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BICONSERVATIVE LORENTZ HYPERSURFACES IN $\mathbb{E}_1^{n+1}$ WITH COMPLEX EIGENVALUES

RAM SHANKAR GUPTA AND AHMAD SHARFUDDIN

Abstract. We prove that every biconservative Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ having complex eigenvalues has constant mean curvature. Moreover, every biharmonic Lorentz hypersurface M_1^n having complex eigenvalues in $\mathbb{E}_1^{n+1}$ must be minimal. Also, we provide some examples of such hypersurfaces.

1. Introduction

The classification of constant mean curvature (CMC) hypersurfaces plays an important role in relativity theory [20, 23], and such type of hypersurfaces are associated with the problem of eigenvalues of the shape operator or differential equations arising from the Laplacian operator.

In 1964, Eells and Sampson [11] introduced the notion of poly-harmonic maps as a natural generalization of the well-known harmonic maps. Thus, while harmonic maps between Riemannian manifolds $\phi : (M, g) \to (N, h)$ are critical points of the energy functional $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$, the biharmonic maps are critical points of the bienergy functional $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$, where $\tau = \operatorname{trace} \nabla d\phi$ is the tension field of ϕ.

In 1924, Hilbert pointed that the stress-energy tensor associated to a functional E is a conservative symmetric 2-covariant tensor S at the critical points of E, i.e., $\operatorname{div} S = 0$ [17]. For the bienergy functional E_2, Jiang defined the stress-bienergy tensor S_2 and proved that it satisfies $\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi\rangle$ [18]. Thus, if ϕ is biharmonic, then $\operatorname{div} S_2 = 0$. For biharmonic submanifolds, from the above relation, we see that $\operatorname{div} S_2 = 0$ if and only if the tangential part of the bitension field vanishes. In particular, an isometric immersion $\phi : (M, g) \to (N, h)$ is called biconservative if $\operatorname{div} S_2 = 0$.

Biconservative submanifolds were studied and classified in $\mathbb{E}^4$ by Hasanis and Vlachos [16], where the biconservative hypersurfaces were called H-hypersurfaces. In [24], the complete classification of H-hypersurfaces with three distinct curvatures

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in Euclidean space of arbitrary dimension was obtained and some explicit example was given. Upadhyay and Turgay classified biconservative hypersurfaces in E_2^5 with diagonal shape operator having three distinct principal curvatures [25]. Further, they have constructed the example of biconservative hypersurfaces with four distinct principal curvatures. Recently in [6], it was proved that every biconservative Lorentz hypersurface in E_1^{n+1} with complex eigenvalues having at most five distinct principal curvatures has constant mean curvature. Further, it was proved that a biconservative Lorentz hypersurface with constant length of second fundamental form and whose shape operator has complex eigenvalues with six distinct principal curvatures has constant mean curvature [6]. For more work on biconservative hypersurfaces in pseudo-Euclidean spaces, see references in [6, 25]. For work on biharmonic submanifolds, see [2, 12] and references therein.

In this paper, we study biconservative Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$ whose shape operator has complex eigenvalues. The shape operator of Lorentz hypersurfaces with complex eigenvalues takes the form [21, 19]

$$\mathcal{A} = \begin{pmatrix} \lambda & \mu & \\ -\mu & \lambda & \\ & & D_{n-2} \end{pmatrix}, \tag{1.1}$$

with respect to a suitable orthonormal base field of the tangent bundle $\{e_1, e_2, \ldots, e_n\}$ of $T_pM_1^n$, which satisfies

$$g(e_1, e_1) = -1, \qquad g(e_i, e_i) = 1, \quad i = 2, 3, \ldots, n, \tag{1.2}$$

and

$$g(e_i, e_j) = 0, \quad i, j = 1, 2, \ldots, n, \ i \neq j, \tag{1.3}$$

where $D_{n-2} = \mathrm{diag}\{\lambda_3, \lambda_4, \ldots, \lambda_n\}$ and $\mu \neq 0$.

We prove:

Theorem 1.1. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ with complex eigenvalues. Then it has constant mean curvature.*

A submanifold satisfying

$$\triangle \vec{H} = 0$$

is called biharmonic submanifold [2].

The study of biharmonic submanifolds in Euclidean spaces was initiated by Chen in the mid 1980s. In particular, he posed the following well-known conjecture in 1991:

The only biharmonic submanifolds of Euclidean spaces are the minimal ones.

The conjecture was later studied by many researchers and so far it is found to be true for hypersurfaces in Euclidean spaces [8, 10, 12, 14, 15, 16]. Chen's conjecture is not always true for submanifolds of semi-Euclidean spaces (see [3, 4, 5]). However, for hypersurfaces in semi-Euclidean spaces, Chen's conjecture is also right (see [1, 4, 5, 7, 9, 13]).

Since every biconservative hypersurface is a biharmonic hypersurface, using Theorem 1.1 and (2.5a) we find:

Theorem 1.2. *Every biharmonic Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ with complex eigenvalues must be minimal.*

2. Preliminaries

Let (M_1^n, g) be an n-dimensional Lorentz hypersurface isometrically immersed in $(\mathbb{E}_1^{n+1}, \overline{g})$ and $g = \overline{g}_{|M_1^n}$. We denote by ξ the unit normal vector to M_1^n, where $\overline{g}(\xi, \xi) = 1$. A vector X in $\mathbb{E}_1^{n+1}$ is called spacelike, timelike or lightlike according as $\overline{g}(X, X) > 0$, $\overline{g}(X, X) < 0$, or $\overline{g}(X, X) = 0$, respectively.

The mean curvature H of M_1^n is given by

$$H = \frac{1}{n} \operatorname{trace} \mathcal{A}, \tag{2.1}$$

where $\mathcal{A}$ is the shape operator of M_1^n.

Let ∇ denote the Levi-Civita connection on M_1^n. Then the Gauss and Codazzi equations are given respectively by

$$R(X, Y)Z = g(\mathcal{A}Y, Z)\mathcal{A}X - g(\mathcal{A}X, Z)\mathcal{A}Y, \tag{2.2}$$

$$(\nabla_X \mathcal{A})Y = (\nabla_Y \mathcal{A})X, \tag{2.3}$$

where R is the curvature tensor and

$$(\nabla_X \mathcal{A})Y = \nabla_X \mathcal{A}Y - \mathcal{A}(\nabla_X Y) \tag{2.4}$$

for all $X, Y, Z \in \Gamma(TM_1^n)$.

The biharmonic equation can be decomposed into its normal and tangential parts. The necessary and sufficient conditions for M_1^n to be biharmonic in $\mathbb{E}_1^{n+1}$ are:

$$\triangle H + H \operatorname{trace} A^2 = 0, \tag{2.5a}$$

$$\mathcal{A}(\operatorname{grad} H) + \frac{n}{2} H \operatorname{grad} H = 0. \tag{2.5b}$$

Then, the submanifold satisfying the tangential part of the biharmonic equation is called biconservative, and therefore the biconservative Lorentz hypersurfaces M_1^n in $\mathbb{E}_1^{n+1}$ are characterized by (2.5b).

3. Biconservative Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$

In this section, we study biconservative Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$ with complex eigenvalues. Since every hypersurface with constant mean curvature is always biconservative, we assume that the mean curvature is not constant and $\operatorname{grad} H \neq 0$. Assuming non-constant mean curvature implies the existence of an open connected subset U of M_1^n with $\operatorname{grad}_x H \neq 0$, for all $x \in U$. From (2.5b), it is easy to see that $\operatorname{grad} H$ is an eigenvector of the shape operator $\mathcal{A}$ with the corresponding principal curvature $-\frac{n}{2}H$. Therefore, there does not exist a biconservative Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ with two distinct principal curvatures of non-constant mean curvature with complex eigenvalues. Without losing generality, we choose e_n in the direction of $\operatorname{grad} H$. Then, the shape operator $\mathcal{A}$ of hypersurfaces M_1^n

in $\mathbb{E}_1^{n+1}$ will take the following form with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$:

$$\mathcal{A}_H e_1 = \lambda e_1 - \mu e_2, \quad \mathcal{A}_H e_2 = \mu e_1 + \lambda e_2, \quad \mathcal{A}_H e_i = \lambda_i e_i, \quad i \in D, \tag{3.1}$$

where $D = \{3, 4, \ldots, n\}$. Also, we denote the following sets by

$$A = \{1, 2, \ldots, n\}, \quad B = \{1, 2, \ldots, n-1\}, \quad C = \{3, 4, \ldots, n-1\}.$$

The $\operatorname{grad} H$ can be expressed as

$$\operatorname{grad} H = \sum_{i=1}^{n} e_i(H) e_i.$$

As we have taken e_n parallel to $\operatorname{grad} H$, we have

$$e_n(H) \neq 0, \qquad e_i(H) = 0, \quad i \in B. \tag{3.2}$$

We express

$$\nabla_{e_i} e_j = \sum_{m=1}^{n} \omega_{ij}^m e_m, \quad i, j \in A. \tag{3.3}$$

Differentiating (1.2) and (1.3) with respect to e_k and using (3.3), we obtain

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.4}$$

for $i \neq j$ and $i, j, k \in A$. Using (2.1) and (3.1), we obtain that

$$2\lambda + \sum_{j=3}^{n-1} \lambda_j = \frac{3nH}{2} = -3\lambda_n. \tag{3.5}$$

From now onwards we assume that $\operatorname{grad} H \neq 0$ *in an open connected subset* U *of* M_1^n, e_n *is in the direction of* $\operatorname{grad} H$, *and* $e_n(H) \neq 0$.

Now, we obtain results which will be useful to prove Theorem 1.1. Although the proofs of some of the following results were given in [6], to make the paper self-contained we have included them here with a different approach in the proofs.

Lemma 3.1. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. Then,*

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j = (\lambda_j - \lambda_i)\omega_{jj}^i, \tag{3.6}$$

$$(\lambda_i-\lambda_j)\omega_{ki}^j=(\lambda_k-\lambda_j)\omega_{ik}^j, \tag{3.7}$$
$$(\lambda_j-\lambda)\omega_{ij}^1-\mu\omega_{ij}^2=(\lambda_i-\lambda)\omega_{ji}^1-\mu\omega_{ji}^2 \tag{3.8}$$
$$(\lambda_j-\lambda)\omega_{ij}^2+\mu\omega_{ij}^1=(\lambda_i-\lambda)\omega_{ji}^2+\mu\omega_{ji}^1, \tag{3.9}$$
$$(\lambda_i-\lambda_j)\omega_{1i}^j=-\mu\omega_{i2}^j+(\lambda-\lambda_j)\omega_{i1}^j, \tag{3.10}$$
$$(\lambda_j-\lambda_i)\omega_{1j}^i=-\mu\omega_{j2}^i+(\lambda-\lambda_i)\omega_{j1}^i, \tag{3.11}$$
$$(\lambda_i-\lambda)\omega_{1i}^1-\mu\omega_{1i}^2=e_i(\lambda), \tag{3.12}$$
$$(\lambda_i-\lambda)\omega_{1i}^2+\mu\omega_{1i}^1=-e_i(\mu), \tag{3.13}$$
$$e_1(\lambda_i)=-\mu\omega_{i2}^i+(\lambda-\lambda_i)\omega_{i1}^i, \tag{3.14}$$
$$(\lambda_i-\lambda_j)\omega_{2i}^j=\mu\omega_{i1}^j+(\lambda-\lambda_j)\omega_{i2}^j, \tag{3.15}$$
$$(\lambda_j-\lambda_i)\omega_{2j}^i=\mu\omega_{j1}^i+(\lambda-\lambda_i)\omega_{j2}^i, \tag{3.16}$$
$$(\lambda_i-\lambda)\omega_{2i}^1-\mu\omega_{2i}^2=e_i(\mu), \tag{3.17}$$
$$(\lambda_i-\lambda)\omega_{2i}^2+\mu\omega_{2i}^1=e_i(\lambda), \tag{3.18}$$
$$e_2(\lambda_i)=\mu\omega_{i1}^i+(\lambda-\lambda_i)\omega_{i2}^i, \tag{3.19}$$
$$e_1(\mu)=e_2(\lambda), \tag{3.20}$$
$$e_1(\lambda)=-e_2(\mu), \tag{3.21}$$
$$(\lambda-\lambda_i)\omega_{12}^i+\mu\omega_{11}^i=(\lambda-\lambda_i)\omega_{21}^i-\mu\omega_{22}^i, \tag{3.22}$$

for distinct $i,j,k\in D$ *such that* $\lambda_k\neq\lambda_j\neq\lambda_i$.

Proof. Taking $X=e_i$, $Y=e_j$ in (2.4) for $i,j\in D$, and using (3.1) and (3.3), we get

$$(\nabla_{e_i}\mathcal{A})e_j=e_i(\lambda_j)e_j+\lambda_j(\omega_{ij}^1e_1+\omega_{ij}^2e_2)-(\omega_{ij}^1\mathcal{A}e_1+\omega_{ij}^2\mathcal{A}e_2)+\sum_{k=3}^n(\lambda_j-\lambda_k)\omega_{ij}^ke_k.$$

Putting the value of $(\nabla_{e_i}\mathcal{A})e_j$ in (2.3), we find

$$\begin{aligned}&e_i(\lambda_j)e_j+\lambda_j(\omega_{ij}^1e_1+\omega_{ij}^2e_2)-(\omega_{ij}^1\mathcal{A}e_1+\omega_{ij}^2\mathcal{A}e_2)+\sum_{k=3}^n(\lambda_j-\lambda_k)\omega_{ij}^ke_k\\&=e_j(\lambda_i)e_i+\lambda_i(\omega_{ji}^1e_1+\omega_{ji}^2e_2)-(\omega_{ji}^1\mathcal{A}e_1+\omega_{ji}^2\mathcal{A}e_2)+\sum_{k=3}^n(\lambda_i-\lambda_k)\omega_{ji}^ke_k,\end{aligned} \tag{3.23}$$

whereby for $i\neq j=k$ and $i\neq j\neq k$ we obtain (3.6) and (3.7), respectively. Moreover, using (3.1) in (3.23) and comparing the coefficients of e_1 and e_2, we find (3.8) and (3.9), respectively.

Next, using (3.1) and (2.4) in $(\nabla_{e_1}\mathcal{A})e_i = (\nabla_{e_i}\mathcal{A})e_1$, for $i \in D$, we obtain

$$e_1(\lambda_i)e_i + \lambda_i(\omega_{1i}^1 e_1 + \omega_{1i}^2 e_2) - (\omega_{1i}^1 \mathcal{A}e_1 + \omega_{1i}^2 \mathcal{A}e_2) + \sum_{j=3}^{n}(\lambda_i - \lambda_j)\omega_{1i}^j e_j$$
$$= (e_i(\lambda) - \mu\omega_{i2}^1)e_1 + (\lambda\omega_{i1}^2 - e_i(\mu))e_2 - \omega_{i1}^2\mathcal{A}e_2 + \sum_{j=3}^{n}\Big((\lambda - \lambda_j)\omega_{i1}^j - \mu\omega_{i2}^j\Big)e_j,$$

whereby for $i \neq j$ we get (3.10). Further, comparing the coefficients of e_1, e_2, and e_i and using (3.4), we have (3.12), (3.13) and (3.14), respectively.

Also, using (3.1) and (2.4) in $((\nabla_{e_1}\mathcal{A})e_j, e_i) = g((\nabla_{e_j}\mathcal{A})e_1, e_i)$ gives (3.11).

Similarly, using (3.1) and (2.4) in $(\nabla_{e_2}\mathcal{A})e_i = (\nabla_{e_i}\mathcal{A})e_2$, for $i \in D$, we get

$$e_2(\lambda_i)e_i + \lambda_i(\omega_{2i}^1 e_1 + \omega_{2i}^2 e_2) - (\omega_{2i}^1 \mathcal{A}e_1 + \omega_{2i}^2 \mathcal{A}e_2) + \sum_{j=3}^{n}(\lambda_i - \lambda_j)\omega_{2i}^j e_j$$
$$= (e_i(\lambda) + \mu\omega_{i1}^2)e_2 + (\lambda\omega_{i2}^1 + e_i(\mu))e_1 - \omega_{i2}^1\mathcal{A}e_1 + \sum_{j=3}^{n}\Big((\lambda - \lambda_j)\omega_{i2}^j + \mu\omega_{i1}^j\Big)e_j,$$

whereby for $i \neq j$ we get (3.15). Further, comparing the coefficients of e_1, e_2, and e_i and using (3.4), we have (3.18), (3.17) and (3.19), respectively.

Also, using (3.1) and (2.4) in $((\nabla_{e_2}\mathcal{A})e_j, e_i) = g((\nabla_{e_j}\mathcal{A})e_2, e_i)$ gives (3.16).

Now, using (3.1) and (2.4) in $(\nabla_{e_1}\mathcal{A})e_2 = (\nabla_{e_2}\mathcal{A})e_1$, for $i \in D$, we obtain

$$e_1(\mu)e_1 + e_1(\lambda)e_2 + \sum_{i=3}^{n}\Big((\lambda - \lambda_i)\omega_{12}^i + \mu\omega_{11}^i\Big)e_i$$
$$= e_2(\lambda)e_1 - e_2(\mu)e_2 + \sum_{i=3}^{n}\Big((\lambda - \lambda_i)\omega_{21}^i - \mu\omega_{22}^i\Big)e_i,$$

whereby comparing the coefficients of e_1, e_2, and e_i and using (3.4) we have (3.20), (3.21), and (3.22), respectively. This completes the proof of the lemma. □

Next, we have

Lemma 3.2. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. Then,*

$$\lambda_n \neq \lambda_k, \quad \forall k \in C. \tag{3.24}$$

Proof. Let $\lambda_n = \lambda_k$ for $k \in C$; then taking $i = n$ and $j = k$ in (3.6), we get

$$e_n(\lambda_k) = 0 \quad \text{or} \quad e_n(H) = 0, \quad \text{as } \lambda_n = -\frac{nH}{2},$$

which contradicts (3.2), thereby completing the proof of lemma. □

Using (3.2), (3.3) and the fact that $[e_i e_j](H) = 0 = \nabla_{e_i} e_j(H) - \nabla_{e_j} e_i(H) = \omega_{ij}^n e_n(H) - \omega_{ji}^n e_n(H)$, for $i \neq j$, we find

$$\omega_{ij}^n = \omega_{ji}^n, \quad i, j \in B. \tag{3.25}$$

Lemma 3.3. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame* $\{e_1, e_2, \dots, e_n\}$. *Then,*

$$\omega_{nn}^i = 0, \quad \forall i \in A. \tag{3.26}$$

Proof. Putting $i \neq n$, $j = n$ in (3.6) and using (3.2) and (3.4), we find

$$\omega_{nn}^i = 0, \quad i \in D. \tag{3.27}$$

Taking $i = n$ in (3.14) and (3.19) and using (3.2) and (3.4), we find

$$\omega_{nn}^1 = \omega_{nn}^2 = 0. \tag{3.28}$$

Combining (3.27) and (3.28), we get (3.26). □

Lemma 3.4. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame* $\{e_1, e_2, \dots, e_n\}$. *Then,*

$$\omega_{22}^n = \omega_{11}^n = \omega_{12}^n = \omega_{21}^n = 0. \tag{3.29}$$

Proof. Taking $i = n$ in (3.22) and using (3.25), we get

$$\omega_{11}^n = -\omega_{22}^n. \tag{3.30}$$

Taking $i = n$ in (3.12), (3.13), (3.17), (3.18) and using (3.4), (3.25), and (3.30), we find

$$-(\lambda_n - \lambda)\omega_{12}^n + \mu\omega_{22}^n = 0, \quad (\lambda_n - \lambda)\omega_{22}^n + \mu\omega_{12}^n = 0. \tag{3.31}$$

Solving (3.31), we get

$$\omega_{12}^n = \omega_{22}^n = 0. \tag{3.32}$$

Using (3.32), (3.30) and (3.25), we get (3.29). □

Lemma 3.5. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame* $\{e_1, e_2, \dots, e_n\}$. *Then,*

$$\omega_{ij}^1 = \omega_{ij}^2 = \omega_{1i}^j = \omega_{2i}^j = 0, \quad i \neq j, \quad i, j \in D. \tag{3.33}$$

Proof. Using (3.10), (3.11) and (3.4), we get

$$\mu\omega_{ij}^2 - (\lambda - \lambda_j)\omega_{ij}^1 = \mu\omega_{ji}^2 - (\lambda - \lambda_i)\omega_{ji}^1. \tag{3.34}$$

Similarly, using (3.15), (3.16) and (3.4), we find

$$\mu\omega_{ij}^1 + (\lambda - \lambda_j)\omega_{ij}^2 = \mu\omega_{ji}^1 + (\lambda - \lambda_i)\omega_{ji}^2. \tag{3.35}$$

Combining (3.9) and (3.35), we obtain

$$\omega_{ij}^1 = \omega_{ji}^1. \tag{3.36}$$

Combining (3.8) and (3.34), we find

$$(\lambda_j - \lambda)\omega_{ij}^1 = (\lambda_i - \lambda)\omega_{ji}^1. \tag{3.37}$$

Using (3.34), (3.35), (3.36) and (3.37), we get (3.33), thereby completing the proof of the lemma. □

Now, we find the following lemma for covariant derivatives.

Lemma 3.6. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \dots, e_n\}$. Then,*

$$\nabla_{e_1}e_1 = \sum_{m\neq 1,n}\omega_{11}^m e_m, \quad \nabla_{e_1}e_2 = \sum_{m\neq 2,n}\omega_{12}^m e_m, \quad \nabla_{e_1}e_n = 0, \quad \nabla_{e_n}e_1 = \omega_{n1}^2 e_2,$$

$$\nabla_{e_2}e_1 = \sum_{m\neq 1,n}\omega_{21}^m e_m, \quad \nabla_{e_2}e_2 = \sum_{m\neq 2,n}\omega_{22}^m e_m, \quad \nabla_{e_2}e_n = 0, \quad \nabla_{e_n}e_2 = \omega_{n2}^1 e_1,$$

$$\nabla_{e_1}e_i = \omega_{1i}^1 e_1 + \omega_{1i}^2 e_2, \quad \nabla_{e_2}e_i = \omega_{2i}^1 e_1 + \omega_{2i}^2 e_2, \quad \nabla_{e_i}e_i = \sum_{m\neq i}\omega_{ii}^m e_m,$$

$$\nabla_{e_i}e_1 = \omega_{i1}^2 e_2 + \omega_{i1}^i e_i, \quad \nabla_{e_i}e_2 = \omega_{i2}^1 e_1 + \omega_{i2}^i e_i, \quad \nabla_{e_i}e_n = -\omega_{ii}^n e_i, \quad \nabla_{e_n}e_n = 0,$$

for $i \in C$. Moreover,

(a) *If M_1^n has all distinct principal curvatures, then*

$$\nabla_{e_n}e_i = 0, \quad \nabla_{e_i}e_j = \omega_{ij}^i e_i, \qquad \forall\, i,j \in C,\ i \neq j,$$

(b) *If M_1^n has q distinct principal curvatures $\lambda \pm \sqrt{-1}\mu, \lambda_3, \dots, \lambda_{q-1}, \lambda_n$, with multiplicities $p_3, \dots, p_{q-1}$ of $\lambda_3, \dots, \lambda_{q-1}$, respectively, such that $p_3 + p_4 + \cdots + p_{q-2} + p_{q-1} = n-3$, then*

$$\nabla_{e_n}e_i = \sum_{C_{i_1}}\omega_{ni}^m e_m, \quad \forall\, i \in C_{i_1},\ m \neq i,$$

$$\nabla_{e_i}e_j = \sum_{C_{i_1}}\omega_{ij}^m e_m, \quad \forall\, i,j \in C_{i_1},\ i \neq j,\ m \neq j,$$

where $i_1 = 3, 2, \dots, q-1$, and $C_3 = \{3, \dots, p_3+2\}$, $C_4 = \{p_3+3, \dots, p_3+p_4+2\}$, …, $C_{q-1} = \{p_3+p_4+\cdots+p_{q-2}+3, \dots, n-1\}$, and ω_{ij}^i satisfy (3.4) *and* (3.6).

Proof. (a) Let M_1^n have all distinct principal curvatures. Putting $j = n$ and $k = j$ in (3.7) and using (3.25), we get

$$\omega_{ji}^n = \omega_{ij}^n = 0, \quad i,j \in C,\ i \neq j. \tag{3.38}$$

Putting $i = n$ and $k = i$ in (3.7) and using (3.38) and (3.4), we find

$$\omega_{ni}^j = \omega_{in}^j = 0, \quad i,j \in C,\ i \neq j. \tag{3.39}$$

(b) Let M_1^n have q distinct principal curvatures. Putting $i = n$ and $k = i$ in (3.7), we obtain

$$\omega_{in}^j = 0, \quad j \neq i \text{ and } j,i \in C_{i_1},\ i_1 = 3, \dots, q-1. \tag{3.40}$$

Putting $j = n$ and $k = j$ in (3.7) and using (3.25), we get

$$\omega_{ji}^n = \omega_{ij}^n = 0, \quad i \in C_{i_1},\ j \in C_{i_2},\ i_1 \neq i_2,\ i_1, i_2 = 3, \dots, q-1. \tag{3.41}$$

Taking $i \in C_{i_1}$ in (3.7), we have

$$\omega_{ki}^j = 0, \quad j \neq k \text{ and } j,k \in C_{i_2},\ i_1 \neq i_2,\ i_1, i_2 = 3, \dots, q-1. \tag{3.42}$$

Putting $i = n$ and $k = i$ in (3.7) and using (3.41) and (3.4), we find

$$\omega_{ni}^{j} = \omega_{in}^{j} = 0, \quad j \in C_{i_1},\ i \in C_{i_2},\ i_1 \neq i_2,\ i_1, i_2 = 3, \ldots, q-1. \tag{3.43}$$

Now, using Lemma 3.3, Lemma 3.4, Lemma 3.5 and (3.38), (3.39), (3.40), (3.41), (3.42) and (3.43) in (3.3), completes the proof of the lemma. ☐

Next, we have

Lemma 3.7. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame* $\{e_1, e_2, \ldots, e_n\}$. *Then,*

$$\lambda = 0. \tag{3.44}$$

Proof. Evaluating $g(R(e_n, e_1)e_n, e_1)$, using (2.2), (3.1) and Lemma 3.6, we have

$$g(\nabla_{e_n}\nabla_{e_1}e_n - \nabla_{e_1}\nabla_{e_n}e_n - \nabla_{[e_n e_1]}e_n, e_1)$$
$$= g(Ae_1, e_n)g(Ae_n, e_1) - g(Ae_n, e_n)g(Ae_1, e_1),$$

which gives

$$\lambda\lambda_n = 0. \tag{3.45}$$

Since $\lambda_n \neq 0$, from (3.45) we find (3.44). Thus, we complete the proof of the lemma. ☐

Now, using Lemma 3.7, we find the following theorem.

Theorem 3.8. *There does not exist a biconservative Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ with three distinct principal curvatures of non-constant mean curvature with complex eigenvalues.*

Proof. Let M_1^n have three distinct principal curvatures. Then, from (3.5) and (3.44), we get $H = 0$, a contradiction, which completes the proof of the theorem. ☐

Next, we have

Lemma 3.9. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame* $\{e_1, e_2, \ldots, e_n\}$. *Then,*

$$\omega_{11}^{i} = \omega_{22}^{i} = \omega_{12}^{i} = \omega_{21}^{i} = 0, \quad \forall\, i \in C, \tag{3.46}$$

and

$$\mu = constant.$$

Proof. Using (3.44) and (3.4) in (3.12) and (3.18), we find

$$\lambda_i\omega_{11}^{i} = \mu\omega_{12}^{i} \quad \text{and} \quad \lambda_i\omega_{22}^{i} = -\mu\omega_{21}^{i}, \tag{3.47}$$

respectively.

On the other hand, adding (3.13) and (3.17), and therein using (3.44), (3.47) and (3.4), we obtain

$$\omega_{11}^{i} = \omega_{22}^{i}, \tag{3.48}$$

which together with (3.47) gives

$$\omega_{12}^i = -\omega_{21}^i. \tag{3.49}$$

Using (3.44), (3.48) and (3.49) in (3.22), we get

$$\mu\omega_{11}^i = \lambda_i\omega_{12}^i. \tag{3.50}$$

Therefore, from (3.47) and (3.50), we obtain

$$(\mu^2 - \lambda_i^2)\omega_{11}^i = 0.$$

We claim that $\omega_{11}^i = 0$. In fact, if $\omega_{11}^i \neq 0$, then $\mu^2 - \lambda_i^2 = 0$, which gives $\lambda_i = \pm\mu$ for all $i \in C$. In view of Theorem 3.8, we consider the following cases:

Case I. Let M_1^n have four distinct principal curvatures. Then, using (3.44) and Lemma 3.2 in (3.5), we obtain $(n-3)\lambda_i = \frac{3nH}{2}$ or $\pm(n-3)\mu = \frac{3nH}{2}$, which on differentiating with respect to e_n gives $\pm(n-3)e_n(\mu) = \frac{3ne_n(H)}{2}$. Also, using (3.29) in (3.13), we find $e_n(\mu) = 0$. Therefore, we obtain $e_n(H) = 0$, a contradiction.

Case II. Let M_1^n have five distinct principal curvatures $\lambda \pm \sqrt{-1}\mu$, $\lambda_3 = \mu$, $\lambda_4 = -\mu$, λ_n. Then, using (3.44) and Lemma 3.2 in (3.5), we get $(p_3 - p_4)\mu = \frac{3nH}{2}$, where p_3 and p_4 are the multiplicities of λ_3 and λ_4, respectively. Now, proceeding as in Case I, we get a contradiction.

Case III. Let M_1^n have more than five distinct principal curvatures. Then, we get $\lambda_i = \pm\mu$ for all $i \in C$, which contradicts the fact of having more than five distinct principal curvatures.

Hence $\omega_{11}^i = 0$. Using this in (3.48), (3.49) and (3.50), we find (3.46). Using (3.29), (3.44) and (3.46) in (3.13), (3.20) and (3.21), we get

$$e_1(\mu) = e_2(\mu) = e_i(\mu) = 0, \quad \forall\, i \in D. \tag{3.51}$$

Hence μ is constant in all directions. This completes the proof of the lemma. □

Lemma 3.10. *Let M_1^n be a biconservative Lorentz hypersurface in $\mathbb{E}_1^{n+1}$ having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. Then, $g(R(e_n, e_i)e_n, e_i)$, $g(R(e_n, e_i)e_i, e_1)$, $g(R(e_n, e_i)e_i, e_2)$, $g(R(e_i, e_1)e_i, e_n)$, $g(R(e_i, e_2)e_i, e_n)$, $g(R(e_i, e_2)e_i, e_1)$ and $g(R(e_i, e_1)e_i, e_2)$ give the following:*

$$e_n(\omega_{ii}^n) - (\omega_{ii}^n)^2 = \lambda_n\lambda_i, \tag{3.52}$$

$$e_n(\omega_{ii}^1) = \omega_{ii}^n\omega_{ii}^1 - \omega_{n2}^1\omega_{ii}^2, \tag{3.53}$$

$$e_n(\omega_{ii}^2) = \omega_{ii}^n\omega_{ii}^2 - \omega_{n1}^2\omega_{ii}^1, \tag{3.54}$$

$$e_1(\omega_{ii}^n) = \omega_{ii}^n\omega_{ii}^1, \tag{3.55}$$

$$e_2(\omega_{ii}^n) = \omega_{ii}^n\omega_{ii}^2, \tag{3.56}$$

$$e_2(\omega_{ii}^1) + \omega_{ii}^2(\omega_{22}^1 - \omega_{ii}^1) = \mu\lambda_i, \tag{3.57}$$

$$e_1(\omega_{ii}^2) + \omega_{ii}^1(\omega_{11}^2 - \omega_{ii}^2) = \mu\lambda_i, \tag{3.58}$$

respectively, for all $i \in C$.

Proof. Here, we give the proof of the first two relations (3.52) and (3.53). The proof of the other relations can be obtained in a similar way.

Using (2.2) and (3.1), we have

$$g(R(e_n,e_i)e_n,e_i) = g(Ae_i,e_n)g(Ae_n,e_i) - g(Ae_n,e_n)g(Ae_i,e_i) = -\lambda_n\lambda_i, \quad (3.59)$$

$$g(R(e_n,e_i)e_i,e_1) = g(Ae_i,e_i)g(Ae_n,e_1) - g(Ae_n,e_i)g(Ae_i,e_1) = 0, \quad (3.60)$$

for all $i \in C$.

(i) Let M_1^n have all the distinct principal curvatures. Then, using Lemma 3.6, we get

$$\begin{aligned} g(R(e_n,e_i)e_n,e_i) &= g(\nabla_{e_n}\nabla_{e_i}e_n - \nabla_{e_i}\nabla_{e_n}e_n - \nabla_{[e_ne_i]}e_n, e_i) \\ &= g(\nabla_{e_n}(-\omega_{ii}^n e_i) - \omega_{ii}^n\nabla_{e_i}e_n, e_i) = g(-e_n(\omega_{ii}^n)e_i + (\omega_{ii}^n)^2 e_i, e_i) \\ &= -e_n(\omega_{ii}^n) + (\omega_{ii}^n)^2, \end{aligned} \quad (3.61)$$

for all $i \in C$.

Therefore, from (3.59) and (3.61), we get (3.52).

Next, we know that

$$g(R(e_n,e_i)e_i,e_1) = g(\nabla_{e_n}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_n}e_i - \nabla_{[e_ne_i]}e_i, e_1), \quad (3.62)$$

for all $i \in C$.

Now, using Lemma 3.6 and Lemma 3.9, we have

$$\nabla_{e_n}\nabla_{e_i}e_i = \nabla_{e_n}\left(\sum_{m\neq i,\, m=1}^{n}\omega_{ii}^m e_m\right) = \sum_{m\neq i,\, m=1}^{n}\big(e_n(\omega_{ii}^m)e_m + \omega_{ii}^m\nabla_{e_n}e_m\big),$$

$$\nabla_{e_i}\nabla_{e_n}e_i = 0, \nabla_{\nabla_{e_n}e_i}e_i = 0,$$

$$\nabla_{\nabla_{e_i}e_n}e_i = -\omega_{ii}^n\nabla_{e_i}e_i = -\omega_{ii}^n\left(\sum_{m\neq i,\, m=1}^{n}\omega_{ii}^m e_m\right).$$

Hence, using the above in (3.62), we get

$$g(R(e_n,e_i)e_i,e_1) = -e_n(\omega_{ii}^1) - \omega_{ii}^2\omega_{n2}^1 + \omega_{ii}^n\omega_{ii}^1 \quad (3.63)$$

for all $i \in C$.

Therefore, from (3.60) and (3.63), we get (3.53) for $i \in C$.

(ii) Let M_1^n have q distinct principal curvatures. Then, using Lemma 3.6, we find

$$\begin{aligned} g(R(e_n,e_i)e_n,e_i) &= g(\nabla_{e_n}\nabla_{e_i}e_n - \nabla_{e_i}\nabla_{e_n}e_n - \nabla_{[e_ne_i]}e_n, e_i) \\ &= g(\nabla_{e_n}(-\omega_{ii}^n e_i) - \sum_{m\neq i}^{C_3}\omega_{ni}^m(\nabla_{e_m}e_n) - \omega_{ii}^n\nabla_{e_i}e_n, e_i) \\ &= g(-e_n(\omega_{ii}^n)e_i - \sum_{m\neq i}^{C_3}\omega_{ni}^m\left(\sum_{l\neq n}^{C_3}\omega_{mn}^l e_l\right) + (\omega_{ii}^n)^2 e_i, e_i), \end{aligned}$$

wherein using (3.40) gives

$$g(R(e_n, e_i)e_n, e_i) = -e_n(\omega_{ii}^n) + (\omega_{ii}^n)^2, \tag{3.64}$$

for all $i \in C_3$.

Therefore, from (3.59) and (3.64), we get (3.52) for $i \in C_3$. Similarly, for all $i \in C$, we find (3.52).

Next, using Lemma 3.6 and Lemma 3.9 for $i \in C_3$, we have

$$\nabla_{e_n}\nabla_{e_i}e_i = \nabla_{e_n}\Big(\sum_{m\neq i,m=1}^{n} \omega_{ii}^m e_m\Big) = \sum_{m\neq i,m=1}^{n} \big(e_n(\omega_{ii}^m)e_m + \omega_{ii}^m \nabla_{e_n}e_m\big),$$

$$\nabla_{e_i}\nabla_{e_n}e_i = \nabla_{e_i}\Big(\sum_{m\neq i,m=3}^{p_3+2} \omega_{ni}^m e_m\Big) = \sum_{m\neq i,m=3}^{p_3+2} \Big(e_i(\omega_{ni}^m)e_m + \omega_{ni}^m \sum_{l\neq m,m=3}^{p_3+2} \omega_{im}^l e_l\Big),$$

$$\nabla_{\nabla_{e_n}e_i}e_i = \sum_{m\neq i,m=3}^{p_3+2} \omega_{ni}^m \nabla_{e_m}e_i = \sum_{m\neq i,m=3}^{p_3+2} \omega_{ni}^m\Big(\sum_{l\neq i,l=3}^{p_3+2} \omega_{mi}^l e_l\Big),$$

$$\nabla_{\nabla_{e_i}e_n}e_i = -\omega_{ii}^n \nabla_{e_i}e_i = -\omega_{ii}^n\Big(\sum_{m\neq i,m=1}^{n} \omega_{ii}^m e_m\Big).$$

Hence, using the above in (3.62), we get

$$g(R(e_n, e_i)e_i, e_1) = -e_n(\omega_{ii}^1) - \omega_{ii}^2\omega_{n2}^1 + \omega_{ii}^n\omega_{ii}^1 \tag{3.65}$$

for all $i \in C_3$.

Therefore, from (3.60) and (3.65), we get (3.53) for $i \in C_3$. Similarly, for all $i \in C$, we find (3.53). □

4. Proof of Theorem 1.1

Proof. Using Lemma 3.6, we get

$$e_1e_n - e_ne_1 = \nabla_{e_1}e_n - \nabla_{e_n}e_1 = -\omega_{n1}^2 e_2. \tag{4.1}$$

Operating ω_{ii}^n on both sides in (4.1), we find

$$e_1e_n(\omega_{ii}^n) - e_ne_1(\omega_{ii}^n) = -\omega_{n1}^2 e_2(\omega_{ii}^n). \tag{4.2}$$

Using (3.52), (3.53), (3.55), (3.56), (3.14), (3.2) and Lemma 3.7 in (4.2), we obtain

$$\mu\lambda_n\omega_{ii}^2 = 0, \tag{4.3}$$

whereby we find

$$\omega_{ii}^2 = 0. \tag{4.4}$$

Now, using Lemma 3.6, we get

$$e_2e_n(\omega_{ii}^n) - e_ne_2(\omega_{ii}^n) = -\omega_{n2}^1 e_1(\omega_{ii}^n). \tag{4.5}$$

Using (3.52), (3.54), (3.55), (3.56), (3.19), (3.2) and Lemma 3.7 in (4.5), we obtain

$$\omega_{ii}^1 = 0. \tag{4.6}$$

Using (4.4) and (4.6) in (3.57), we find $\lambda_i = 0$ for all $i \in C$. Using this in (3.5), we get $H = 0$, a contradiction, and hence the proof of Theorem 1.1 is complete. □

5. Examples of Lorentz hypersurfaces with complex eigenvalues in $\mathbb{E}^{n+1}_1$

In this section, we give some examples of Lorentz hypersurfaces in support of the results obtained in previous sections. We denote the metric tensor g in $\mathbb{E}^{n+1}_1$ by

$$g = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2, \tag{5.1}$$

where $(x_1, x_2, \ldots, x_n, x_{n+1})$ is a rectangular coordinate system in $\mathbb{E}^{n+1}_1$.

Example 5.1. *Consider the hypersurface M in $\mathbb{E}^{n+1}_1$ given by*

$$f(t, s, t_1, t_2, \ldots, t_{n-2}) = (\phi(t)\cosh s, \phi(t)\sinh s, s, t_1, t_2, \ldots, t_{n-2}),$$

where $\phi(t)$ is a smooth function such that $\phi'(t) \neq 0$, and $s, t_1, \ldots, t_{n-3} \in \mathbb{R}$.

By a direct computation, we find that the vector fields

$$e_1 = \frac{1}{\phi'(t)}\partial_t,\ e_2 = \frac{1}{\sqrt{1+\phi^2(t)}}\partial_s,\ e_3 = \partial_{t_1},\ e_4 = \partial_{t_2}, \ldots,\ e_n = \partial_{t_{n-2}}$$

form an orthonormal base for the tangent bundle of M and the unit normal vector field of M is given by

$$N = \frac{1}{\sqrt{1+\phi^2(t)}}(-\sinh s, -\cosh s, \phi(t), 0, 0, \ldots, 0).$$

A further computation shows that

$$\overline{\nabla}_{e_1} N = -\frac{1}{1+\phi^2(t)} e_2, \quad \overline{\nabla}_{e_2} N = \frac{1}{1+\phi^2(t)} e_1, \quad \overline{\nabla}_{e_j} N = 0,\ j = 3, 4, \ldots, n. \tag{5.2}$$

In view of (5.1) and (5.2), the shape operator $\mathcal{A}$ of M has the following form:

$$\mathcal{A} = \begin{pmatrix} 0 & -\frac{1}{1+\phi^2(t)} & \\ \frac{1}{1+\phi^2(t)} & 0 & \\ & & D_{n-2} \end{pmatrix},$$

where $D_{n-2} = \operatorname{diag}\{0, 0, \ldots, 0\}$.

If M is biconservative, then $\operatorname{grad} H$ must be in the direction of any one of the vector fields $e_3, \ldots, e_n$ with corresponding eigenvalue $\frac{-nH}{2}$. From (5.2), we can see that the eigenvalue corresponding to each of the vector fields $e_3, \ldots, e_n$ is zero. Therefore, we get $H = 0$. This fact can be verified as the hypersurface M has three distinct eigenvalues $\frac{\sqrt{-1}}{1+\phi^2(t)}, -\frac{\sqrt{-1}}{1+\phi^2(t)}, 0$.

Example 5.2. *Consider the hypersurface M in $\mathbb{E}_1^{n+1}$ given by*

$$f(t,s,\theta,t_1,t_2,\dots,t_{n-3}) = (\psi(t)\cosh\theta\cosh s, \psi(t)\cosh\theta\sinh s, s, \psi(t)\sinh\theta, t_1,t_2,\dots,t_{n-3}), \quad (5.3)$$

where $\psi(t)$ is a smooth function such that $\psi(t),\psi'(t)\neq 0$, and $s,\theta,t_1,\dots,t_{n-3}\in\mathbb{R}$.

We find that the vector fields

$$e_1 = \Big(\frac{\cosh\theta}{\psi'(t)}\Big)\partial_t - \Big(\frac{\sinh\theta}{\psi(t)}\Big)\partial_\theta, \quad e_2 = \Big(\frac{1}{\sqrt{1+\psi^2(t)\cosh^2\theta}}\Big)\partial_s,$$

$$e_3 = \Big(\frac{\sinh\theta}{\psi'(t)}\Big)\partial_t - \Big(\frac{\cosh\theta}{\psi(t)}\Big)\partial_\theta, \quad e_4=\partial_{t_1}, \quad \dots, \quad e_n = \partial_{t_{n-3}},$$

form an orthonormal base for the tangent bundle of M such that

$$g(e_1,e_1)=-1, \quad g(e_i,e_i)=1, \quad \text{and} \quad g(e_k,e_l)=0, \tag{5.4}$$

for $i=2,3,\dots,n$ and $k,l=1,2,\dots,n$, with $k\neq l$.

The unit normal vector field of M is given by

$$N = \frac{1}{\sqrt{1+\psi^2(t)\cosh^2\theta}}(-\sinh s, -\cosh s, \psi(t)\cosh\theta, 0,0,\dots,0).$$

Also, we obtain

$$\overline{\nabla}_{e_1}N = \frac{1}{1+\psi^2(t)\cosh^2\theta}e_2, \quad \overline{\nabla}_{e_2}N = -\frac{1}{1+\psi^2(t)\cosh^2\theta}e_1, \quad \overline{\nabla}_{e_j}N = 0, \tag{5.5}$$

for $j=3,4,\dots,n$.

In view of (5.4) and (5.5), the shape operator $\mathcal{A}$ of M has the following form:

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{1}{1+\psi^2(t)\cosh^2\theta} & \\ -\frac{1}{1+\psi^2(t)\cosh^2\theta} & 0 & \\ & & D_{n-2} \end{pmatrix}, \tag{5.6}$$

where $D_{n-2} = \operatorname{diag}\{0,0,\dots,0\}$.

Further, we can see that M is biconservative with $H=0$ as the hypersurface M has three distinct eigenvalues $\frac{\sqrt{-1}}{1+\psi^2(t)\cosh^2\theta}, -\frac{\sqrt{-1}}{1+\psi^2(t)\cosh^2\theta}, 0$.

Example 5.3. *Consider the hypersurface M in $\mathbb{E}_1^{n+1}$ given by*

$$f(t,s,\theta,t_1,t_2,\dots,t_{n-3}) = (t\cosh s, t\sinh s, s+\cos\theta, \sin\theta, t_1,t_2,\dots,t_{n-3}),$$

where $\theta\in[0,2\pi)$ and $s,t,t_1,\dots,t_{n-3}\in\mathbb{R}$.

We can see that the vector fields $e_1=\partial_t$, $e_2 = \frac{1}{\sqrt{1+t^2)}}\partial_s$, $e_3 = \Big(\sqrt{\frac{1+t^2}{t^2+\cos^2\theta}}\Big)\Big(\partial_\theta + (\frac{\sin\theta}{1+t^2})\partial_s\Big)$, $e_4=\partial_{t_1}$, $\dots$, $e_n=\partial_{t_{n-3}}$, form an orthonormal base for the tangent bundle of M such that

$$g(e_1,e_1)=-1, \quad g(e_i,e_i)=1, \quad \text{and} \quad g(e_k,e_l)=0, \tag{5.7}$$

for $i=2,3,\dots,n$ and $k,l=1,2,\dots,n$, with $k\neq l$.

The unit normal vector field of M is given by

$$N = \frac{1}{\sqrt{t^2 + \cos^2\theta}}(-\cos\theta \sinh s, -\cos\theta \cosh s, t\cos\theta, t\sin\theta, 0, \dots, 0).$$

Also, we obtain

$$\begin{aligned}
\overline{\nabla}_{e_1} N &= \frac{\cos\theta}{\sqrt{(1+t^2)(t^2+\cos^2\theta)}} e_2 + \frac{\sin\theta\cos\theta}{\sqrt{(1+t^2)(t^2+\cos^2\theta)}} e_3, \\
\overline{\nabla}_{e_3} N &= -\frac{\sin\theta\cos\theta}{\sqrt{(1+t^2)(t^2+\cos^2\theta)}} e_1 + \frac{t(1+t^2)}{(t^2+\cos^2\theta)^{\frac{3}{2}}} e_3, \\
\overline{\nabla}_{e_2} N &= -\frac{\cos\theta}{\sqrt{(1+t^2)(t^2+\cos^2\theta)}} e_1, \\
\overline{\nabla}_{e_j} N &= 0, \quad \text{for } j = 4, 5, \dots, n.
\end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we can see that M has four distinct eigenvalues for $\theta = 0$ and its shape operator $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{1}{1+t^2} & & \\ -\frac{1}{1+t^2} & 0 & & \\ & & -\frac{t}{\sqrt{1+t^2}} & \\ & & & D_{n-3} \end{pmatrix},$$

where $D_{n-3} = \operatorname{diag}\{0, 0, \dots, 0\}$.

Further, we can see that M is biconservative with $H = 0$ for $t = 0$.

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R. S. Gupta ✉
University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Sector-16C, Dwarka, New Delhi-110078, India
ramshankar.gupta@gmail.com

A. Sharfuddin
Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi-110025, India
sharfuddin_ahmad12@yahoo.com

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