

ON CONVERGENCE OF SUBSPACES GENERATED BY DILATIONS OF POLYNOMIALS. AN APPLICATION TO BEST LOCAL APPROXIMATION

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ABSTRACT. We study the convergence of a net of subspaces generated by dilations of polynomials in a finite dimensional subspace. As a consequence, we extend the results given by Zó and Cuenya [*Advanced Courses of Mathematical Analysis II (Granada, 2004)*, 193–213, World Scientific, 2007] on a general approach to the problems of best vector-valued approximation on small regions from a finite dimensional subspace of polynomials.

1. INTRODUCTION

Suppose that $\{a_j\}$ is a data set. These data are values of a function and its derivatives at a point. If we want to approximate these data using a polynomial of degree at most l , which will be the best algorithm to use? A Taylor polynomial of degree l is probably the most natural procedure to use.

The problem of finding an optimal algorithm to approximate a finite number of data corresponding to a function is developed in the best local approximation theory.

In 1934, Walsh proved in [11] that the Taylor polynomial of degree l for an analytic function f can be obtained by taking the limit as $\epsilon \rightarrow 0$ of the best Chebyshev approximation to f from Π^l on the disk $|z| \leq \epsilon$. This paper was the first association between the best local approximation to a function f from Π^l in 0 and the Taylor polynomial for f at the origin. However, the concept of best local approximation has been introduced and developed more recently by Chui, Shisha, and Smith in [2]. Later, several authors [3, 4, 5, 6, 8, 9, 10, 12] have studied this problem.

We consider a family of function seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$, acting on Lebesgue measurable functions $F : B \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$, where B is the unit ball centered at the origin in \mathbb{R}^n . We will use the notation $F^\epsilon(x) = F(\epsilon x)$ and $\|F\|_\epsilon^* = \|F^\epsilon\|_\epsilon$. For $l \in \mathbb{N} \cup \{0\}$,

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we will denote by Π^l the class of algebraic polynomials in n variables of degree at most l , and Π_k^l the set $\{P = (p_1, \dots, p_k) : p_s \in \Pi^l\}$.

Let \mathcal{A} be a subspace of Π_k^l and let $\{P_\epsilon\}_{\epsilon>0}$ be a net of best approximants to F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$, i.e.,

$$\|F - P_\epsilon\|_\epsilon^* \leq \|F - P\|_\epsilon^*, \quad \text{for all } P \in \mathcal{A}. \tag{1.1}$$

If the net $\{P_\epsilon\}_{\epsilon>0}$ has a limit in \mathcal{A} as $\epsilon \rightarrow 0$, this limit is called the *best local approximation to F from \mathcal{A} in 0*. According to (1.1), we observe that P_ϵ^ϵ is a polynomial in

$$\mathcal{A}^\epsilon := \{P^\epsilon : P \in \mathcal{A}\} \subset \Pi_k^l \tag{1.2}$$

of best approximation to F^ϵ by elements of the class \mathcal{A}^ϵ , with respect to the seminorm $\|\cdot\|_\epsilon$. We write it briefly as $P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(F^\epsilon)$. Note that \mathcal{A}^ϵ is a subspace generated by dilations of polynomials in \mathcal{A} .

From now on, we assume the following properties for the family of function seminorms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$.

- (1) For $F = (f_1, \dots, f_k)$ and $G = (g_1, \dots, g_k)$, we have $\|F\|_\epsilon \leq \|G\|_\epsilon$, for every $\epsilon > 0$, whenever $|f_s| \leq |g_s|$, $s = 1, \dots, k$.
- (2) If 1 is the function $F(x) = (1, \dots, 1)$, we have $\|1\|_\epsilon < \infty$, for all $\epsilon > 0$.
- (3) For every $F \in C_k(B)$, we have $\|F\|_\epsilon \rightarrow \|F\|_0$, as $\epsilon \rightarrow 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$. Moreover, $\|\cdot\|_0$ is a norm on $C_k(B)$.

An important point to note here is that there exist positive constants $C = C(m, k)$ and $\epsilon(m, k)$ such that for every $0 < \epsilon \leq \epsilon(m, k)$,

$$\frac{1}{C} \|P\|_0 \leq \|P\|_\epsilon \leq C \|P\|_0, \quad \text{for every } P \in \Pi_k^m \tag{1.3}$$

[13, Proposition 3.1].

In order to give an example of norms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$, with the properties (1)–(3), we recall a definition of convergence of measures given in [6]. See also [1] for the notion of weak convergence of measures in general.

Definition 1.1. Let μ_ϵ , $0 \leq \epsilon \leq 1$, be a family of probability measures on B . We say that the measures μ_ϵ converge weakly in the proper sense to the measure μ_0 if we have

$$\int_B f(x) d\mu_\epsilon(x) \rightarrow \int_B f(x) d\mu_0(x), \quad f \in C_1(B),$$

and $\mu_0(B') > 0$ for any ball $B' \subset B$.

The assumption on the measure μ_0 implies that

$$\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = \left(\int_B \|F\|^p d\mu_\epsilon \right)^{\frac{1}{p}}$$

is actually a norm on $C_k(B)$ for $\epsilon = 0$ and $1 \leq p < \infty$, where $\|\cdot\|$ stands for any monotone norm on \mathbb{R}^k . We use a monotone norm on \mathbb{R}^k to ensure property (1) for the family of seminorms $\|\cdot\|_\epsilon$, $0 \leq \epsilon \leq 1$.

Let F be in $C_k(B)$; it is readily seen, by using the definition of weak convergence of measures, that there exists $\epsilon_0 = \epsilon_0(F) > 0$ such that if $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = 0$, for some $0 < \epsilon \leq \epsilon_0$, then $F = 0$. Moreover we have that $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)}$ converges as $\epsilon \rightarrow 0$ to the norm $\|F\|_0 = \|F\|_{L^p(\mu_0)}$ if $F \in C_k(B)$.

For more examples of nets of seminorms fulfilling conditions (1)–(3), we refer the reader to [13].

We say that $F : B \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$ has a Taylor polynomial of degree m at 0 if there exists $P \in \Pi_k^m$ such that

$$\|F - P\|_\epsilon^* = o(\epsilon^m), \quad \text{as } \epsilon \rightarrow 0.$$

It is well known that if a Taylor polynomial exists, it is unique [13, Proposition 3.3]; we denote it by $T_m = T_m(F)$. We write $F \in t^m$ if the function F has the Taylor polynomial of degree m at 0. Moreover, if $F \in t^m$ and $T_m(F) = \sum_{|\alpha| \leq m} C_\alpha x^\alpha$, then the Taylor polynomial of degree $l \leq m$ for F at 0 is given by $T_l(F) = \sum_{|\alpha| \leq l} C_\alpha x^\alpha$ [13, Proposition 3.5], where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. We set $\partial^\alpha F(0)$ for the vector $\alpha! C_\alpha$ with $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$.

The problem of best local approximation with a family of function seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon > 0}$ satisfying (1)–(3) was considered in [13] for two types of approximation class \mathcal{A} fulfilling $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ and

- (c1) $\mathcal{A}^\epsilon = \mathcal{A}$, for each $\epsilon > 0$, or
- (c2) if $P \in \mathcal{A}$ and $T_{m+1}(P) = 0$, then $P = 0$.

Firstly, the authors studied the asymptotic behavior of a normalized error function as $\epsilon \rightarrow 0$ [13, Theorems 4.2 and 4.5]. Secondly, they showed that there exists the best local approximation to F in 0 and is associated with a Taylor polynomial for F in 0 [13, Theorem 5.1]. In particular, if $\mathcal{A} = \Pi_k^m$ and $F \in t^m$, they proved that $P_\epsilon \rightarrow T_m(F)$ as $\epsilon \rightarrow 0$ [13, Theorem 3.1].

In this work we generalize the results found in [13], without the restrictions (c1) or (c2) given above. For this, it is essential to study the convergence of the net $\{\mathcal{A}^\epsilon\}$ as $\epsilon \rightarrow 0$.

This paper is organized as follows. In Section 2, we investigate the asymptotic behavior of $\{\mathcal{A}^\epsilon\}$. In Section 3, we study the asymptotic behavior of the error function $\epsilon^{-m-1}(F_\epsilon - P_\epsilon)^\epsilon$ for a suitable integer, and we show some results about the best local approximation in the origin which generalizes those of [13].

2. ASYMPTOTIC BEHAVIOR OF THE NET $\{\mathcal{A}^\epsilon\}$

In this section, we study the asymptotic behavior of the net $\{\mathcal{A}^\epsilon\}$ given in (1.2). We begin with the following definition.

Definition 2.1. Let $\mathcal{A} \subset \Pi_k^l$ be a subspace. We say that $P \in \lim_{\epsilon \rightarrow 0} \mathcal{A}^\epsilon$ if there exists a net $\{P_\epsilon\} \subset \mathcal{A}$ such that $\lim_{\epsilon \rightarrow 0} \|P - P_\epsilon\|_0 = 0$. We denote $\mathcal{B} = \lim_{\epsilon \rightarrow 0} \mathcal{A}^\epsilon$.

Remark 2.2. If $\mathcal{A} \subset \Pi_k^l$ is a subspace, then the sets \mathcal{A}^ϵ and \mathcal{B} are also subspaces of Π_k^l . Furthermore, if $\mathcal{A}^\epsilon = \mathcal{A}$, for all $\epsilon > 0$, we have that $\mathcal{B} = \mathcal{A}$.

Next, we show a simple example of $\mathcal{A}^\epsilon = \mathcal{A}$.

Example 2.3. Set $n = 3$ and $\mathcal{A} = \text{span}\{(x_1, x_1 + x_2 + x_3, x_1^2 + x_2^2)\}$. Then, clearly we obtain $\mathcal{A}^\epsilon = \text{span}\{(\epsilon x_1, \epsilon(x_1 + x_2 + x_3), \epsilon^2(x_1^2 + x_2^2))\} = \mathcal{A}$.

Proposition 2.4. Let \mathcal{A} be a subspace of polynomials such that $\Pi_k^m \subset \mathcal{A}$ for some $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Then $\Pi_k^m \subset \mathcal{A}^\epsilon$ for all $\epsilon > 0$. Moreover, $\Pi_k^m \subset \mathcal{B}$.

Proof. Set $R_{\alpha,i}(x) = x^\alpha e_i$, $|\alpha| \leq m$, $1 \leq i \leq k$, where $\{e_i\}_{i=1}^k$ is the canonical basis of \mathbb{R}^k . Then

$$\{R_{\alpha,i} : |\alpha| \leq m, 1 \leq i \leq k\} \tag{2.1}$$

is a basis of the space Π_k^m . Since \mathcal{A}^ϵ is a subspace, we have $R_{\alpha,i} = \frac{1}{\epsilon^{|\alpha|}} R_{\alpha,i}^\epsilon \in \mathcal{A}^\epsilon$, and so $\Pi_k^m \subset \mathcal{A}^\epsilon$, for all $\epsilon > 0$. Finally, using the definition of \mathcal{B} , we obtain $\Pi_k^m \subset \mathcal{B}$. \square

From now on, for any Lebesgue measurable function $F : B \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$ we denote $T_{-1}(F) = 0$.

Proposition 2.5. Let \mathcal{A} be a subspace of Π_k^l and let $0 \leq s + 1 \leq l$ be an integer. If $P \in \mathcal{A}$ satisfies $T_s(P) = 0$ and $T_{s+1}(P) \neq 0$, then $T_{s+1}(P) \in \mathcal{B}$.

Proof. For each $\epsilon > 0$ we define $Q_\epsilon = \frac{P}{\epsilon^{s+1}} \in \mathcal{A}$. Since $T_s(P) = 0$, it follows that $\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = \frac{\|(T_{s+1}(P) - P)^\epsilon\|_0}{\epsilon^{s+1}}$. So $\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = o(1)$ as $\epsilon \rightarrow 0$, and thus $T_{s+1}(P) \in \mathcal{B}$. \square

The following sets will be needed throughout the paper. Let \mathcal{A} be a non-zero subspace of Π_k^l . We define

$$A_{-1} := \mathcal{A} \quad \text{and} \quad A_j := \{P \in \mathcal{A} : T_j(P) = 0\}, \text{ for } 0 \leq j \leq l. \tag{2.2}$$

We note that

$$A_j \subset A_i, \quad \text{whenever } i < j.$$

Since $A_l \subset \{P \in \Pi_k^l : T_l(P) = 0\} = \{0\}$, we have

$$\{j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A}\} \neq \emptyset \quad \text{and} \quad \{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\} \neq \emptyset.$$

Set

$$s_0 = \min \{j : 0 \leq j \leq l \text{ and } A_j \neq \mathcal{A}\}$$

and

$$r_0 = \min \{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\}.$$

It is easy to see that $0 \leq s_0 \leq r_0 \leq l$, and

$$s_0, r_0 \in \{j : s_0 \leq j \leq r_0 \text{ and } A_j \subsetneq A_{j-1}\} =: J.$$

We can now formulate our main result which describes the limit set \mathcal{B} .

Theorem 2.6. Let \mathcal{A} be a non-zero subspace of Π_k^l . Then \mathcal{B} is a subspace of $\Pi_k^{r_0}$ isomorphic to \mathcal{A} . Furthermore, under the above notation the following holds:

- (a) if $s_0 < r_0$ and $J \setminus \{r_0\} = \{s_0, \dots, s_N\}$ with $s_i < s_{i+1}$ for $N > 0$, then $\mathcal{B} = T_{r_0}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \dots \oplus T_{s_0}(S_{s_0})$, where $A_{s_i} \oplus S_{s_i} = A_{s_{i-1}}$, $0 \leq i \leq N$;

(b) if $s_0 = r_0$, then $\mathcal{B} = T_{r_0}(\mathcal{A})$.

Proof. (a) Assume $s_0 < r_0$. Since every subspace of $A_{s_{i-1}}$, $0 \leq i \leq N$, has a complement, there exists a subspace $S_{s_i} \subset A_{s_{i-1}}$ such that

$$A_{s_i} \oplus S_{s_i} = A_{s_{i-1}}, \quad 0 \leq i \leq N. \tag{2.3}$$

In consequence,

$$\mathcal{A} = A_{s_N} \oplus S_{s_N} \oplus S_{s_{N-1}} \oplus \cdots \oplus S_{s_0}. \tag{2.4}$$

As $S_{s_i} \subset A_{s_{i-1}}$, $0 \leq i \leq N$, and $A_{r_0-1} = A_{s_N}$, we obtain

$$Q(x) = \begin{cases} \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in S_{s_i}, \ 0 \leq i \leq N; \\ \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in A_{s_N}, \end{cases} \tag{2.5}$$

where $s_{N+1} = r_0$. Let $T_i : S_{s_i} \rightarrow \Pi_k^{s_i}$ be a linear operator defined by $T_i(P) = T_{s_i}(P)$, $0 \leq i \leq N$, and $T_{N+1} : \mathcal{A} \rightarrow \Pi_k^{s_{N+1}}$ be the linear operator given by $T_{N+1}(P) = T_{s_{N+1}}(P)$. We claim that

- (i) T_i is an injective operator, $0 \leq i \leq N + 1$.
- (ii) $T_{s_{N+1}}(A_{s_N}) \cap \sum_{i=0}^N T_{s_i}(S_{s_i}) = \{0\}$.
- (iii) If $N > 0$ then $T_{s_l}(S_{s_l}) \cap \left(T_{s_{N+1}}(A_{s_N}) + \sum_{i=0, i \neq l}^N T_{s_i}(S_{s_i}) \right) = \{0\}$ whenever $l \neq i$.

Indeed, let $0 \leq i \leq N$. If $T_{s_i}(P) = T_{s_i}(Q)$ for some $P, Q \in S_{s_i}$, then $P - Q \in A_{s_i} \cap S_{s_i}$. So (2.3) implies that $P = Q$. On the other hand, if $T_{s_{N+1}}(P) = T_{s_{N+1}}(Q)$ with $P, Q \in \mathcal{A}$, then $P - Q \in A_{s_{N+1}} = \{0\}$, which proves (i). To prove (ii) we consider $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ such that $P = T_{s_{N+1}}(Q_{N+1}) = \sum_{i=0}^N T_{s_i}(Q_i)$. From (2.5) we see that

$$T_{s_{N+1}}(Q_{N+1})(x) = \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha Q_{N+1}(0)}{\alpha!} x^\alpha \quad \text{and} \quad \sum_{i=0}^N T_{s_i}(Q_i) \in \Pi_k^{s_N}. \tag{2.6}$$

Therefore $P = 0$. Now, let $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ be such that

$$P = T_{s_l}(Q_l) = T_{s_{N+1}}(Q_{N+1}) + \sum_{i=0, i \neq l}^N T_{s_i}(Q_i). \tag{2.7}$$

From (2.5) it follows that

$$T_{s_i}(Q_i) = \sum_{|\alpha|=s_i} \frac{\partial^\alpha Q_i(0)}{\alpha!} x^\alpha, \quad 0 \leq i \leq N.$$

According to (2.6) and (2.7) we have $P = 0$, and (iii) is proved. Using (i)–(iii), we deduce that the subspace

$$T_{s_{N+1}}(A_{s_N}) + T_{s_N}(S_{s_N}) + T_{s_{N-1}}(S_{s_{N-1}}) + \cdots + T_{s_0}(S_{s_0})$$

is a direct sum isomorphic to \mathcal{A} . The proof concludes by proving

$$\mathcal{B} = T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0}).$$

We observe that if $P \in S_{s_i} \setminus \{0\}$, then $T_{s_i}(P) \neq 0$ and $T_{s_{i-1}}(P) = 0$ by (2.3). So, Proposition 2.5 implies that $T_{s_i}(P) \in \mathcal{B}$. On the other hand, if $P \in A_{s_N} \setminus \{0\}$, we get $T_{s_N}(P) = 0$. Moreover, we have $T_{s_{N+1}}(P) \neq 0$. In fact, on the contrary, we see that $P \in A_{s_{N+1}} = \{0\}$. Proposition 2.5 now gives $T_{s_{N+1}}(P) \in \mathcal{B}$. Therefore,

$$T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \dots \oplus T_{s_0}(S_{s_0}) \subset \mathcal{B}.$$

On the other hand, if $P \in \mathcal{B}$, there exists $\{P_\epsilon\} \subset \mathcal{A}$ such that

$$\lim_{\epsilon \rightarrow 0} \|P - P_\epsilon^\epsilon\|_0 = 0. \tag{2.8}$$

Let $d_{N+1} = \dim(A_{s_N})$ and $d_i = \dim(S_{s_i})$, $0 \leq i \leq N$. We take $\{v_l\}_{l=1}^{d_{N+1}}$ and $\{w_{ir}\}_{r=1}^{d_i}$ bases of A_{s_N} and S_{s_i} , respectively. It is easy to check that for each $0 < \epsilon \leq 1$, $\{\epsilon^{-s_{N+1}}v_l\}_{l=1}^{d_{N+1}}$ is a basis of A_{s_N} and $\{\epsilon^{-s_i}w_{ir}\}_{r=1}^{d_i}$ is a basis of S_{s_i} , $0 \leq i \leq N$. According to (2.4), we have that there exist real numbers $D_{l,\epsilon}$ and $C_{i,r,\epsilon}$ such that

$$P_\epsilon = \sum_{l=1}^{d_{N+1}} \epsilon^{-s_{N+1}} D_{l,\epsilon} v_l + \sum_{i=0}^N \sum_{r=1}^{d_i} \epsilon^{-s_i} C_{i,r,\epsilon} w_{ir}.$$

From (2.5) it follows that

$$v_l(x) = \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \quad \text{and} \quad w_{ir}(x) = \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha. \tag{2.9}$$

So,

$$\begin{aligned} P_\epsilon^\epsilon(x) &= \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \epsilon^{-s_{N+1}} v_l^\epsilon(x) + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \epsilon^{-s_i} w_{ir}^\epsilon(x) \\ &= \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha| > s_{N+1}} \epsilon^{|\alpha|-s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \\ &\quad + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha|=s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| > s_i} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha. \end{aligned}$$

Consequently

$$\begin{aligned} T_{s_j}(P_\epsilon^\epsilon)(x) &= \sum_{i=0}^j \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha|=s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \\ &\quad + \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_i < |\alpha| \leq s_j} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \end{aligned}$$

if $0 \leq j \leq N$, and

$$\begin{aligned}
 T_{s_{N+1}}(P_\epsilon^\epsilon)(x) &= \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha|=s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \\
 &\quad + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_i < |\alpha| \leq s_{N+1}} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.
 \end{aligned}$$

From (2.9) it follows that

$$T_{s_{N+1}}(v_\ell)(x) = \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_\ell(0)}{\alpha!} x^\alpha \quad \text{and} \quad T_{s_j}(w_{j,r})(x) = \sum_{|\alpha|=s_j} \frac{\partial^\alpha w_{j,r}(0)}{\alpha!} x^\alpha.$$

Thus, a straightforward computation yields

$$T_{s_0}(P_\epsilon^\epsilon)(x) = \sum_{r=1}^{d_0} C_{0,r,\epsilon} T_{s_0}(w_{0,r})(x), \tag{2.10}$$

$$\begin{aligned}
 T_{s_j}(P_\epsilon^\epsilon)(x) &= T_{s_{j-1}}(P_\epsilon^\epsilon)(x) + \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_{j-1} < |\alpha| \leq s_j} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha \\
 &\quad + \sum_{r=1}^{d_j} C_{j,r,\epsilon} T_{s_j}(w_{j,r})(x)
 \end{aligned} \tag{2.11}$$

if $1 \leq j \leq N$, and

$$\begin{aligned}
 T_{s_{N+1}}(P_\epsilon^\epsilon)(x) &= T_{s_N}(P_\epsilon^\epsilon)(x) + \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} T_{s_{N+1}}(v_\ell)(x) \\
 &\quad + \sum_{i=0}^N \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_N < |\alpha| \leq s_{N+1}} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha.
 \end{aligned} \tag{2.12}$$

From (2.8) and (2.10), we deduce that $T_{s_0}(P_\epsilon^\epsilon)(x) = \sum_{r=1}^{d_0} C_{0,r,\epsilon} T_{s_0}(w_{0,r})(x)$ is convergent as $\epsilon \rightarrow 0$. Since $\{T_{s_0}(w_{0,r})\}_{r=1}^{d_0}$ is a basis of $T_{s_0}(S_{s_0})$, there are real numbers $C_{0,r}$, $1 \leq r \leq d_0$, such that $C_{0,r,\epsilon} \rightarrow C_{0,r}$ as $\epsilon \rightarrow 0$. According to (2.8) and (2.11) it follows that $\sum_{r=1}^{d_1} C_{1,r,\epsilon} T_{s_1}(w_{1,r})(x)$ is convergent as $\epsilon \rightarrow 0$. Hence, there are real numbers $C_{1,r}$, $1 \leq r \leq d_1$, such that $C_{1,r,\epsilon} \rightarrow C_{1,r}$ as $\epsilon \rightarrow 0$, because $\{T_{s_1}(w_{1r})\}_{r=1}^{d_1}$ is a basis of $T_{s_1}(S_{s_1})$. Similarly, as $\{T_{s_{N+1}}(v_l)\}_{l=1}^a$ is a basis of $T_{s_{N+1}}(A_{s_N})$ and $\{T_{s_i}(w_{ir})\}_{r=1}^{d_i}$ is a basis of $T_{s_i}(S_{s_i})$, $0 \leq i \leq N$, (2.8) and (2.10)–(2.12) show that there are real numbers D_l and $C_{i,r}$ such that $D_{l,\epsilon} \rightarrow D_l$ and $C_{i,r,\epsilon} \rightarrow C_{i,r}$ as $\epsilon \rightarrow 0$. In consequence,

$$P = \sum_{l=1}^a D_l T_{s_{N+1}}(v_l) + \sum_{i=0}^N \left(\sum_{r=1}^{d_i} C_{i,r} T_{s_i}(w_{ir}) \right),$$

and so $P \in T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \dots \oplus T_{s_0}(S_{s_0})$.

(b) Now assume $s_0 = r_0$, i.e., $A_{s_0} = \{0\}$. Then \mathcal{A} has the form (2.4) with $N = 0$, $A_{s_0} = \{0\}$ and $S_{s_0} = \mathcal{A}$. An analysis similar to the proof of (a) shows that T_{r_0} is an isomorphism and $\mathcal{B} = T_{s_0}(S_{s_0}) = T_{r_0}(\mathcal{A})$. \square

The following corollary follows immediately from the proof of Theorem 2.6.

Corollary 2.7. *Let \mathcal{A} be a non-zero subspace of Π_k^l . Then $\lim_{n \rightarrow \infty} \mathcal{A}^{\epsilon_n} = \mathcal{B}$ for any sequence $\{\epsilon_n\}$ of the net $\epsilon \downarrow 0$.*

Remark 2.8. \mathcal{B} is isomorphic to $T_{r_0}(\mathcal{A})$.

Corollary 2.9. *Let $s \geq m+1$ and let $\mathcal{A} = \Pi_k^m \oplus A_{s-1}$ be such that $A_s = \{0\}$. Then $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1})$ and the linear operator $T : \mathcal{A} \rightarrow \Pi_k^s$ given by $T(P) = T_s(P)$ defines an isomorphism between \mathcal{A} and \mathcal{B} .*

Proof. We first claim that T is an injective operator. Indeed, if $T(P) = T(Q)$ for $P, Q \in \mathcal{A}$, then $T_s(P-Q) = 0$ and so $P-Q \in A_s$. Since $A_s = \{0\}$, we have $P = Q$.

Since \mathcal{A} is isomorphic to $T(\mathcal{A})$, the proof concludes by proving $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1}) = T_s(\mathcal{A})$.

Let A_j be the sets defined in (2.2). Since

$$\{0\} = A_s \subsetneq A_{s-1} = \dots = A_m \subsetneq A_{m-1} \subsetneq \dots \subsetneq A_0 \subsetneq \mathcal{A},$$

then $\mathcal{A} = A_{s-1} \oplus B_m \oplus B_{m-1} \oplus \dots \oplus B_0$, where $A_i \oplus B_i = A_{i-1}$, $0 \leq i \leq m$. Therefore Π_k^m is isomorphic to $B_m \oplus \dots \oplus B_0$. On the other hand, since $s_0 = 0$, $r_0 = s$ and $J \setminus \{r_0\} = \{0, 1, \dots, m\}$, by Proposition 2.6 (a),

$$\mathcal{B} = T_s(A_{s-1}) \oplus T_m(B_m) \oplus \dots \oplus T_0(B_0).$$

From the proof of Theorem 2.6, we obtain that $B_m \oplus \dots \oplus B_0$ is isomorphic to $T_m(B_m) \oplus \dots \oplus T_0(B_0)$, and consequently Π_k^m is isomorphic to $T_m(B_m) \oplus \dots \oplus T_0(B_0) \subset \Pi_k^m$. Hence, $T_m(B_m) \oplus \dots \oplus T_0(B_0) = \Pi_k^m$ and so $\mathcal{B} = T_s(A_{s-1}) \oplus \Pi_k^m = T_s(A_{s-1}) \oplus T_s(\Pi_k^m) = T_s(\mathcal{A})$. \square

3. AN APPLICATION TO BEST LOCAL APPROXIMATION

Let $\{P_\epsilon\}$ be a net of best approximants to F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$, and let E_ϵ be the error function

$$E_\epsilon(F) = \frac{F^\epsilon - P_\epsilon^\epsilon}{\epsilon^{m+1}}.$$

If $F \in t^{m+1}$, then

$$F^\epsilon = T_{m+1}^\epsilon + \epsilon^{m+1} R_{m+1}^\epsilon,$$

where $R_{m+1}^\epsilon = \frac{F - T_{m+1}}{\epsilon^{m+1}}$, $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$, and T_{m+1} is the Taylor polynomial of F of degree $m+1$ at 0. Moreover,

$$\lambda P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(\lambda F^\epsilon) \quad \text{and} \quad P^\epsilon + P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}((P + F)^\epsilon), \quad \text{for } P \in \mathcal{A}.$$

The following proposition may be proved in much the same way as [13, Proposition 4.1]. However, we repeat the proof for completeness.

Proposition 3.1. *Let \mathcal{A} be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon),$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \rightarrow 0$.

Proof. Since $R_{m+1}^\epsilon = \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}}$, then

$$\begin{aligned} \phi_{m+1} + R_{m+1}^\epsilon &= T_{m+1} - T_m + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} = \frac{T_{m+1}^\epsilon - T_m^\epsilon}{\epsilon^{m+1}} + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} \\ &= \frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}}. \end{aligned}$$

As $T_m \in \mathcal{A}$, we have

$$\begin{aligned} \phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon) &= \frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}} - \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon} \left(\frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}} \right) \\ &= \frac{F^\epsilon - P_\epsilon^\epsilon}{\epsilon^{m+1}} = E_\epsilon(F). \quad \square \end{aligned}$$

Next, we give a new result about the asymptotic behavior of the error without the conditions (c1) or (c2), which generalizes Theorems 4.2 and 4.5 of [13].

Theorem 3.2. *Let \mathcal{A} be a non-zero subspace of Π_k^l with $l > m$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$\|E_\epsilon(F)\|_\epsilon \rightarrow \inf_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. By Proposition 3.1,

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon), \tag{3.1}$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \rightarrow 0$. We first prove

$$\overline{\lim}_{\epsilon \rightarrow 0} \|E_\epsilon(F)\|_\epsilon \leq \inf_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0. \tag{3.2}$$

In fact, let $P \in \mathcal{B}$. By the definition of \mathcal{B} , there exists a net $\{Q_\epsilon\} \subset \mathcal{A}$ such that $\|P - Q_\epsilon^\epsilon\|_0 \rightarrow 0$, as $\epsilon \rightarrow 0$. In consequence, $\|P - Q_\epsilon^\epsilon\|_\epsilon = o(1)$, as $\epsilon \rightarrow 0$, by (1.3). Since $Q_\epsilon^\epsilon \in \mathcal{A}^\epsilon$ and $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$, from (3.1) we obtain

$$\|E_\epsilon(F)\|_\epsilon \leq \|\phi_{m+1} + R_{m+1}^\epsilon - Q_\epsilon^\epsilon\|_\epsilon \leq \|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon + o(1), \quad \text{as } \epsilon \rightarrow 0. \tag{3.3}$$

By Property (3), $\|\phi_{m+1} - P\|_\epsilon \rightarrow \|\phi_{m+1} - P\|_0$, as $\epsilon \rightarrow 0$. Hence, using the triangle inequality we have

$$\begin{aligned} \|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_0 &\leq \|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_\epsilon \\ &\quad + \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0 \\ &\leq \|P - Q_\epsilon^\epsilon\|_\epsilon + \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0 = o(1) \end{aligned}$$

as $\epsilon \rightarrow 0$. Now, according to (3.3) we get (3.2).

The proof finishes by observing that

$$\liminf_{\epsilon \rightarrow 0} \|E_\epsilon(F)\|_\epsilon \geq \inf_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0. \tag{3.4}$$

Let $\epsilon \downarrow 0$ be a sequence such that $\lim_{\epsilon \rightarrow 0} \|E_\epsilon(F)\|_\epsilon = \lim_{\epsilon \rightarrow 0} \|E_\epsilon(F)\|_\epsilon$. We consider $P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon, \epsilon}(\phi_{m+1} + R_{m+1}^\epsilon)$. We claim that there exist constants $M, \epsilon_0 > 0$ such that

$$\|P_\epsilon^\epsilon\|_0 \leq M, \quad 0 < \epsilon \leq \epsilon_0. \tag{3.5}$$

Indeed, as $0 \in \mathcal{A}^\epsilon$ we get

$$\begin{aligned} \|P_\epsilon^\epsilon\|_\epsilon &\leq \|P_\epsilon^\epsilon - (\phi_{m+1} + R_{m+1}^\epsilon)\|_\epsilon + \|\phi_{m+1} + R_{m+1}^\epsilon\|_\epsilon \\ &\leq 2\|\phi_{m+1} + R_{m+1}^\epsilon\|_\epsilon \\ &\leq 2\|\phi_{m+1}\|_\epsilon + 2\|R_{m+1}^\epsilon\|_\epsilon, \end{aligned} \tag{3.6}$$

for $0 < \epsilon \leq 1$. By Proposition 3.1 and Property (3), we see that $2\|\phi_{m+1}\|_\epsilon + 2\|R_{m+1}^\epsilon\|_\epsilon \rightarrow 2\|\phi_{m+1}\|_0$, as $\epsilon \rightarrow 0$. So, from (1.3) and (3.6), we obtain (3.5).

In consequence, there exists a subsequence of $\{P_\epsilon^\epsilon\}$, which is denoted in the same way, and $P_0 \in \Pi_k^l$ such that $P_\epsilon^\epsilon \rightarrow P$ uniformly on B , as $\epsilon \rightarrow 0$. Since $\|\phi_{m+1} - P_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_0 \leq \|\phi_{m+1} - P_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_\epsilon + \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0 \leq \|P - P_\epsilon^\epsilon\|_\epsilon + \|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0$, using Property (3) we get

$$\|\phi_{m+1} - P\|_0 = \|\phi_{m+1} - P_\epsilon^\epsilon\|_\epsilon + o(1), \quad \text{as } \epsilon \rightarrow 0.$$

We observe that $P \in B$ by Corollary 2.7. Therefore, by Proposition 3.1,

$$\begin{aligned} \inf_{Q \in \mathcal{B}} \|\phi_{m+1} - Q\|_0 &\leq \|\phi_{m+1} - P\|_0 = \|\phi_{m+1} - P_\epsilon^\epsilon\|_\epsilon + o(1) \\ &\leq \|\phi_{m+1} + R_{m+1}^\epsilon - P_\epsilon^\epsilon\|_\epsilon + \|R_{m+1}^\epsilon\|_\epsilon \\ &= \|E_\epsilon(F)\|_\epsilon + \|R_{m+1}^\epsilon\|_\epsilon. \end{aligned}$$

So, $\inf_{Q \in \mathcal{B}} \|\phi_{m+1} - Q\|_0 \leq \lim_{\epsilon \rightarrow 0} (\|E_\epsilon(F)\|_\epsilon + \|R_{m+1}^\epsilon\|_\epsilon) = \lim_{\epsilon \rightarrow 0} \|E_\epsilon(F)\|_\epsilon$, and (3.4) is proved. □

The following result provides us with a useful and important property for a net of best approximants to F from \mathcal{A} .

Theorem 3.3. *Let \mathcal{A} be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. Assume $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$. If \mathcal{C} is the cluster point set of the net $\left\{ \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\}$, as $\epsilon \rightarrow 0$, then $\mathcal{C} \neq \emptyset$. Moreover, each polynomial in \mathcal{C} is a solution of the minimization problem*

$$\min_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0. \tag{3.7}$$

Proof. We observe that

$$\begin{aligned} E_\epsilon(F) &= \frac{(F - P_\epsilon)^\epsilon}{\epsilon^{m+1}} = \frac{(T_{m+1} - T_m)^\epsilon + (F - T_{m+1})^\epsilon - (P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \\ &= \frac{\phi_{m+1}^\epsilon - (P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} + \frac{(F - T_{m+1})^\epsilon}{\epsilon^{m+1}} \\ &= \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} + \frac{(F - T_{m+1})^\epsilon}{\epsilon^{m+1}}. \end{aligned}$$

Then

$$\begin{aligned} \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon - \frac{\|(F - T_{m+1})^\epsilon\|_\epsilon}{\epsilon^{m+1}} \\ \leq \|E_\epsilon(F)\|_\epsilon \leq \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon + \frac{\|(F - T_{m+1})^\epsilon\|_\epsilon}{\epsilon^{m+1}}, \end{aligned}$$

and consequently,

$$\|E_\epsilon(F)\|_\epsilon = \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon + o(1), \quad \text{as } \epsilon \rightarrow 0,$$

since $F \in t^{m+1}$. By Theorem 3.2,

$$\inf_{P \in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon \rightarrow 0} \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon. \tag{3.8}$$

According to (1.3), there exist constants $\epsilon_0, M > 0$ such that

$$\left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_0 \leq M,$$

for all $0 < \epsilon \leq \epsilon_0$. The equivalence of the norms in Π_k^l implies that the net $\left\{ \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\}_{0 < \epsilon \leq \epsilon_0}$ is uniformly bounded on B . So, there exists a subsequence of $\left\{ \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\}_{0 < \epsilon \leq \epsilon_0}$, which is denoted in the same way, and a polynomial P_0 such that

$$\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \text{ converges to } P_0, \text{ uniformly on } B, \text{ as } \epsilon \rightarrow 0.$$

In consequence, $\mathcal{C} \neq \emptyset$.

On the other hand, if $P_0 \in \mathcal{C}$, there is a sequence $\epsilon \downarrow 0$ such that $\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \rightarrow P_0$. Since $T_m \in \mathcal{A}$, we have $P_\epsilon - T_m \in \mathcal{A}$, and so $P_0 \in \mathcal{B}$ by Corollary 2.7. Finally, from Property (3) and (3.8) we conclude that

$$\inf_{P \in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon \rightarrow 0} \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon = \|\phi_{m+1} - P_0\|_0,$$

i.e., P_0 is a solution of (3.7). □

The following theorem is an extension of [13, Theorem 5.1].

Theorem 3.4. *Let \mathcal{A} be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. Assume $m + 1 = \min \{j : 0 \leq j \leq l \text{ and } A_j = \{0\}\}$, $F \in t^{m+1}$ with $T_m \in \mathcal{A}$, and set $\phi_{m+1} = T_{m+1} - T_m$. If the minimization problem (3.7) has a unique solution P_0 , then $P_\epsilon \rightarrow T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

Proof. Since (3.7) has a unique solution P_0 , Theorem 3.3 implies that

$$\lim_{\epsilon \rightarrow 0} \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} = P_0.$$

In consequence, $\partial^\alpha(P_\epsilon - T_m)(0) \rightarrow 0$, $|\alpha| \leq m$, and $\partial^\alpha(P_\epsilon - T_m)(0) \rightarrow \partial^\alpha P_0(0)$, $|\alpha| = m + 1$, as $\epsilon \rightarrow 0$. Therefore

$$T_{m+1}(P_\epsilon - T_m)(x) \rightarrow \sum_{|\alpha|=m+1} \frac{\partial^\alpha P_0(0)}{\alpha!} x^\alpha =: R(x), \quad x \in B, \text{ as } \epsilon \rightarrow 0. \quad (3.9)$$

Let $T : \mathcal{A} \rightarrow \Pi_k^{m+1}$ be the linear operator defined by $T(P) = T_{m+1}(P)$. As $A_{m+1} = \{0\}$, an analysis similar to that in the proof of Corollary 2.9 shows that T is an injective operator. Since $T(\mathcal{A})$ is a closed subspace and $\{T_{m+1}(P_\epsilon - T_m)\} \subset T(\mathcal{A})$, (3.9) implies that there exists a unique $P \in \mathcal{A}$ such that $T_{m+1}(P) = R$. Hence $T_{m+1}(P_\epsilon - T_m - P) \rightarrow 0$ as $\epsilon \rightarrow 0$. As $A_{m+1} = \{0\}$ we see that $\|Q\| := \|T_{m+1}(Q)\|_0$ is a norm on \mathcal{A} , and so $P_\epsilon \rightarrow T_m + P$ as $\epsilon \rightarrow 0$. Finally, by Theorem 2.6, $\mathcal{B} \subset \Pi_k^{m+1}$, and consequently $P_0 - T_m(P_0) = T_{m+1}(P_0) - T_m(P_0) = R$. The proof is complete. \square

Remark 3.5. If \mathcal{A} satisfies the condition (c2), then $\mathcal{A} = \Pi_k^m \oplus A_m$ with $A_{m+1} = \{0\}$. By Corollary 2.9, $\mathcal{B} = \Pi_k^m \oplus T_{m+1}(A_m)$ and each element $P \in \mathcal{A}$ is uniquely determined by $T_{m+1}(P)$. So, we can rewrite the problem (3.7) in the following (equivalent) form:

$$\min_{Q+U \in \Pi_k^m \oplus A_m} \|\phi_{m+1} - (Q + T_{m+1}(U))\|_0. \quad (3.10)$$

The following result has been proved in [13, Theorem 5.1] and it is a consequence of Theorem 3.4.

Corollary 3.6. *Let $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ be a non-zero subspace that satisfies the condition (c2) and let $\{P_\epsilon\}$ be a net of best approximants of F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. Assume $F \in t^{m+1}$. If the minimization problem (3.10) has a unique solution P_0 , then $P_\epsilon \rightarrow T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

In the following example we present a function $F \in \bigcap_{m=0}^\infty t^m$ such that $T_2(F) \notin \mathcal{A}$ and the net $\{T_i(P_\epsilon)\}$ does not converge for the same $i > m + 1$.

Example 3.7. Set $B = [-1, 1]$, $\|G\|_\epsilon = \left(\int_{-1}^1 |G(x)|^2 dx \right)^{\frac{1}{2}}$, $\mathcal{A} = \text{span}\{1, x^2, x^3\}$, and $F(x) = x$. So

$$\|G\|_\epsilon^* = \left(\frac{1}{\epsilon} \int_{-\epsilon}^\epsilon |G(x)|^2 dx \right)^{\frac{1}{2}},$$

$A_0 = A_1 = \text{span}\{x^2, x^3\}$, $A_2 = \text{span}\{x^3\}$ and $A_3 = \{0\}$. Since $T_1(x^2) = 0$, we observe that the subspace \mathcal{A} does not satisfy the condition (c2). Moreover, an straightforward computation shows that

$$\frac{\|F - T_0\|_\epsilon^*}{\epsilon^0} = \frac{\sqrt{6}}{3} \epsilon \quad \text{and} \quad \frac{\|F - T_s\|_\epsilon^*}{\epsilon^s} = 0, \quad s \in \mathbb{N},$$

where $T_0(x) = 0$ and $T_s(x) = x$. In consequence, $F \in t^m$ for all $m \in \mathbb{N} \cup \{0\}$, and $T_2(F) \notin \mathcal{A}$. Since $\int_{-\epsilon}^\epsilon (x - \frac{7}{5\epsilon^2}x^3) x^i dx = 0$, $i = 0, 2, 3$, then $P_\epsilon(x) = \frac{7}{5\epsilon^2}x^3$ is the best approximant to F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. Therefore $T_i(P_\epsilon)(x) \rightarrow 0$, for $i = 0, 1, 2$, but $T_3(P_\epsilon)(x)$ does not converge, as $\epsilon \rightarrow 0$. So, the best local approximation to F from \mathcal{A} in 0 does not exist, and

$$\|E_\epsilon(F)\|_\epsilon = \frac{\|F - P_\epsilon\|_\epsilon^*}{\epsilon^3} = \frac{2\sqrt{6}}{15\epsilon^2} \rightarrow \infty, \quad \text{as } \epsilon \rightarrow 0.$$

We now give another example which shows that the condition $T_m \in \mathcal{A}$ is not necessary for the existence of the best local approximation.

Example 3.8. Set B , $\|\cdot\|_\epsilon^*$ and F as in Example 3.7, and we consider the subspace $\mathcal{A} = \text{span}\{1, x^2\}$. It is clear that $A_0 = A_1 = \text{span}\{x^2\}$, $A_2 = \{0\}$ and $\mathcal{B} = \mathcal{A}$. Moreover, we have $F \in t^2$, $T_1 \notin \mathcal{A}$, and \mathcal{A} does not satisfy the condition (c2) since $T_1(x^2) = 0$. As $\int_{-\epsilon}^\epsilon (x - 0) x^i dx = 0$, $i = 0, 2$, then $P_\epsilon(x) = 0$ is the best approximant to F from \mathcal{A} with respect to $\|\cdot\|_\epsilon^*$. Therefore, the polynomial 0 is the best local approximation to F from \mathcal{A} in 0.

REFERENCES

[1] Billingsley, P., *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968. MR 0233396.
 [2] Chui, C.K., Shisha, O., Smith, P.W., Best local approximation, *J. Approx. Theory* 15 (1975), no. 4, 371–381. MR 0433101.
 [3] Chui, C.K., Smith, P.W., Ward, J.D., Best L_2 approximation, *J. Approx. Theory* 22 (1978), no. 3, 254–261. MR 0510758.
 [4] Chui, C.K., Diamond, H., Raphael, L.A., Best local approximation in several variables, *J. Approx. Theory* 40 (1984), no. 4, 343–350. MR 0740646.
 [5] Cuenya, H.H., Ferreyra, D.E., C^p condition and the best local approximation, *Anal. Theory Appl.* 31 (2015), no. 1, 58–67. MR 3338788.
 [6] Favier, S., Convergence of function averages in Orlicz spaces, *Numer. Funct. Anal. Optim.* 15 (1994), no. 3-4, 263–278. MR 1272205.
 [7] Feller, W., *An Introduction to Probability and Its Applications*, Vol. II, John Wiley & Sons, New York, 1966. MR 0210154.

- [8] Headley, V.B., Kerman, R.A., Best local approximations in $L^p(\mu)$, *J. Approx. Theory* 62 (1990), no. 3, 277–281. MR 1070281.
- [9] Macías, R.A., Zó, F., Weighted best local L^p approximation, *J. Approx. Theory* 42 (1984), no. 2, 181–192. MR 0760741.
- [10] Maehly, H., Witzgall, Ch., Tschebyscheff-Approximationen in kleinen Intervallen. I. Approximation durch Polynome, *Numer. Math.* 2 (1960), 142–150. MR 0119002.
- [11] Walsh, J.L., On approximation to an analytic function by rational functions of best approximation. *Math. Z.* 38 (1934), no. 1, 163–176. MR 1545445.
- [12] Wolfe, J.M., Interpolation and best L_p local approximation. *J. Approx. Theory* 32 (1981), no. 2, 96–102. MR 0633695.
- [13] Zó, F., Cuenya, H.H., Best approximations on small regions. A general approach. In: *Advanced Courses of Mathematical Analysis II (Granada, 2004)*, 193–213. World Scientific, Hackensack, NJ, 2007. MR 2343602.

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