisfies $A = A^H$ (its conjugate transpose), and the set of Hermitian matrices is a real vector space. We say A is positive definite if $u^H A u > 0$ for every nonzero vector $u \in \mathbb{C}^l$, and positive semi-definite if the inequality is not strict. For $0 \leq k \leq j - 1$, let

$$\chi_k^{(j)} := [\mathbf{0} \cdots \mathbf{0} \mathbf{I} \mathbf{0} \cdots \mathbf{0}]^H \in \mathcal{M}_{jl \times l},$$

where \mathbf{I} is in the k -th position, and define $\chi_k^{(j)} := [\mathbf{0}, \dots, \mathbf{0}]^H$ for $j \geq k$ or $k < 0$. We also define

$$\Lambda_{n,m}^{(j+1)} := [\chi_n^{(j+1)} \chi_{n+1}^{(j+1)} \cdots \chi_m^{(j+1)}] \in \mathcal{M}_{(j+1)l \times (m-n+1)l},$$

with $n \leq m$, for $n, m \in \mathbb{Z}$.

An $l \times l$ matrix-valued measure (or simply, matrix measure) $\sigma = (\sigma_{i,j})_{i,j=1}^l$ supported on $E \subseteq \mathbb{R}$ is said to be Hermitian (resp. positive semi-definite, positive definite), if for every Borel subset \mathcal{B} on E , the matrix $\sigma(\mathcal{B})$ is a Hermitian (resp. positive semi-definite, positive definite) matrix. Notice that the assumption of positive definiteness implies the Hermitian character of the measure. For more information about matrix measures, see [8]

2.1. Matrix orthogonal polynomials on the real line. Let $\mu = (\mu_{i,j})_{i,j=1}^l$ be a Hermitian matrix measure supported on $E \subseteq \mathbb{R}$ and denote by $\mathbb{P}^l[x]$ the linear space of polynomials in $x \in \mathbb{R}$ with coefficients in \mathcal{M}_l . If the scalar measures $\mu_{i,j}$, $i, j = 1, \dots, l$, are absolutely continuous with respect to the scalar Lebesgue measure dx , then according to the Radon–Nikodym theorem, it can always be expressed as $d\mu_{i,j}(x) = \omega_{i,j}(x)dx$. Since μ is Hermitian, the matrix $\omega(x) = (\omega_{i,j}(x))_{i,j=1}^l$ is also a Hermitian matrix, and thus

$$d\mu(x) = \omega(x)dx. \tag{2.1}$$

Notice that μ induces right and left sesquilinear forms, defined respectively by

$$\langle f, g \rangle_{R,\mu} = \int_E f(x)^H d\mu(x)g(x) \in \mathcal{M}_l, \quad f, g \in \mathbb{P}^l[x], \tag{2.2}$$

$$\langle f, g \rangle_{L,\mu} = \int_E f(x)d\mu(x)g(x)^H \in \mathcal{M}_l, \quad f, g \in \mathbb{P}^l[x]. \tag{2.3}$$

On the other hand, the j -th moment associated to the matrix measure μ is defined by

$$\mu_j = \langle \mathbf{I}, x^j \mathbf{I} \rangle_{R,\mu} = \langle x^j \mathbf{I}, \mathbf{I} \rangle_{L,\mu} = \int_E x^j d\mu(x) \in \mathcal{M}_l, \quad j \geq 0.$$

Notice that every moment is a Hermitian matrix, i.e., $\mu_j^H = \mu_j$ for $j \geq 0$, since μ is Hermitian. We denote by $H_j \in \mathcal{M}_{lj}$ and $F_j \in \mathcal{M}_{l(j+1)}$ the block matrices

$$H_j = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{j-1} \\ \mu_1 & \mu_2 & \cdots & \mu_j \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{j-1} & \mu_j & \cdots & \mu_{2j-2} \end{bmatrix}, \quad F_j = \begin{bmatrix} & & & \mu_j \\ & & & \vdots \\ & H_j & & \\ & & & \mu_{2j-1} \\ \mathbf{I} & x\mathbf{I} & \cdots & x^j \mathbf{I} \end{bmatrix},$$

with $x \in \mathbb{R}$ and $j \geq 1$, and the block vector $v_{j,2j-1} = [\mu_j \ \mu_{j+1} \ \cdots \ \mu_{2j-1}]^H \in \mathcal{M}_{lj \times l}$. Notice that H_j is a Hankel invertible and Hermitian matrix for $j \geq 1$. With these matrices, it is shown in [27] that the sequence of (right) matrix polynomials $(P_j^R(x))_{j \geq 0}$ defined by

$$P_j^R(x) = x^j \mathbf{I} - [\mathbf{I} \ x\mathbf{I} \ \cdots \ x^{j-1} \mathbf{I}] H_j^{-1} v_{j,2j-1}, \quad \text{with } P_0^R(x) = \mathbf{I} \tag{2.4}$$

(i.e., each P_j^R is defined as the Schur complement of $x^j \mathbf{I}$ in the matrix F_j), is orthogonal with respect to the sesquilinear form (2.2), that is,

$$\langle P_m^R, P_j^R \rangle_{R,\mu} = \delta_{m,j} s_j, \quad m = 0, \dots, j,$$

where s_j is the Schur complement of H_{j+1} with respect to μ_{2j} , i.e.

$$s_j = \mu_{2j} - v_{j,2j-1}^H H_j^{-1} v_{j,2j-1}, \quad \text{with } s_0 = \mu_0. \tag{2.5}$$

Conversely, any sequence of matrix moments $(\mu_j)_{j \geq 0}$ satisfying that the corresponding Hankel matrices are invertible and Hermitian defines a Hermitian matrix measure. In this situation, we will say $(\mu_j)_{j \geq 0}$ is a Hermitian sequence. The positiveness of μ can also be characterized in terms of the norms $s_j, j \geq 0$, as follows.

Theorem 2.1 ([27]). *Let μ be a matrix measure supported on $E \subseteq \mathbb{R}$. Then, μ is positive semi-definite for every $x \in E$ if and only if s_j is positive semi-definite for every $j \geq 0$.*

In a similar way, a sequence of left matrix polynomials $(P_n^L(x))_{n \geq 0}$ can be defined such that it is orthogonal with respect to the sesquilinear form (2.3). Since $P_j^L(x) = P_j^R(x)^H$ for all $x \in \mathbb{R}$, it suffices to consider $P_j(x) = P_j^R(x)$.

Let $P_j^{(k)}(x)$ be the k -th derivative of $P_j(x)$. From (2.4) we can deduce

$$\frac{P_j^{(k)}(0)}{k!} = \begin{cases} -(\chi_k^{(j)})^H H_j^{-1} v_{j,2j-1}, & \text{if } 0 \leq k \leq j - 1, \\ \mathbf{I}, & \text{if } k = j. \end{cases} \tag{2.6}$$

The kernel matrix polynomial of degree j for $x, y \in \mathbb{R}$ is defined by

$$K_j(y, x) = \sum_{k=0}^j P_k(x) s_k^{-1} P_k(y)^H,$$

which can be expressed also in terms of the moments by (see [27])

$$K_j(y, x) = [\mathbf{I} \ x \mathbf{I} \ \dots \ x^j \mathbf{I}] H_{j+1}^{-1} [\mathbf{I} \ y \mathbf{I} \ \dots \ y^j \mathbf{I}]^T. \tag{2.7}$$

Denoting $K_j^{(i,r)}(y, x)$ the i -th (resp. r -th) derivative of $K_j(y, x)$ with respect to the variable y (resp. x), we have from (2.7), for $0 \leq i \leq j$ and $0 \leq r \leq j$,

$$\begin{aligned} \frac{K_j^{(i,0)}(0, x)}{i!} &= [\mathbf{I} \ x \mathbf{I} \ \dots \ x^j \mathbf{I}] H_{j+1}^{-1} \chi_i^{(j+1)}, \\ \frac{K_j^{(0,r)}(y, 0)}{r!} &= (\chi_r^{(j+1)})^H H_{j+1}^{-1} [\mathbf{I} \ y \mathbf{I} \ \dots \ y^j \mathbf{I}]^T, \\ \frac{K_j^{(i,r)}(0, 0)}{i!r!} &= (\chi_r^{(j+1)})^H H_{j+1}^{-1} \chi_i^{(j+1)}. \end{aligned}$$

2.2. Matrix orthogonal polynomials on the unit circle. Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle and unit disc, respectively, where $z \in \mathbb{T}$ is parametrized as $z = e^{i\theta}$ with $\theta \in (-\pi, \pi]$. In what follows, $\sigma = (\sigma_{i,j})_{i,j=1}^l$ will be a Hermitian matrix measure supported on \mathbb{T} (on all or part of \mathbb{T}) and $\mathbb{P}^l[z]$ will denote the linear space of polynomials in $z \in \mathbb{C}$ with coefficients in \mathcal{M}_l . A Hermitian matrix measure σ can be decomposed as

$$d\sigma(\theta) = \mathcal{W}(\theta) \frac{d\theta}{2\pi} + d\sigma_s(\theta),$$

where the Hermitian matrix $\mathcal{W}(\theta)$ denotes the Radon–Nikodym derivative of the absolutely continuous part of σ with respect to the scalar Lebesgue measure, and σ_s represents the singular part.

As in the real line case, a matrix measure σ induces right and left sesquilinear forms, defined respectively as

$$\langle p, q \rangle_{R,\sigma} = \int_{-\pi}^{\pi} p(e^{i\theta})^H d\sigma(\theta) q(e^{i\theta}) \in \mathcal{M}_l, \quad p, q \in \mathbb{P}^l[z], \tag{2.8}$$

$$\langle p, q \rangle_{L,\sigma} = \int_{-\pi}^{\pi} p(e^{i\theta}) d\sigma(\theta) q(e^{i\theta})^H \in \mathcal{M}_l, \quad p, q \in \mathbb{P}^l[z]. \tag{2.9}$$

On the other hand, the j -th (matrix) moment associated with the matrix measure σ is

$$c_j = \langle \mathbf{I}, z^j \mathbf{I} \rangle_{R,\sigma} = \langle z^j \mathbf{I}, \mathbf{I} \rangle_{L,\sigma} = \int_{-\pi}^{\pi} e^{ij\theta} d\sigma(\theta) \in \mathcal{M}_l, \quad j \in \mathbb{Z}, j \neq 0,$$

and $c_0 + c_0^H := \int_{-\pi}^{\pi} d\sigma(\theta)$. Notice that every matrix moment satisfies $c_{-j}^H = c_j$ for $j \in \mathbb{Z}$, since σ is an Hermitian measure. Let $T_j \in \mathcal{M}_{l_j}$ and $M_j \in \mathcal{M}_{l(j+1)}$ be defined as

$$T_j = \begin{bmatrix} c_0^H + c_0 & c_1 & \cdots & c_{j-1} \\ c_1^H & c_0^H + c_0 & \cdots & c_{j-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{j-1}^H & c_{j-2}^H & \cdots & c_0^H + c_0 \end{bmatrix}, \quad G_j = \begin{bmatrix} & & & c_j \\ & & & \vdots \\ & T_j & & c_1 \\ \mathbf{I} & z\mathbf{I} & \cdots & z^j \mathbf{I} \end{bmatrix},$$

and the vector block $\nu_j = [c_j^H \ c_{j-1}^H \ \cdots \ c_1^H]^H \in \mathcal{M}_{l_j \times l}$. Notice that T_j is a Toeplitz invertible and Hermitian matrix for $j \geq 1$. As in the real line case, any sequence $(c_j)_{j \in \mathbb{Z}}$ such that T_j is invertible and Hermitian for $j \geq 1$ will be called a Hermitian sequence, and has an associated Hermitian matrix measure σ supported on \mathbb{T} . The sequence of monic matrix polynomials $(\Phi_j^R(z))_{j \geq 0}$ defined by (see [26])

$$\Phi_j^R(z) = z^j \mathbf{I} - [\mathbf{I} \ z\mathbf{I} \ \cdots \ z^{j-1} \mathbf{I}] T_j^{-1} \nu_j, \quad \text{with } \Phi_0^R(z) = \mathbf{I}, \tag{2.10}$$

is orthogonal with respect to the sesquilinear form (2.8) and satisfies

$$\langle \Phi_m^R, \Phi_j^R \rangle_{R,\sigma} = \delta_{m,j} S_j^R, \quad m = 0, \dots, j,$$

where the Hermitian matrix S_j^R is the Schur complement of T_{j+1} with respect to $c_0 + c_0^H$, i.e.,

$$S_j^R = c_0^H + c_0 - \nu_j^H T_j^{-1} \nu_j, \quad \text{with } S_0^R = c_0 + c_0^H. \tag{2.11}$$

In a similar way, a sequence $(\Phi_n^L(z))_{n \geq 0}$ can be constructed such that it is orthogonal with respect to the sesquilinear form (2.9). Unlike the real line case, where the right-orthogonal matrix polynomial is just the conjugate transpose of the left-orthogonal matrix polynomial, on the unit circle there is not a direct relation between the right and left-orthogonal polynomials. This is a consequence of the fact that $\langle zf, g \rangle_{R,\sigma} = \langle f, \bar{z}g \rangle_{R,\sigma}$. Nevertheless, we will consider only the right matrix polynomials, denoted by $\Phi_j = \Phi_j^R$ and the corresponding norms $S_j = S_j^R$ for $j \geq 0$, since the results for the left polynomials are analogous.

Let $\Phi_j^{(k)}(x)$ be the k -th derivative of $\Phi_j(x)$. From (2.10) we can deduce

$$\frac{\Phi_j^{(k)}(0)}{k!} = \begin{cases} -(\chi_k^{(j)})^H T_j^{-1} \nu_j, & \text{if } 0 \leq k \leq j - 1, \\ \mathbf{I}, & \text{if } k = j. \end{cases} \tag{2.12}$$

Theorem 2.1 is valid also for the matrix measure σ with support on \mathbb{T} . The kernel matrix polynomial of degree j for $z, w \in \mathbb{C}$ is defined in this case by

$$K_j(w, z) = \sum_{k=0}^j \Phi_k(z) S_k^{-1} \Phi_k(w)^H,$$

and its corresponding expression in terms of the matrix moments is (see [26])

$$K_j(w, z) = [\mathbf{I} \ z \mathbf{I} \ \dots \ z^j \mathbf{I}] T_{j+1}^{-1} [\mathbf{I} \ \bar{w} \mathbf{I} \ \dots \ \bar{w}^j \mathbf{I}]^T. \tag{2.13}$$

As in the real case, $K_j^{(i,r)}(w, z)$ is the i -th (resp. r -th) derivative of $K_j(w, z)$ with respect to the variable w (resp. z). Thus, from (2.13), for $0 \leq i \leq j$ and $0 \leq r \leq j$, we have

$$\begin{aligned} \frac{K_j^{(i,0)}(0, z)}{i!} &= [\mathbf{I} \ x \mathbf{I} \ \dots \ z^j \mathbf{I}] T_{j+1}^{-1} \chi_i^{(j+1)}, \\ \frac{K_j^{(0,r)}(w, 0)}{r!} &= (\chi_r^{(j+1)})^H T_{j+1}^{-1} [\mathbf{I} \ \bar{w} \mathbf{I} \ \dots \ \bar{w}^j \mathbf{I}]^T, \\ \frac{K_j^{(i,r)}(0, 0)}{i!r!} &= (\chi_r^{(j+1)})^H T_{j+1}^{-1} \chi_i^{(j+1)}. \end{aligned}$$

3. MATRIX MOMENTS PERTURBATION AND ITS CONNECTION THROUGH THE INVERSE SZEGŐ TRANSFORMATION

In this section, we define the Szegő matrix transformation of a matrix measure supported on the real line, as well as some of its properties. Later on, we will define a perturbation of a matrix measure supported on \mathbb{T} that consists in the perturbation of a single matrix moment of such measure, and analyze the resulting perturbation on the real line when the measures are related through the Szegő matrix transformation.

3.1. Szegő matrix transformation. A matrix measure $\sigma(\theta)$ supported on \mathbb{T} that is invariant under $\theta \mapsto -\theta$ (i.e., $z \mapsto \bar{z} = z^{-1}$) is called *symmetric*. If $\sigma(\theta)$ is symmetric and positive semi-definite, then its associated matrix moments are Hermitian matrices, since

$$c_n = \int_{-\pi}^{\pi} e^{in\theta} d\sigma(\theta) = \int_{-\pi}^{\pi} e^{-in\theta} d\sigma(\theta) = c_{-n} = c_n^H, \quad \text{for } n \in \mathbb{Z}.$$

On the other hand, if $\mu(x)$ is a positive semi-definite matrix measure supported on $[-1, 1]$, it is possible to define a symmetric positive semi-definite matrix measure

$\sigma(\theta)$ supported on $(-\pi, \pi]$ by

$$d\sigma(\theta) = \begin{cases} \frac{1}{2}d\mu(\cos \theta), & -\pi \leq \theta \leq 0, \\ -\frac{1}{2}d\mu(\cos \theta), & 0 \leq \theta \leq \pi. \end{cases} \tag{3.1}$$

This is the so-called Szegő matrix transformation (see [8, 9, 33]). The scalar version of the Szegő transformation can be found in [30]. We will write $d\sigma = Sz(d\mu)$ if the matrix measures μ and σ satisfy

$$\int_{-1}^1 f(x) d\mu(x) = \int_{-\pi}^{\pi} f(\cos \theta) d\sigma(\theta)$$

for any integrable function f on $[-1, 1]$. In a similar way, the inverse Szegő matrix transformation can be defined by the relation

$$\int_{-\pi}^{\pi} g(\theta) d\sigma(\theta) = \int_{-1}^1 g(\arccos x) d\mu(x)$$

for any integrable function g on \mathbb{T} such that $g(\theta) = g(-\theta)$. We will denote this relation by $d\mu = Sz^{-1}(d\sigma)$. Notice that if μ has the form (2.1), then (3.1) becomes

$$d\sigma(\theta) = \begin{cases} -\frac{1}{2}\omega(\cos \theta) \sin \theta d\theta, & -\pi \leq \theta \leq 0, \\ \frac{1}{2}\omega(\cos \theta) \sin \theta d\theta, & 0 \leq \theta \leq \pi, \end{cases}$$

which can be written as

$$d\sigma(\theta) = \frac{1}{2}|\sin \theta|\omega(\cos \theta) d\theta.$$

If $d\mu(x) = \omega(x) dx$, and $d\sigma(\theta) = \mathcal{W}(\theta)\frac{d\theta}{2\pi}$, then they are related through the Szegő matrix transformation when

$$\mathcal{W}(\theta) = \pi|\sin \theta|\omega(\cos \theta), \quad \omega(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \mathcal{W}(\arccos x), \tag{3.2}$$

$$Sz \left(\frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} \mathcal{W}(\arccos x) dx \right) = \pi|\sin \theta|\omega(\cos \theta) \frac{d\theta}{2\pi}. \tag{3.3}$$

In particular, for $\mathcal{W}(\theta) = \cos^k \theta$, with $k \geq 0$, we have

$$Sz^{-1} \left(\cos^k \theta \frac{d\theta}{2\pi} \right) = \frac{1}{\pi} \frac{x^k}{\sqrt{1-x^2}} dx. \tag{3.4}$$

The proof is analogous to the one given in [30, Proposition 13.1.1].

3.2. Matrix moments on real line and the unit circle related through the inverse Szegő transformation. Let $\langle \cdot, \cdot \rangle_{R,\theta}$ be a right sesquilinear form defined by

$$\langle p, q \rangle_{R,\theta} = \int_{-\pi}^{\pi} p(e^{i\theta})^H q(e^{i\theta}) \frac{d\theta}{2\pi}, \quad p, q \in \mathbb{P}^l[z].$$

Given a positive semi-definite matrix measure σ supported on \mathbb{T} , we can consider the perturbation

$$d\sigma_j(\theta) = d\sigma(\theta) + z^j m_j \frac{d\theta}{2\pi} + z^{-j} m_j^H \frac{d\theta}{2\pi}, \quad z = e^{i\theta}, \tag{3.5}$$

for a fixed j and $m_j \in \mathcal{M}_l$ such that σ_j is positive semi-definite. In particular, if σ is symmetric and m_j is a Hermitian matrix, then (3.5) becomes

$$d\sigma_j(\theta) = d\sigma(\theta) + 2 \cos(j\theta) m_j \frac{d\theta}{2\pi}, \tag{3.6}$$

and thus σ_j is a symmetric, positive semi-definite matrix measure supported on the unit circle.

Notice that the right sesquilinear form associated with (3.5) is

$$\langle p, q \rangle_{R, \sigma_j} = \langle p, q \rangle_{R, \sigma} + \langle m_j^H p, z^j q \rangle_{R, \theta} + \langle z^j m_j p, q \cdot \rangle_{R, \theta}, \quad p, q \in \mathbb{P}^l[z]. \tag{3.7}$$

From (3.7), one easily sees that the corresponding matrix moment \tilde{c}_k satisfies

$$\tilde{c}_k = \begin{cases} c_k, & \text{if } k \notin \{j, -j\}, \\ c_j + m_j^H, & \text{if } k = j, \\ c_{-j} + m_j, & \text{if } k = -j. \end{cases}$$

In other words, σ_j represents an additive perturbation of the matrix moments c_j and c_{-j} of σ .

The following result shows the matrix measure supported on $[-1, 1]$ that is related with (3.5) through the inverse Szegő matrix transformation. This result constitutes a generalization of the corresponding result on the scalar case studied in [18, 19].

Proposition 3.1. *Let σ be a positive semi-definite matrix measure supported on \mathbb{T} , and let $m_j \in \mathcal{M}_l$ be a Hermitian matrix such that σ_j defined as in (3.5) is positive semi-definite. Assume $d\mu = Sz^{-1}(d\sigma)$. Then, $d\mu_j = Sz^{-1}(d\sigma_j)$ is given by*

$$d\mu_j(x) = d\mu(x) + 2T_j(x)m_j \frac{dx}{\pi\sqrt{1-x^2}}, \quad j \geq 0, \tag{3.8}$$

where $T_j(x) := \cos(j\theta)$ is the j -th degree Chebyshev polynomial of the first kind on \mathbb{R} (see [5]).

Proof. Let $T_j(x) = \sum_{k=0}^j a_k x^k$, $x = \cos \theta$, $z = e^{i\theta} = \cos \theta + i \sin \theta$, and m_j an Hermitian matrix. Then, from (3.5),

$$\begin{aligned} d\sigma_j(\theta) &= d\sigma(\theta) + (\cos \theta + i \sin \theta)^j m_j \frac{d\theta}{2\pi} + (\cos \theta + i \sin \theta)^{-j} m_j \frac{d\theta}{2\pi} \\ &= d\sigma(\theta) + 2T_j(x)m_j \frac{d\theta}{2\pi} \\ &= d\sigma(\theta) + 2 \sum_{k=0}^j a_k \cos^k \theta m_j \frac{d\theta}{2\pi}. \end{aligned}$$

Since $\sigma_j(\theta)$ is a symmetric matrix measure, by applying the inverse Szegő matrix transformation and taking into account (3.4), we have

$$\begin{aligned} d\mu_j(x) &= d\mu(x) + 2 \sum_{k=0}^j a_k x^k m_j \frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\mu(x) + 2T_j(x)m_j \frac{dx}{\pi\sqrt{1-x^2}}, \end{aligned}$$

which is (3.8). □

Recall that the double factorial or semifactorial of a number n (denoted by $n!!$) is the product of all the integers from 1 up to n that have the same parity (odd or even) as n . That is,

$$n!! = \begin{cases} \prod_{k=1}^{n/2} (2k) = n(n-2)(n-4)\cdots(4)(2), & \text{if } n \text{ is even,} \\ \prod_{k=1}^{(n+1)/2} (2k-1) = n(n-2)(n-4)\cdots(3)(1), & \text{if } n \text{ is odd.} \end{cases}$$

The next proposition expresses the considered perturbation in terms of the moments associated with the matrix measure (3.8).

Proposition 3.2. *Let μ be a positive semi-definite matrix measure supported on $[-1, 1]$, μ_j defined as in (3.8), and let $(\mu_n)_{n \geq 0}$, $(\tilde{\mu}_n)_{n \geq 0}$ be their corresponding sequences of matrix moments. Then,*

$$\tilde{\mu}_n = \begin{cases} \mu_n + m_j \mathbb{B}(n, j), & \text{if } j \leq n \text{ and } n + j \text{ is even,} \\ \mu_n, & \text{otherwise,} \end{cases} \tag{3.9}$$

where

$$\mathbb{B}(n, j) = j \sum_{k=0}^{[j/2]} (-1)^k \frac{(j-k-1)!(2)^{j-2k} (j+n-2k-1)!!}{k!(j-2k)! (j+n-2k)!!}.$$

Proof. Since $T_j(x)$ is orthogonal with respect to $\frac{1}{\sqrt{1-x^2}}$, we have

$$\tilde{\mu}_n = \int_{-1}^1 x^n d\tilde{\mu}_j(x) = \begin{cases} \mu_n + \frac{2}{\pi} m_j \int_{-1}^1 x^n T_j(x) \frac{dx}{\sqrt{1-x^2}}, & \text{if } j \leq n, \\ \mu_n, & \text{otherwise.} \end{cases}$$

Moreover, from the symmetry, $\int_{-1}^1 x^n T_j(x) \frac{dx}{\sqrt{1-x^2}} = 0$ if $n + j$ is odd, and thus

$$\tilde{\mu}_n = \begin{cases} \mu_n + \frac{2}{\pi} m_j \int_{-1}^1 x^n T_j(x) \frac{dx}{\sqrt{1-x^2}}, & \text{if } j \leq n \text{ and } n + j \text{ is even,} \\ \mu_n, & \text{otherwise.} \end{cases} \tag{3.10}$$

Furthermore (see [23]), we have

$$T_j(x) = \frac{j}{2} \sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)!(2x)^{j-2k}}{k!(j-2k)!}, \quad j \geq 1,$$

where $[j/2] = j/2$ if j is even and $[j/2] = (j - 1)/2$ if j is odd. As a consequence, if $j \leq n$ and $n + j$ is even,

$$\begin{aligned} \int_{-1}^1 x^n T_j(x) \frac{dx}{\sqrt{1-x^2}} &= \frac{j}{2} \int_{-1}^1 x^n \sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)! (2x)^{j-2k}}{k!(j-2k)!} \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{j}{2} \sum_{k=0}^{[j/2]} \frac{(-1)^k (j-k-1)! (2)^{j-2k}}{k!(j-2k)!} \int_{-1}^1 x^{j+n-2k} \frac{dx}{\sqrt{1-x^2}}, \end{aligned}$$

and, since

$$\int_{-1}^1 x^n \frac{dx}{\sqrt{1-x^2}} = \pi \frac{(n-1)!!}{n!!}$$

for n even, and equals zero if n is odd, (3.10) becomes (3.9). □

This means that the perturbation of the matrix moments c_j and c_{-j} of a matrix measure σ supported on the unit circle results in a perturbation, defined by (3.9), of the moments μ_n associated with a measure μ supported in $[-1, 1]$, when both measures are related through the inverse matrix Szegő transformation.

4. CONNECTION FORMULAS ON THE REAL LINE AND FURTHER RESULTS

Let μ be a Hermitian matrix measure supported on $E \subseteq \mathbb{R}$ with an associated sequence of monic matrix orthogonal polynomials $(P_n(x))_{n \geq 0}$, and a Hermitian sequence of matrix moments $(\mu_n)_{n \geq 0}$. Define a new matrix moments sequence $(\tilde{\mu}_n)_{n \geq 0}$ by $\tilde{\mu}_n = \mu_n + \mathbf{M}_n \in \mathcal{M}_l$ with \mathbf{M}_n Hermitian, in such a way that $(\tilde{\mu}_n)_{n \geq 0}$ is also a Hermitian sequence. Then, there exists an associated measure $\tilde{\mu}$ that is Hermitian, with a corresponding sequence of matrix orthogonal polynomials $(\tilde{P}_n(x))_{n \geq 0}$. Also, define

$$s_j = \langle P_j, P_j \rangle_{R, \mu} \quad \text{and} \quad \tilde{s}_j = \langle \tilde{P}_j, \tilde{P}_j \rangle_{R, \tilde{\mu}}.$$

In this section, we will show the relation between the matrix polynomials \tilde{P}_j and P_j , as well as the relation between \tilde{s}_j and s_j , in terms of the sequence of matrix moments $(\mu_n)_{n \geq 0}$. Notice that the perturbation of a matrix measure μ supported on $E = [-1, 1]$ defined by (3.8) constitutes a particular case with

$$\mathbf{M}_n = \begin{cases} m_j \mathbb{B}(n, j), & \text{if } j \leq n \text{ and } n + j \text{ is even,} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

We will use the following notation:

$$\begin{aligned}
 \mathbf{K}_{j-1}^k(0, x) &:= \left[K_{j-1}^{(k,0)}(0, x), K_{j-1}^{(k-1,0)}(0, x), \dots, K_{j-1}^{(0,0)}(0, x) \right] \in \mathcal{M}_{l \times (k+1)l}, \\
 \mathbf{P}_{j,k}(0) &:= [P_j^{(0)}(0)^T, P_j^{(1)}(0)^T, \dots, P_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l \times l}, \\
 \tilde{\mathbf{P}}_{j,k}(0) &:= [\tilde{P}_j^{(0)}(0)^T, \tilde{P}_j^{(1)}(0)^T, \dots, \tilde{P}_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l \times l}, \\
 \mathbf{D}_{0,k} &:= \text{diag}\{0!\mathbf{I}, 1!\mathbf{I}, \dots, k!\mathbf{I}\} \in \mathcal{M}_{(k+1)l}, \\
 \mathbf{D}_{k,0} &:= \text{diag}\{k!\mathbf{I}, (k-1)!\mathbf{I}, \dots, 0!\mathbf{I}\} \in \mathcal{M}_{(k+1)l}, \\
 \mathbf{D}_{\mathbf{M}_k} &:= \text{diag}\{\mathbf{M}_k, \dots, \mathbf{M}_k\} \in \mathcal{M}_{(k+1)l}.
 \end{aligned} \tag{4.1}$$

For a fixed k , with $0 \leq k \leq 2j - 1$, it has been proved in [7] that

$$\begin{aligned}
 \tilde{P}_j(x) &= P_j(x) - \sum_{p=0}^k \frac{K_{j-1}^{(k-p,0)}(0, x)}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!} \\
 &= P_j(x) - \mathbf{K}_{j-1}^k(0, x) \mathbf{D}_{k,0}^{-1} \mathbf{D}_{\mathbf{M}_k} \mathbf{D}_{0,k}^{-1} \tilde{\mathbf{P}}_{j,k}(0),
 \end{aligned}$$

with

$$\tilde{\mathbf{P}}_{j,k}(0) = \left[\mathbf{I}_{(k+1)l} + \mathbf{D}_{0,k}(\Lambda_{0,k}^{(j)})^H H_j^{-1} \Lambda_{0,k}^{(j)} \mathbf{D}_{k,0}^{-1} \mathbf{D}_{\mathbf{M}_k} \Lambda_{k,0}^{(k+1)} \right]^{-1} \mathbf{P}_{j,k}(0). \tag{4.2}$$

As a consequence we have the next result.

Theorem 4.1 ([7]). *Let μ and $\tilde{\mu}$ be Hermitian matrix measures associated with the Hermitian sequences $(\mu_k)_{k=0}^{2n-1}$ and $(\tilde{\mu}_k)_{k=0}^{2n-1}$, respectively, where $\tilde{\mu}_k = \mu_k + \mathbf{M}_k$, with $0 \leq k \leq 2n - 1$. For $1 \leq j \leq n$, $(\tilde{P}_j(x))_{j=0}^n$ is orthogonal with respect to $\tilde{\mu}$ if*

$$\tilde{P}_j(x) = P_j(x) - \sum_{k=0}^{2n-1} \mathbf{K}_{j-1}^k(0, x) \mathbb{M}(k, j) \mathbf{P}_{j,k}(0), \quad \text{with } \tilde{P}_0(x) = P_0(x) = \mathbf{I},$$

where

$$\mathbb{M}(k, j) = \mathbf{D}_{k,0}^{-1} \mathbf{D}_{\mathbf{M}_k} \mathbf{D}_{0,k}^{-1} \left[\mathbf{I}_{(k+1)l} + \mathbf{D}_{0,k}(\Lambda_{0,k}^{(j)})^H H_j^{-1} \Lambda_{0,k}^{(j)} \mathbf{D}_{k,0}^{-1} \mathbf{D}_{\mathbf{M}_k} \Lambda_{k,0}^{(k+1)} \right]^{-1}. \tag{4.3}$$

Now, we will deduce a relation for the corresponding norms. We will need the following auxiliary results. For $0 \leq j \leq n$, let $\tilde{\mathbf{M}}_j \in \mathcal{M}_{jl}$ be defined as

$$\tilde{\mathbf{M}}_j := \begin{bmatrix} \mathbf{M}_0 & \cdots & \mathbf{M}_k & \cdots & \mathbf{M}_{j-1} \\ \vdots & \mathbf{M}_k & & & \vdots \\ \mathbf{M}_k & & \vdots & & \mathbf{M}_{2j-2-k} \\ \vdots & & & \mathbf{M}_{2j-2-k} & \vdots \\ \mathbf{M}_{j-1} & \cdots & \mathbf{M}_{2j-2-k} & \cdots & \mathbf{M}_{2j-2} \end{bmatrix} = \sum_{k=0}^{j-1} \mathbf{M}_{j,k} + \sum_{k=j}^{2j-2} \mathbf{M}_{j,k}^t,$$

where for $k = 0, \dots, j - 1$ (resp. $k = j, \dots, 2j - 2$), the matrices $\mathbf{M}_{j,k} \in \mathcal{M}_{jl}$ (resp. $\mathbf{M}_{j,k}^t \in \mathcal{M}_{jl}$) are given by

$$\mathbf{M}_{j,k} := \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{M}_k & \cdots & \mathbf{0} \\ \vdots & \mathbf{M}_k & & & \vdots \\ \mathbf{M}_k & & \vdots & & \vdots \\ \vdots & & & \ddots & \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix}, \quad \mathbf{M}_{j,k}^t := \begin{bmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \vdots & & \vdots & \mathbf{M}_k \\ \vdots & & & \mathbf{M}_k \\ \mathbf{0} & \cdots & \mathbf{M}_k & \cdots & \mathbf{0} \end{bmatrix},$$

and for $k \geq 2j - 1$, $\mathbf{M}_{j,k} = \mathbf{M}_{j,k}^t = \mathbf{0} \in \mathcal{M}_{jl}$.

Lemma 4.2. *Let*

$$\tilde{H}_{j,k} = \begin{cases} H_j + \mathbf{M}_{j,k}, & \text{if } k = 0, \dots, j - 1; \\ H_j + \mathbf{M}_{j,k}^t, & \text{if } k = j, \dots, 2j - 2. \end{cases} \tag{4.4}$$

(1) *If* $k = 0, \dots, j - 1$, *then*

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = - \sum_{p=0}^k H_j^{-1} \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H \tilde{H}_{j,k}^{-1}. \tag{4.5}$$

(2) *If* $k = j, \dots, 2j - 2$, *then*

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = - \sum_{p=k-(j-1)}^{j-1} H_j^{-1} \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H \tilde{H}_{j,k}^{-1}. \tag{4.6}$$

Proof. For $k = 0, \dots, j - 1$, we have

$$\tilde{H}_{j,k}^{-1} - H_j^{-1} = H_j^{-1} (H_j - \tilde{H}_{j,k}) \tilde{H}_{j,k}^{-1} = -H_j^{-1} \mathbf{M}_{j,k} \tilde{H}_{j,k}^{-1}, \tag{4.7}$$

and thus

$$\mathbf{M}_{j,k} = \chi_k^{(j)} \mathbf{M}_k (\chi_0^{(j)})^H + \cdots + \chi_0^{(j)} \mathbf{M}_k (\chi_k^{(j)})^H = \sum_{p=0}^k \chi_{k-p}^{(j)} \mathbf{M}_k (\chi_p^{(j)})^H.$$

Substituting $\mathbf{M}_{j,k}$ in (4.7), (4.5) follows. A similar procedure can be used to prove (4.6). \square

Let $\tilde{H}_j \in \mathcal{M}_{lj}$ be the block Hankel matrix associated with the perturbed matrix moments sequence, i.e.

$$\tilde{H}_j = \begin{bmatrix} \tilde{\mu}_0 & \tilde{\mu}_1 & \cdots & \tilde{\mu}_{j-1} \\ \tilde{\mu}_1 & \tilde{\mu}_2 & \cdots & \tilde{\mu}_j \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{j-1}^{\tilde{}} & \tilde{\mu}_j & \cdots & \mu_{2j-2}^{\tilde{}} \end{bmatrix} = H_j + \tilde{\mathbf{M}}_j.$$

Thus, from (4.4) for $1 \leq j \leq n$,

$$\tilde{H}_j = \tilde{H}_{j,0} + \cdots + \tilde{H}_{j,2j-2}. \tag{4.8}$$

Theorem 4.3. *Let μ be a Hermitian matrix measure with matrix moments $(\mu_k)_{k=0}^{2n-1}$. Define a new Hermitian sequence of matrix moments $(\tilde{\mu}_k)_{k=0}^{2n-1}$ by $\tilde{\mu}_k = \mu_k + \mathbf{M}_k$ with $0 \leq k \leq 2n - 1$, and denote by $\tilde{\mu}$ its corresponding Hermitian matrix measure. Then, for $1 \leq j \leq n$, we have*

$$\tilde{s}_j = s_j + \sum_{k=0}^{2n-1} \mathbf{P}_{j,k}(0)^H \Lambda_{k,0}^{(k+1)} \mathbb{M}(k, j) \mathbf{P}_{j,k}(0), \tag{4.9}$$

with $\tilde{s}_0 = \tilde{\mu}_0 = \mu_0 + \mathbf{M}_0$ and $\mathbb{M}(k, j)$ as in (4.3).

Proof. Let us assume first that only the k -th moment is perturbed. From (2.5) we have that if $2j < k$, then $\tilde{s}_j = s_j$. We will show that if $k \leq 2j$, we have

$$\tilde{s}_j = s_j + \sum_{p=0}^k \frac{(P_j^{(k-p)}(0))^H}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!}. \tag{4.10}$$

For $0 \leq k \leq j - 1$, notice that

$$\tilde{\mu}_{2j} = \mu_{2j} \quad \text{and} \quad \tilde{v}_{j,2j-1} = v_{j,2j-1},$$

and thus from (2.5) we get

$$\begin{aligned} \tilde{s}_j &= s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} \\ &= s_j - v_{j,2j-1}^H (\tilde{H}_{j,k}^{-1} - H_j^{-1}) \tilde{v}_{j,2j-1}. \end{aligned}$$

From (4.5) and using (2.6), we deduce (4.10). On the other hand, for $j \leq k \leq 2j - 1$ we have

$$\tilde{\mu}_{2j} = \mu_{2j} \quad \text{and} \quad \tilde{v}_{j,2j-1} = v_{j,2j-1} + \chi_{k-j}^{(j)} \mathbf{M}_k.$$

From (2.5) and (2.6), proceeding as above, we get

$$\begin{aligned} \tilde{s}_j &= s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} \\ &= s_j - v_{j,2j-1}^H (\tilde{H}_{j,k}^{-1} - H_j^{-1}) \tilde{v}_{j,2j-1} + \frac{(P_j^{(k-j)}(0))^H}{(k-j)!} \mathbf{M}_k + \mathbf{M}_k \frac{\tilde{P}_j^{(k-j)}(0)}{(k-j)!}, \end{aligned}$$

and from (4.6) and using (2.6),

$$\begin{aligned} \tilde{s}_j &= s_j + \sum_{p=k-(j-1)}^{j-1} \frac{(P_j^{(k-p)}(0))^H}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!} + \frac{(P_j^{(k-j)}(0))^H}{(k-j)!} \mathbf{M}_k \frac{\tilde{P}_j^{(j)}(0)}{j!} \\ &\quad + \frac{(P_j^{(j)}(0))^H}{j!} \mathbf{M}_k \frac{\tilde{P}_j^{(k-j)}(0)}{(k-j)!} \\ &= s_j + \sum_{p=k-j}^j \frac{(P_j^{(k-p)}(0))^H}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!}. \end{aligned}$$

Since

$$\begin{cases} \tilde{P}_j^{(p)}(0) = \mathbf{0}, & \text{if } j < p, \\ P_j^{(k-p)}(0) = \mathbf{0}, & \text{if } p < k - j, \end{cases}$$

we get (4.10). Finally, for $k = 2j$ (notice that if $j = n$ the result is also valid with $\mathbf{M}_{2j} = \mathbf{0}$),

$$\tilde{\mu}_{2j} = \mu_{2j} + \mathbf{M}_{2j} \quad \text{and} \quad \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} = v_{j,2j-1}^H H_j^{-1} v_{j,2j-1},$$

and thus from (2.5),

$$\begin{aligned} \tilde{s}_j &= s_j - \mu_{2j} + v_{j,2j-1}^H H_j^{-1} v_{j,2j-1} + \tilde{\mu}_{2j} - \tilde{v}_{j,2j-1}^H \tilde{H}_{j,k}^{-1} \tilde{v}_{j,2j-1} \\ &= s_j - \mu_{2j} + \tilde{\mu}_{2j} = s_j + \mathbf{M}_{2j} = s_j + \sum_{p=0}^k \frac{(P_j^{(k-p)}(0))^H}{(k-p)!} \mathbf{M}_k \frac{\tilde{P}_j^{(p)}(0)}{p!}. \end{aligned}$$

On the other hand, using the notation given in (4.1), (4.10) can be written as

$$\tilde{s}_j = s_j + [P_j^{(k)}(0)^H P_j^{(k-1)}(0)^H \dots P_j^{(0)}(0)^H] \mathbf{D}_{k,0}^{-1} \mathbf{D}_{\mathbf{M}_k} \mathbf{D}_{0,k}^{-1} \tilde{\mathbf{P}}_{j,k}(0),$$

which, using (4.2) and (4.3), becomes

$$\tilde{s}_j = s_j + \mathbf{P}_{j,k}(0)^H \Lambda_{k,0}^{(k+1)} \mathbb{M}(k, j) \mathbf{P}_{j,k}(0).$$

The general case, i.e. when all moments are perturbed ($k = 0, \dots, 2n - 1$) is easy to deduce, since we have (4.8) and

$$\tilde{v}_{j,2j-1} = v_{j,2j-1} + \sum_{k=0}^{2j-1} \chi_{k-j}^{(j)} \mathbf{M}_k,$$

so (4.9) follows. □

5. CONNECTION FORMULAS FOR THE UNIT CIRCLE AND FURTHER RESULTS

Let σ be a Hermitian matrix measure supported on \mathbb{T} , with associated monic matrix orthogonal polynomials $(\Phi_n(z))_{n \geq 0}$ and matrix moments $(c_n)_{n \geq 0}$. We define a new Hermitian sequence of matrix moments by $(\tilde{c}_n)_{n \geq 0}$, where $\tilde{c}_n = c_n + m_n \in \mathcal{M}_l$ for $n \geq 0$, and denote its corresponding Hermitian matrix measure and monic matrix orthogonal polynomials by $\tilde{\sigma}$ and $(\tilde{\Phi}_n(z))_{n \geq 0}$, respectively. Furthermore, we define

$$S_j = \langle \Phi_j, \Phi_j \rangle_{R, \sigma} \quad \text{and} \quad \tilde{S}_j = \langle \tilde{\Phi}_j, \tilde{\Phi}_j \rangle_{R, \tilde{\sigma}}.$$

In this section, in a similar way as in the real case, we will show the relation between the matrix polynomials $\tilde{\Phi}_j$ and Φ_j , as well as the relation between \tilde{S}_j and S_j , in terms of the matrix moments $(c_n)_{n \geq 0}$. Notice that (3.5) constitutes a particular case, when only one of the moments is perturbed. Let us define the following notation:

$$\begin{aligned} \mathbb{K}_{j-1}^k(0, z) &:= [K_{j-1}^{(0,0)}(0, z) K_{j-1}^{(1,0)}(0, z) \dots K_{j-1}^{(k,0)}(0, z)] \in \mathcal{M}_{l \times (k+1)l}, \\ \Upsilon_j^k(0) &:= [\Phi_j^{(0)}(0)^T \Phi_j^{(1)}(0)^T \dots \Phi_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l \times l}, \\ \tilde{\Upsilon}_j^k(0) &:= [\tilde{\Phi}_j^{(0)}(0)^T \tilde{\Phi}_j^{(1)}(0)^T \dots \tilde{\Phi}_j^{(k)}(0)^T]^T \in \mathcal{M}_{(k+1)l \times l}, \\ \mathbf{D}_{m_k}^{j+1} &:= \text{diag}\{m_k \dots m_k\} \in \mathcal{M}_{(j+1)l}, \\ \mathbf{D}_{k,j+k} &:= \text{diag}\{k! \mathbf{I} \dots (j+k)! \mathbf{I}\} \in \mathcal{M}_{(j+1)l}. \end{aligned} \tag{5.1}$$

As in the real line case, for a fixed k , with $0 \leq k \leq j - 1$, it has been proved in [6] that

$$\begin{aligned} \tilde{\Phi}_j(z) &= \Phi_j(z) - \sum_{p=0}^{j-k} \frac{K_{j-1}^{(p,0)}(0, z)}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} - \sum_{p=0}^{j-k-1} \frac{K_{j-1}^{(p+k,0)}(0, z)}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} \\ &= \Phi_j(z) \\ &\quad - \mathbb{K}_{j-1}^j(0, z) \left(\mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \tilde{\Upsilon}_j^j(0), \end{aligned}$$

with

$$\begin{aligned} \tilde{\Upsilon}_j^j(0) &= \\ &\left(\mathbf{I}_{(j+1)l} + \mathbf{D}_{0,j}(\Lambda_{0,j}^{(j)})^H T_j^{-1} \Lambda_{0,j}^{(j)} \left(\mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \right)^{-1} \\ &\quad \times \Upsilon_j^j(0). \end{aligned} \tag{5.2}$$

As a consequence, we have the following result.

Theorem 5.1 ([6]). *Let σ be a Hermitian matrix measure supported on \mathbb{T} with matrix moments $(c_k)_{k=0}^n$. Define a new Hermitian sequence of matrix moments $(\tilde{c}_k)_{k=0}^n$ by $\tilde{c}_k = c_k + m_k$, for $0 \leq k \leq n$. Then, if $(\Phi_j(z))_{j=0}^n$ and $(\tilde{\Phi}_j(z))_{j=0}^n$ are the corresponding monic matrix orthogonal polynomials, we have*

$$\tilde{\Phi}_j(z) = \Phi_j(z) - \sum_{k=0}^j \mathbb{K}_{j-1}^j(0, z) \mathbb{N}(k, j) \Upsilon_j^j(0), \quad \text{with } \tilde{\Phi}_0(x) = \Phi_0(x) = \mathbf{I},$$

where

$$\begin{aligned} \mathbb{N}(k, j) &= \left(\mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \\ &\times \left(\mathbf{I}_{(j+1)l} + \mathbf{D}_{0,j}(\Lambda_{0,j}^{(j)})^H T_j^{-1} \Lambda_{0,j}^{(j)} \left(\mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \right)^{-1}. \end{aligned} \tag{5.3}$$

The above theorem is a generalization to the matrix case of what was studied in [4, 19]. The particular case of a perturbation of a single matrix moment given by (3.5) is shown in the following corollary.

Corollary 5.2. *Let $k \in \mathbb{N}$ (fixed) with $0 \leq k \leq n$, and let σ be a Hermitian matrix measure supported on \mathbb{T} . Define σ_k as a perturbation of σ given by (3.5). Then,*

$$\tilde{\Phi}_j(z) = \begin{cases} \Phi_j(z), & 0 \leq j < k, \\ \Phi_j(z) - \mathbb{K}_{j-1}^j(0, z) \mathbb{N}(k, j) \Upsilon_j^j(0), & k \leq j \leq n, \end{cases}$$

with $\mathbb{N}(k, j)$ as in (5.3).

On the other hand, for $1 \leq j \leq n$, let $\tilde{\mathbf{N}}_j \in \mathcal{M}_{jl}$ be defined as

$$\tilde{\mathbf{N}}_j := \begin{bmatrix} m_0^H + m_0 & \cdots & m_k & \cdots & m_{j-1} \\ \vdots & \ddots & & \ddots & \\ m_k^H & & m_0^H + m_0 & & m_k \\ \vdots & \ddots & & \ddots & \\ m_{j-1}^H & \cdots & m_k^H & \cdots & m_0^H + m_0 \end{bmatrix} = \mathbf{N}_{j,0} + \cdots + \mathbf{N}_{j,j-1},$$

where for $k = 0, \dots, j - 1$, we have

$$\mathbf{N}_{j,k} := \begin{bmatrix} \mathbf{0} & \cdots & m_k & \cdots & \mathbf{0} \\ \vdots & \ddots & & \ddots & \vdots \\ \mathbf{0} & & \vdots & & m_k \\ \vdots & & & \ddots & \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & & \ddots & \vdots \\ m_k^H & & \vdots & & \vdots \\ \vdots & \ddots & & \ddots & \\ \mathbf{0} & \cdots & m_k^H & \cdots & \mathbf{0} \end{bmatrix},$$

and for $k \geq j$, $\mathbf{N}_{j,k} = \mathbf{0} \in \mathcal{M}_{lj}$.

Lemma 5.3. For $1 \leq j \leq n$ and $0 \leq k \leq j - 1$, define

$$\tilde{T}_{j,k} = T_j + \mathbf{N}_{j,k}. \tag{5.4}$$

Then,

$$T_j^{-1} - \tilde{T}_{j,k}^{-1} = \sum_{p=0}^{j-k-1} T_j^{-1} \chi_p^{(j)} m_k (\chi_{p+k}^{(j)})^H \tilde{T}_{j,k}^{-1} + \sum_{p=0}^{j-k-1} T_j^{-1} \chi_{p+k}^{(j)} m_k^H (\chi_p^{(j)})^H \tilde{T}_{j,k}^{-1}. \tag{5.5}$$

Proof. We have

$$\tilde{T}_{j,k}^{-1} - T_j^{-1} = T_j^{-1} (T_j - \tilde{T}_{j,k}) \tilde{T}_{j,k}^{-1} = -T_j^{-1} \mathbf{N}_{j,k} \tilde{T}_{j,k}^{-1}. \tag{5.6}$$

Notice that

$$\mathbf{N}_{j,k} = \sum_{p=0}^{j-k-1} \chi_p^{(j)} m_k (\chi_{p+k}^{(j)})^H + \sum_{p=0}^{j-k-1} \chi_{p+k}^{(j)} m_k^H (\chi_p^{(j)})^H,$$

and thus substituting $\mathbf{N}_{j,k}$ in (5.6), (5.5) follows. □

Let $\tilde{T}_j \in \mathcal{M}_{lj}$ be the block Toeplitz matrix with every moment perturbed, i.e.,

$$\tilde{T}_j = \begin{bmatrix} \tilde{c}_0^H + \tilde{c}_0 & \tilde{c}_1 & \cdots & \tilde{c}_{j-1} \\ \tilde{c}_1^H & \tilde{c}_0^H + \tilde{c}_0 & \cdots & \tilde{c}_{j-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{j-1}^H & \tilde{c}_{j-2}^H & \cdots & \tilde{c}_0^H + \tilde{c}_0 \end{bmatrix} = T_j + \tilde{\mathbf{N}}_j.$$

Then, from (5.4), for $1 \leq j \leq n$, we get

$$\tilde{T}_j = \tilde{T}_{j,0} + \cdots + \tilde{T}_{j,j-1}. \tag{5.7}$$

Theorem 5.4. *Let σ be a Hermitian matrix measure supported on \mathbb{T} with matrix moments $(c_k)_{k=0}^n$. Define a Hermitian sequence $(\tilde{c}_k)_{k=0}^n$ with $\tilde{c}_k = c_k + m_k$, for $0 \leq k \leq n$, and denote by $\tilde{\sigma}$ the associated Hermitian matrix measure. Then, for $0 \leq j \leq n$, we have*

$$\tilde{S}_j = S_j + \sum_{k=0}^j \Upsilon_j^j(0)^H \mathbb{N}(k, j) \Upsilon_j^j(0), \tag{5.8}$$

with $\tilde{S}_0 = \tilde{c}_0^H + \tilde{c}_0 = c_0 + m_0 + (c_0 + m_0)^H$ and $\mathbb{N}(k, j)$ as in (5.3).

Proof. Let us first assume that only a single k -th matrix moment is perturbed. From (2.11), we have that if $j < k$, then $\tilde{S}_j = S_j$. Let us show that for $0 \leq k \leq j$,

$$\tilde{S}_j = S_j + \sum_{p=0}^{j-k} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \sum_{p=0}^{j-k} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!}. \tag{5.9}$$

For $k = 0$, $\tilde{\nu}_j = \nu_j$, and from (2.11) we get

$$\begin{aligned} \tilde{S}_j &= \tilde{c}_0^H + \tilde{c}_0 - \tilde{\nu}_j^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j = S_j - c_0^H - c_0 + \nu_j^H T_j^{-1} \nu_j + \tilde{c}_0^H + \tilde{c}_0 - \tilde{\nu}_j^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j \\ &= S_j + m_0^H + m_0 - \nu_j (\tilde{T}_{j,k}^{-1} - T_j^{-1}) \tilde{\nu}_j. \end{aligned}$$

By using (2.12) and (5.5), we obtain

$$\begin{aligned} \tilde{S}_j &= S_j + m_0^H + m_0 \\ &\quad + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!}, \end{aligned}$$

and since for $k = 0$ we have

$$m_0 = \frac{(\Phi_j^{(j)}(0))^H}{j!} m_k \frac{\tilde{\Phi}_{j-k+k}^{(j)}(0)}{(j-k+k)!}, \quad m_0^H = \frac{(\Phi_j^{(j-k+k)}(0))^H}{(j-k+k)!} m_k^H \frac{\tilde{\Phi}_j^{(j)}(0)}{j!},$$

we deduce (5.9).

For $1 \leq k \leq j$, $\tilde{\nu}_j = \nu_j + \chi_{j-k}^{(j)} m_k$, and proceeding as above we get

$$\tilde{S}_j = S_j - \nu_j^H (\tilde{T}_{j,k}^{-1} - T_j^{-1}) \tilde{\nu}_j - \nu_j^H T_j^{-1} \chi_{j-k}^{(j)} m_k - m_k^H (\chi_{j-k}^{(j)})^H \tilde{T}_{j,k}^{-1} \tilde{\nu}_j.$$

By using (2.12) and (5.5), we obtain

$$\begin{aligned} \tilde{S}_j &= S_j + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} + \frac{(\Phi_j^{(j-k)}(0))^H}{(j-k)!} m_k \frac{\tilde{\Phi}_j^{(j)}(0)}{j!} \\ &\quad + \sum_{p=0}^{j-k-1} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} + \frac{(\Phi_j^{(j)}(0))^H}{j!} m_k^H \frac{\tilde{\Phi}_j^{(j-k)}(0)}{(j-k)!}, \end{aligned}$$

so (5.9) follows.

On the other hand, using (5.1), the first sum of (5.9) can be written as

$$\begin{aligned} \sum_{p=0}^{j-k} \frac{(\Phi_j^{(p)}(0))^H}{p!} m_k \frac{\tilde{\Phi}_j^{(p+k)}(0)}{(p+k)!} &= \sum_{p=k}^j \frac{(\Phi_j^{(p-k)}(0))^H}{(p-k)!} m_k \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} \\ &= \sum_{p=0}^j \frac{(\Phi_j^{(p-k)}(0))^H}{(p-k)!} m_k \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} \\ &= \Upsilon_j^j(0)^H \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} \tilde{\Upsilon}_j^j(0), \end{aligned}$$

and in a similar way, the second sum of (5.9) becomes

$$\sum_{p=0}^{j-k} \frac{(\Phi_j^{(p+k)}(0))^H}{(p+k)!} m_k^H \frac{\tilde{\Phi}_j^{(p)}(0)}{p!} = \Upsilon_j^j(0)^H \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \tilde{\Upsilon}_j^j(0).$$

As a consequence, (5.9) becomes

$$\tilde{S}_j = S_j + \Upsilon_j^j(0)^H \left(\mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \Lambda_{-k,j-k}^{(j+1)} \mathbf{D}_{m_k}^{j+1} + \Lambda_{k,j+k}^{(j+1)} \mathbf{D}_{0,j}^{-1} \mathbf{D}_{k,j+k}^{-1} \mathbf{D}_{m_k^H}^{j+1} \right) \tilde{\Upsilon}_j^j(0),$$

which, using (5.2) and (5.3), can be written as

$$\tilde{S}_j = S_j + \Upsilon_j^j(0)^H \mathbb{N}(k, j) \Upsilon_j^j(0).$$

The general case, when all moments are perturbed, follows immediately taking into account (5.7) and

$$\tilde{\nu}_j = \nu_j + \sum_{k=1}^j \chi_{j-k}^{(j)} m_k,$$

so (5.8) follows. □

For the particular case of (3.5), we have

Corollary 5.5. *Let $k \in \mathbb{N}$ (fixed) with $0 \leq k \leq n$, and let σ be a Hermitian matrix measure supported on \mathbb{T} . Let σ_k be a perturbation of σ defined by (3.5). Then, for $0 \leq j \leq n$,*

$$\tilde{S}_j = \begin{cases} S_j, & 0 \leq j < k, \\ S_j + \Upsilon_j^j(0)^H \mathbb{N}(k, j) \Upsilon_j^j(0), & k \leq j \leq n, \end{cases}$$

with $\mathbb{N}(k, j)$ as in (5.3).

6. EXAMPLE

To illustrate the results of the above subsections, consider the matrix measure that appears in [3, Section 8]:

$$d\sigma(\theta) = \frac{1}{\sqrt{\cos 2\theta}} \begin{bmatrix} \sqrt{1 + \cos 2\theta} & \pm 1 \\ \pm 1 & \sqrt{1 + \cos 2\theta} \end{bmatrix} \frac{d\theta}{2\pi}, \quad \text{if } \begin{cases} \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}], \\ \theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]. \end{cases} \tag{6.1}$$

Its corresponding matrix moments are given by

$$c_0 = \mathbf{I}_2, \quad c_{4m} = c_{4m+2} = \frac{\alpha_m}{2} \mathbf{I}_2, \quad c_{4m+1} = \frac{\alpha_m}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c_{4m+3} = \mathbf{0}_2, \quad (6.2)$$

with

$$\alpha_0 = 1, \quad \alpha_n = (-1)^n \prod_{k=1}^n \left(1 - \frac{1}{2k}\right), \quad n \geq 1.$$

Now, we consider the problem of determining the obtained matrix measure if some of the moments given by (6.2) change. For instance, let us introduce a perturbation of the matrix moment c_{4m+1} , for some $m \in \mathbb{Z}$, given by

$$\tilde{c}_{4m+1} = \frac{\tilde{\alpha}_m}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\alpha_m + \epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = c_{4m+1} + \frac{\epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

i.e., $m_{4m+1} = \frac{\epsilon}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The perturbed matrix measure, obtained by using (3.6) for $j = 4m + 1$, is

$$d\sigma_{4m+1}(\theta) = d\sigma(\theta) + \sqrt{2}\epsilon \begin{bmatrix} 0 & \cos(4m + 1)\theta \\ \cos(4m + 1)\theta & 0 \end{bmatrix} \frac{d\theta}{2\pi}, \quad (6.3)$$

where σ is as in (6.1). Notice that $\sqrt{2}\epsilon \begin{bmatrix} 0 & \cos(4m + 1)\theta \\ \cos(4m + 1)\theta & 0 \end{bmatrix}$ is a positive semi-definite matrix for $\theta \in (-\pi, \pi]$ if $\epsilon < 0$. In particular, it is positive semi-definite in the same support as σ and thus σ_{4m+1} is positive semi-definite if $\epsilon < 0$. Furthermore, the matrix measure (6.1) and its perturbation (6.3) are Hermitian and thus there exists an associated sequence of orthogonal matrix polynomials, as well as their norms, given by Theorems 5.1 and 5.4.

On the other hand, to illustrate the results in Section 3, by using (3.2), (3.3) and taking into account that $\cos 2\theta = 2 \cos^2 \theta - 1$, we get

$$d\mu(x) = Sz^{-1}(d\sigma(\theta)) = \frac{1}{\pi} \frac{1}{\sqrt{2x^2 - 1}} \begin{bmatrix} \sqrt{2}|x| & \pm 1 \\ \pm 1 & \sqrt{2}|x| \end{bmatrix} \frac{dx}{\sqrt{1 - x^2}}, \quad (6.4)$$

where the plus sign $+1$ holds in the interval $[\frac{1}{\sqrt{2}}, 1]$ and the minus sign -1 holds in the interval $[-1, -\frac{1}{\sqrt{2}}]$. (6.4) is the measure associated with the measure (6.1), connected through the inverse Szegő matrix transformation. Furthermore, from Proposition 3.1 with $\epsilon < 0$ we get

$$d\mu_{4m+1}(x) = d\mu(x) + \frac{\sqrt{2}\epsilon}{\pi} \begin{bmatrix} 0 & T_{4m+1}(x) \\ T_{4m+1}(x) & 0 \end{bmatrix} \frac{dx}{\sqrt{1 - x^2}}, \quad (6.5)$$

where μ is as in (6.4). Notice that the matrix measure (6.4) and its perturbation (6.5) are positive semi-definite, so there exists a corresponding sequence of matrix orthogonal polynomials on the real line and their norms, that can be obtained from Theorems 4.1 and 4.3.

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