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SECOND COHOMOLOGY SPACE OF $\mathfrak{sl}(2)$ ACTING ON THE SPACE OF BILINEAR BIDIFFERENTIAL OPERATORS

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Abstract. We consider the $\mathfrak{sl}(2)$-module structure on the spaces of bilinear bidifferential operators acting on the spaces of weighted densities. We compute the second cohomology group of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators that act on tensor densities $\mathcal{D}_{\lambda,\nu,\mu}$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. We shall associate a cochain complex known as the *Chevalley–Eilenberg differential.* The n-th space of this complex will be denoted by $C^n(\mathfrak{g}, M)$.

For $n > 0$, it is the space of n-linear antisymmetric mappings of $\mathfrak{g}$ into M: they will be called n-cochains of $\mathfrak{g}$ with coefficients in M. The space of 0-cochains $C^0(\mathfrak{g}, M)$ reduces to M. The differential δ^n will be defined by the following formula: for $c \in C^n(\mathfrak{g}, M)$, the $(n+1)$-cochain $\delta^n(c)$ evaluated on $g_1, g_2, \ldots, g_{n+1} \in \mathfrak{g}$ gives:

$$\begin{aligned}\delta^n c(g_1, \ldots, g_{n+1}) = &\sum_{1\le s<t\le n+1} (-1)^{s+t-1} c([g_s, g_t], g_1, \ldots, \hat{g}_s, \ldots, \hat{g}_t, \ldots, g_{q+1}) \\ &+ \sum_{1\le s\le n+1} (-1)^s g_s c(g_1, \ldots, \hat{g}_s, \ldots, g_{n+1});\end{aligned}$$

the notation $\hat{g}_i$ indicates that the i-th term is omitted.

We check that $\delta^{n+1} \circ \delta^n = 0$, so we have a complex:

$$0 \to C^0(\mathfrak{g}, M) \to \cdots \to C^{n-1}(\mathfrak{g}, M) \overset{\delta^{n-1}}{\to} C^n(\mathfrak{g}, M) \to \cdots$$

We denote by $\mathrm{H}^n(\mathfrak{g}, M) = \ker d^n / \operatorname{Im} d^{n-1}$ the quotient space. This space is called the space of n-cohomology of $\mathfrak{g}$ with coefficients in M. We denote by:

$Z^n(\mathfrak{g}, M) = \ker \delta_n$: the space of n-cocycles;

$B^n(\mathfrak{g}, M) = \operatorname{Im} \delta_{n-1}$: the space of n-coboundaries.

For $M = \mathbb{R}$ (or $\mathbb{C}$) considered as a trivial module, we denote the cohomologies, in this case, by $\mathrm{H}^n(\mathfrak{g})$.

We shall now recall classical interpretations of cohomology spaces of low degrees.

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• The space $\mathrm{H}^0(\mathfrak{g}, M) \simeq \mathrm{Inv}_{\mathfrak{g}}(M) := \{m \in M :\ \forall X \in \mathfrak{g},\ X.m = 0\}$.

• The space $\mathrm{H}^1(\mathfrak{g}, M)$ classifies derivations of $\mathfrak{g}$ with values in M modulo inner ones (see [1]). This result is particularly useful when $M = \mathfrak{g}$ with the adjoint representation. In this case, a derivation is a map $\varrho : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

$$\varrho([X, Y]) - [\varrho(X), Y] - [X, \varrho(Y)] = 0,$$

while an inner derivation is given by the adjoint action of some element $Z \in \mathfrak{g}$.

◦ If $M = \mathrm{Hom}(\mathcal{N}, \mathcal{M})$, the nontrivial extensions of $\mathfrak{g}$-modules are classified by the first cohomology group $\mathrm{H}^1(\mathfrak{g}, \mathrm{Hom}(\mathcal{N}, \mathcal{M}))$ (see e.g. [4, 5]). Any 1-cocycle Υ generates a new action on $\mathcal{M} \oplus \mathcal{N}$ as follows: for all $g \in \mathfrak{g}$ and for all $(\phi, \varphi) \in \mathcal{M} \oplus \mathcal{N}$, we define

$$g^*(\phi, \varphi) := (g^*\phi + \Upsilon(\varphi), g^*\varphi).$$

◦ Let $\rho_0 : \mathfrak{g} \to \mathrm{End}(V)$ be an action of a Lie algebra $\mathfrak{g}$ on a vector space V. It is well known that the first cohomology space $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$ determines and classifies infinitesimal deformations up to equivalence. Thus, if $\dim \mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V)) = m$, then choose 1-cocycles $\Upsilon_1, \ldots, \Upsilon_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$ and consider the infinitesimal deformation

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i \Upsilon_i,$$

where $t_1, \ldots, t_m$ are independent parameters.

• The space $\mathrm{H}^2(\mathfrak{g}, M)$ classifies central extensions of $\mathfrak{g}$ by M (see [8, 7]), i.e. short exact sequences of Lie algebras

$$0 \to M \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

in which M is considered as an abelian Lie algebra. We shall mainly consider two particular cases of this situation which will be extensively studied in the sequel:

◦ If M is a trivial $\mathfrak{g}$-module (typically $M = \mathbb{R}$ or $\mathbb{C}$), $\mathrm{H}^2(\mathfrak{g}, M)$ classifies central extensions modulo trivial ones. Recall that a central extension of $\mathfrak{g}$ by $\mathbb{R}$ produces a new Lie bracket on $\hat{\mathfrak{g}} = \mathfrak{g} \oplus M$ by setting

$$[(X, \lambda), (Y, \mu)] = ([X, Y], c(X, Y)).$$

It is trivial if the cocycle $c = dl$ is a coboundary of a 1-cochain l, in which case the map $(X, \lambda) \to (X, \lambda - l(X))$ yields a Lie isomorphism between $\hat{\mathfrak{g}}$ and $\mathfrak{g} \oplus M$ considered as a direct sum of Lie algebras.

◦ If $M = \mathfrak{g}$ with the adjoint representation, then $\mathrm{H}^2(\mathfrak{g}, \mathfrak{g})$ classifies infinitesimal deformations modulo trivial ones. By definition, a (formal) series

$$(X, Y) \to \Phi_\lambda(X, Y) := [X, Y] + \lambda f_1(X, Y) + \lambda^2 f_2(X, Y) + \cdots \tag{1.1}$$

is a deformation of Lie bracket $[\,,]$ if Φ_λ is a Lie bracket for every λ, i.e. it is an antisymmetric bilinear form in X, Y and satisfies Jacobi's identity. If one sets simply

$$[X, Y]_\lambda = [X, Y] + \lambda c(X, Y), \tag{1.2}$$

c being a 2-cochain with values in $\mathfrak{g}$ and λ being a scalar, then this bracket satisfies Jacobi's identity modulo terms of order $O(\lambda^2)$ if and only if c is a 2-cocycle.

Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X_h = h\frac{d}{dx}$, where $h \in \mathcal{C}^\infty(\mathbb{R})$ on $\mathbb{R}$. Consider the 1-parameter deformation of the $\mathrm{Vect}(\mathbb{R})$ action on $\mathcal{C}^\infty(\mathbb{R})$:

$$L^\lambda_{X_h}(f) = hf' + \lambda h' f,$$

where f', h' are respectively $\frac{df}{dx}$, $\frac{dh}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}(\mathbb{R})$-module structure on $\mathcal{C}^\infty(\mathbb{R})$ defined by L^λ for a fixed λ.

Each bilinear bidifferential operator A on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $\lambda, \nu, \mu \in \mathbb{R}$, by $f dx^\lambda \otimes g dx^\nu \mapsto A(f \otimes g)dx^\mu$,

$$A(f dx^\lambda \otimes g dx^\nu) = \sum_{k=0}^{m} \sum_{i+j=k} a_{i,j} f^i g^j dx^\mu,$$

where the coefficients $a_{i,j}$ are constants.

The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$ as follows:

$$X_h.A = L^\mu_{X_h} \circ A - A \circ L^{(\lambda,\nu)}_{X_h}, \tag{1.3}$$

where $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ defined by the Leibniz rule:

$$L^{(\lambda,\nu)}_{X_h}(f \otimes g) = L^\lambda_{X_h}(f) \otimes g + f \otimes L^\nu_{X_h}(g).$$

If we restrict ourselves to the Lie algebra $\mathfrak{sl}(2)$, which is isomorphic to the Lie subalgebra of $\mathrm{Vect}(\mathbb{R})$ spanned by

$$\{X_1, X_x, X_{x^2}\},$$

we have a family of infinite dimensional $\mathfrak{sl}(2)$-modules still denoted by $\mathcal{D}_{\lambda,\nu,\mu}$. Bouarroudj, in [5], computes the cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu})$, where $\mathrm{H}^1_{\mathrm{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered (see e.g. [6]). In this paper we compute the second cohomology space $\mathrm{H}^2_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu})$ of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$. Moreover, we give explicit formulae for non trivial 2-cocycles which generate these spaces.

2. Vect(ℝ)-module structures on the space of bilinear bidifferential operators

The Lie algebra $\mathfrak{sl}(2)$ is realized as subalgebra of the Lie algebra $\mathrm{Vect}(\mathbb{R})$,

$$\mathfrak{sl}(2) = \mathrm{Span}\left(X_1 = \frac{d}{dx},\, X_x = x\frac{d}{dx},\, X_{x^2} = x^2\frac{d}{dx}\right), \tag{2.1}$$

corresponding to the fraction-linear transformations

$$x \mapsto \frac{ax+b}{cx+d}, \qquad ad - bc = 1.$$

A *projective structure* on $\mathbb{R}$ (or S^1) is given by an atlas with fraction-linear coordinate transformations (in other words, by an atlas such that the $\mathfrak{sl}(2)$-action (2.1) is well-defined).

The commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_x] = 0, \qquad [X_1, X_1] = 0,$$
$$[X_1, X_{x^2}] = 2X_x, \qquad [X_x, X_{x^2}] = X_{x^2}, \qquad [X_{x^2}, X_{x^2}] = 0.$$

2.1. **The space of tensor densities on $\mathbb{R}$.** Let $\mathrm{Vect}(\mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}$. Consider the 1-parameter deformation of the $\mathrm{Vect}(\mathbb{R})$ action on $\mathcal{C}^\infty(\mathbb{R})$:

$$L^\lambda_{X_h}(f) = hf' + \lambda h' f,$$

where f', h' are respectively $\frac{df}{dx}$, $\frac{dh}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}(\mathbb{R})$-module structure on $\mathcal{C}^\infty(\mathbb{R})$ defined by L^λ for a fixed λ. Geometrically, $\mathcal{F}_\lambda = \left\{ f dx^\lambda : f \in \mathcal{C}^\infty(\mathbb{R}) \right\}$ is the space of weighted densities of weight $\lambda \in \mathbb{R}$, so its elements can be represented as $f(x)dx^\lambda$, where $f(x)$ is a function and dx^λ is a formal (for the time being) symbol. This space coincides with the space of vector fields, functions, and differential forms for $\lambda = -1$, 0, and 1, respectively.

The space $\mathcal{F}_\lambda$ is a $\mathrm{Vect}(\mathbb{R})$-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(f dx^\lambda) = (gf' + \lambda g' f) dx^\lambda. \tag{2.2}$$

2.2. **The space of bilinear bidifferential operators as a $\mathrm{Vect}(\mathbb{R})$-module.** We are interested in defining a cohomology of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ with coefficients in the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$. The counterpart $\mathrm{Vect}(\mathbb{R})$-modules of the space of linear differential operators is a classical object (see e.g. [9]).

Consider bilinear bidifferential operators that act on tensor densities:

$$A : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \longrightarrow \mathcal{F}_\mu. \tag{2.3}$$

The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space of bilinear bidifferential operators as follows. For all $\phi \in \mathcal{F}_\lambda$ and for all $\psi \in \mathcal{F}_\nu$,

$$L^{\lambda,\nu,\mu}_X(A)(\phi, \psi) = L^\mu_X \circ A(\phi, \psi) - A(L^\lambda_X(\phi), \psi) - A(\phi, L^\nu_X(\psi)), \tag{2.4}$$

where L^λ_X is the action (2.2). We denote by $\mathcal{D}_{\lambda,\nu,\mu}$ the space of bilinear bidifferential operators (2.3) endowed with the defined $\mathrm{Vect}(\mathbb{R})$-module structure (2.4).

3. The second differentiable cohomology space of $\mathfrak{sl}(2)$ acting on $\mathcal{D}_{\lambda,\upsilon,\mu}$

In this section, we investigate the second space differentiable cohomology of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators that act on tensor densities $\mathcal{D}_{\lambda,\nu,\mu}$. Following Sofiane Bouarroudj, we give explicit expressions of the basis cocycles. Namely, we consider only cochains that are given by differentiable maps.

3.1. **The main theorem.**

Theorem 3.1. *The second differentiable cohomology space of the $\mathfrak{sl}(2)$-module $\mathcal{D}_{\lambda,\upsilon,\mu}$ has the following structure:*

(1) *If $\mu-\lambda-\upsilon=0$, then*

$$\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})=\mathbb{R}.$$

(2) *If $\mu-\lambda-\upsilon=k$, where k is a positive integer, then*

$$\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})\simeq\begin{cases}\mathbb{R}^4, & \text{if } (\lambda,\mu)=(-\frac{s}{2},-\frac{t}{2}), \text{ where } 0\le s,\ k-s-2<t\le k-1;\\ \mathbb{R}, & \text{otherwise.}\end{cases}$$

(3) *If $\mu-\lambda-\upsilon=k$, where k is not a positive integer, then*

$$\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})\simeq 0.$$

Before proving the theorem, we are required to prove the following two lemmas.

Lemma 3.2. *Let $C:\mathcal{F}_\lambda\otimes\mathcal{F}_\upsilon\to\mathcal{F}_\mu$ be a bilinear bidifferential operator defined as follows: for all $\phi\in\mathcal{F}_\lambda$ and for all $\psi\in\mathcal{F}_\upsilon$,*

$$\begin{aligned}C(\phi\otimes\psi)=&\sum_{i+j=k}a_{i,j}(XY'-X'Y)\phi^{(i)}\psi^{(j)}+\sum_{i+j=k-1}b_{i,j}(XY''-X''Y)\phi^{(i)}\psi^{(j)}\\&+\sum_{i+j=k-2}c_{i,j}(X'Y''-X''Y')\phi^{(i)}\psi^{(j)},\end{aligned}$$

where the superscript $'$ stands for $\frac{d}{dx}$ and $a_{i,j}$, $b_{i,j}$, and $c_{i,j}$ are constants, and let the 2-cocycle condition read as follows: for all vector fields $X\frac{d}{dx}$, $Y\frac{d}{dx}$, and $Z\frac{d}{dx}$ in $\mathfrak{sl}(2)$,

$$\begin{aligned}\delta C(\phi\otimes\psi)=&\left(L_X^{\lambda,\upsilon,\mu}C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)-L_Y^{\lambda,\upsilon,\mu}C\left(X\frac{d}{dx},Z\frac{d}{dx}\right)\right.\\&\left.-L_{Z\frac{d}{dx}}^{\lambda,\upsilon,\mu}C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)\right)(\phi\otimes\psi)\\&-\left(C\left(\left[X\frac{d}{dx},Y\frac{d}{dx}\right],Z\frac{d}{dx}\right)+C\left(\left[X\frac{d}{dx},Z\frac{d}{dx}\right],Y\frac{d}{dx}\right)\right.\\&\left.-C\left(\left[Y\frac{d}{dx},Y\frac{d}{dx}\right],Z\frac{d}{dx}\right)\right)(\phi\otimes\psi)\\=&\ 0.\end{aligned}$$

Then we have

$$\begin{aligned}\delta C(\phi\otimes\psi)=&\frac{1}{2}\sum_{i+j=k-1}\big(X(Y''Z'-Y'Z'')+Y(Z''X'-Z'X'')+Z(X''Y'-X'Y'')\\&\times\big((i+1)(i+2\lambda)\,a_{i+1,j}+(j+1)(j+2\upsilon)\,a_{i,j+1}\big)\\&+(\mu-\lambda-\upsilon-i-j)b_{i,j}\big)\phi^{(i)}\psi^{(j)}.\end{aligned}\tag{3.1}$$

Proof. Straightforward computation using the definition (2.2). □

Lemma 3.3. *Let $b : \mathcal{F}_\lambda \otimes \mathcal{F}_\upsilon \to \mathcal{F}_\mu$ be a bilinear bidifferential operator defined as follows. For all $\phi \in \mathcal{F}_\lambda$ and for all $\psi \in \mathcal{F}_\upsilon$:*

$$b\Big(X\frac{d}{dx}\Big)(\phi\otimes\psi) = \sum_{i+j=k}\alpha_{i,j}X\phi^{(i)}\psi^{(j)} + \sum_{i+j=k-1}\beta_{i,j}X'\phi^{(i)}\psi^{(j)}, \qquad (3.2)$$

where $\alpha_{i,j}$, $\beta_{i,j}$ are constants. For all $X\frac{d}{dx}, Y\frac{d}{dx} \in \mathfrak{sl}(2)$, we have

$$\begin{aligned}\delta b(\phi\otimes\psi) = {}& \frac{1}{2}\sum_{i+j=k-1}(XY''-X''Y)\\ &\times\big((i+1)(i+2\lambda)\alpha_{i+1,j}+(j+1)(j+2\upsilon)\alpha_{i,j+1}\big)\phi^{(i)}\psi^{(j)}\\ &+\frac{1}{2}\sum_{i+j=k-2}(X'Y''-X''Y')\\ &\times\big((i+1)(i+2\lambda)\beta_{i+1,j}+(j+1)(j+2\upsilon)\beta_{i,j+1}\big)\phi^{(i)}\psi^{(j)}.\end{aligned} \qquad (3.3)$$

Proof. Straightforward computation using the definition (2.2). □

3.2. **Proof of Theorem 3.1.** Using Lemma 3.2, for all $X\frac{d}{dx}, Y\frac{d}{dx} \in \mathfrak{sl}(2)$, $\phi \in \mathcal{F}_\lambda$, and $\psi \in \mathcal{F}_\upsilon$, we prove that the coefficient of the component $\phi^{(i)}\psi^{(j)}$ in the 2-cocycle condition above is equal to

$$\frac{1}{2}\big((i+1)(i+2\lambda)a_{i+1,j}+(j+1)(j+2\upsilon)a_{i,j+1}+(\mu-\lambda-\upsilon-i-j)b_{i,j}\big)\phi^{(i)}\psi^{(j)}. \qquad (3.4)$$

The annihilation of the 2-cocycle condition requires the annihilation of the formula (3.4). So we have

$$(i+1)(i+2\lambda)a_{i+1,j}+(j+1)(j+2\upsilon)a_{i,j+1}+(\mu-\lambda-\upsilon-i-j)b_{i,j}=0. \qquad (3.5)$$

We distinguish many cases:

- For $\mu-\lambda-\upsilon=0$, the 2-cocycle on $\mathfrak{sl}(2)$ has the following form:

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi) = a(XY'-X'Y)\phi\psi,$$

  where $X\frac{d}{dx} \in \mathfrak{sl}(2)$, $\phi \in \mathcal{F}_\lambda$, $\psi \in \mathcal{F}_\upsilon$, and a is a constant. The 2-cocycle condition is proved by a direct computation:

$$\delta C\left(X\frac{d}{dx},Y\frac{d}{dx},Z\frac{d}{dx}\right)(\phi,\psi)=0.$$

  Thus the space $\mathrm{Z}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})$ is one-dimensional. Now we are going to study the triviality of the general cocycle (3.2). Every trivial 2-cocycle of $\mathfrak{sl}(2)$ in $\mathcal{D}_{\lambda,\upsilon,\lambda+\upsilon}$ must be of the form δQ, where Q is an element of $\mathcal{D}_{\lambda,\upsilon,\lambda+\upsilon}$ defined as follows:

$$Q\left(X\frac{d}{dx}\right)(\phi,\psi) = X\alpha\phi\psi + X'\beta\phi\psi,$$

where α and β are constants. We have

$$\begin{aligned}\delta Q\Big(X\frac{d}{dx},Y\frac{d}{dx}\Big)(\phi,\psi)&\\&=L^{\lambda,\nu,\lambda+\nu}_{X\frac{d}{dx}}Q\left(Y\frac{d}{dx}\right)(\phi,\psi)-L^{\lambda,\nu,\lambda+\nu}_{Y\frac{d}{dx}}Q\left(X\frac{d}{dx}\right)(\phi,\psi)\\&\quad-Q\left(\left[X\frac{d}{dx},Y\frac{d}{dx}\right]\right)(\phi,\psi),\end{aligned}$$

After a direct computation, the result will be $\delta Q(X\frac{d}{dx},Y\frac{d}{dx})(\phi,\psi)=0$; then $\delta Q(X\frac{d}{dx},Y\frac{d}{dx})(\phi,\psi)\neq C(X\frac{d}{dx},Y\frac{d}{dx})(\phi,\psi)$ shows that the general cocycle (3.2) cannot be ultimately trivial. Therefore the coboundary space $\mathrm{B}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})$ vanishes. As a consequence,

$$\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\lambda+\nu})=\mathrm{Z}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\lambda+\nu}).$$

- For $\mu-\lambda-\upsilon=k$, where k is a positive integer:
  (1) If $\lambda\neq\frac{-s}{2}$ and $\upsilon\neq\frac{-t}{2}$, where $s,t\in\{0,\dots,k-1\}$, then the space of solutions of the system (3.5) is one-dimensional, generated by $a_{0,k}$. Indeed, in that case $(i+1)(i+2\lambda)\neq 0$ and $(j+1)(j+2\upsilon)\neq 0$; therefore the system (3.4) is equivalent to

$$a_{i+1,j}=-\frac{(j+1)(j+2\upsilon)}{(i+1)(i+2\lambda)}a_{i,j+1},$$

where $i+j=k-1$. By iterations, we get

$$\begin{aligned}a_{1,k-1}&=-\frac{k(k-1+2\upsilon)}{2\lambda}a_{0,k}=-C_k^1\frac{(k-1+2\upsilon)}{2\lambda}a_{0,k},\\a_{2,k-2}&=-\frac{(k-1)(k-2+2\upsilon)}{1+2\lambda}a_{1,k-1}=C_k^2\frac{(k-1+2\upsilon)(k-2+2\upsilon)}{2\lambda(1+2\lambda)}a_{0,k},\\&\vdots\\a_{i,k-i}&=(-1)^{i+1}C_k^{i+1}\\&\quad\times\frac{(k-i+2\upsilon)(k-i+1+2\upsilon)(k-i+2+2\upsilon)\cdots(k-1+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots 2\lambda}a_{0,k}.\end{aligned}$$

Now, we show how the constants $b_{i,j}$ and $c_{i,j}$ can be eliminated from our initial 2-cocycle (3.2). We add the coboundary δb of the equation (3.3) of our 2-cocycle (3.1). The constants $\alpha_{i,j}$ and $\beta_{i,j}$ are chosen such that

$$\begin{cases}b_{i,j}=-\frac{1}{2}((i+1)(i+2\lambda)\alpha_{i+1,j}+(j+1)(j+2\upsilon)\alpha_{i,j+1}),\\c_{i,j}=-\frac{1}{2}((i+1)(i+2\lambda)\beta_{i+1,j}+(j+1)(j+2\upsilon)\beta_{i,j+1}).\end{cases}$$

Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycle

$$\begin{aligned}
&C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi)\\
&\quad=(XY'-X'Y)\phi\psi^{(k)}\\
&\qquad+\sum_{i+j=k-1}(-1)^{(i+1)}C_k^{(i+1)}(XY'-X'Y)\\
&\qquad\times\frac{(k-i+2\upsilon)(k-i+1+2\upsilon)(k-i+2+2\upsilon)\cdots(k-1+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots 2\lambda}\\
&\qquad\times\phi^{(i+1)}\psi^{(j)}.
\end{aligned}$$

(2) If $\lambda\neq\frac{-s}{2}$ and $\upsilon=\frac{-t}{2}$, where $s,t\in\{0,\dots,k-1\}$, then the constants $a_{k-t,k},a_{k-t+1,t-1},\dots,a_{k,0}$ are zero, and the space of solutions of the system (3.5) is one-dimensional, generated by $a_{0,k}$. Two cases should be studied:

(a) If $j\leq t$:

– For $j=t$, we have $(j+1)(j+2\upsilon)=0$. So,

$$(k-t)(k-t-1+2\upsilon)a_{k-t,t}=0.$$

We have $\lambda\neq\frac{-s}{2}$, for all $s\in\{0,\dots,k-1\}$, then $(i+2\lambda)\neq 0$. Thus $a_{k-t,t}=0$.

– For $j\in\{0,\dots,t-1\}$, we have $(j+1)(j+2\upsilon)\neq 0$.
So, $a_{k-t+1,t-1}=-\frac{t(t-1+2\upsilon)}{(k-t+1)(k-1+2\lambda)}a_{k-t,t}=0$.
Thus, $a_{k-t+2,t-2}=-\frac{(t-1)(t-2+2\upsilon)}{(k-t+2)(k-2+2\lambda)}a_{k-t+1,t-1}=0$.

$\vdots$

Finally, $a_{k,0}=0$.

(b) If $j>t$, then

$$a_{i+1,j}=-\frac{(j+1)(j+2\upsilon)}{(i+1)(i+2\lambda)}a_{i,j+1},$$

where $i+j=k-1$. By iterations, we get

$$\begin{aligned}
a_{1,k-1}&=-C_k^1\frac{(k-1+2\upsilon)}{2\lambda}a_{0,k},\\
a_{2,k-2}&=C_k^2\frac{(k-1+2\upsilon)(k-2+2\upsilon)}{2\lambda(1+2\lambda)}a_{0,k},\\
&\vdots\\
a_{i,k-i}&=(-1)^{i+1}C_k^{i+1}\frac{(k-i+2\upsilon)(k-i+1+2\upsilon)(k-i+2+2\upsilon)\cdots(k-1+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots 2\lambda}a_{0,k}.
\end{aligned}$$

The constants $b_{i,j}$ and $c_{i,j}$ can be eliminated by the same method as in Part (1). We have just proved that the cohomology group in question is generated by the 2-cocycle

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi)=(XY'-X'Y)\left(\phi\psi^{(k)}+\sum_{i+j=k-1}a_{i+1,j}\phi^{i+1}\psi^{(j)}\right),$$

where

$$a_{i+1,j}\simeq\begin{cases}0, & \text{if } j\le t;\\ (-1)^{i+1}C_k^{i+1}\frac{(k-i+2\upsilon)(k-i+1+2\upsilon)(k-i+2+2\upsilon)\cdots(k-1+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots 2\lambda}, & \text{otherwise.}\end{cases}$$

(3) If $\lambda=\frac{-s}{2}$ and $\upsilon\neq\frac{-t}{2}$, where $s,t\in\{0,\dots,k-1\}$, then we follow the same steps as in (2) (b). Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycle

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi)=(XY'-X'Y)\left(\phi\psi^{(k)}+\sum_{i+j=k-1}a_{i,j+1}\phi^{i}\psi^{(j+1)}\right),$$

where

$$a_{i,j+1}\simeq\begin{cases}0, & \text{if } j\le t;\\ (-1)^{k-i}C_k^{i+1}\frac{(i+2\lambda)(i+1+2\lambda)\cdots(k-1+2\lambda)}{(j+2\upsilon)(j-1+2\upsilon)\cdots 2\upsilon}, & \text{otherwise.}\end{cases}$$

(4) If $\lambda=\frac{-s}{2}$ and $\upsilon\neq\frac{-k-s-1}{2}$, where $s\in\{0,\dots,k-1\}$, then the space of solutions of the system (3.5) is two dimensional, generated by $a_{s+1,k-s-1}$ and $a_{s,k-s}$.

(a) If $i=s$, $j=k-s-1$, we have

$$\begin{cases}(i+1)(i+2\lambda)=0,\\ (j+1)(j+2\upsilon)=0.\end{cases}$$

(b) If $i\neq s$, we have $(i+1)(i+2\lambda)\neq 0$.
The system (3.5) is equivalent to the system

$$a_{i+1,j}=-\frac{(j+1)(j+2\upsilon)}{(i+1)(i+2\lambda)}a_{i,j+1}.$$

(i) If $i+j=k-1$ for all $i\in\{1,\dots,s-1\}$: by iterations, we get

$$\begin{aligned}a_{1,k-1}&=-C_k^1\frac{(k-1+2\upsilon)}{2\lambda}a_{0,k}\\ a_{2,k-2}&=C_k^2\frac{(k-1+2\upsilon)(k-2+2\upsilon)}{2\lambda(1+2\lambda)}a_{0,k}\\ &\vdots\\ a_{i,k-i}&=(-1)^sC_k^s\frac{(k-s+2\upsilon)(k-s+1+2\upsilon)(k-s+2+2\upsilon)\cdots(k-1+2\upsilon)}{(s-1+2\lambda)(s-2+2\lambda)\cdots 2\lambda}a_{0,k}.\end{aligned}$$

(ii) If $i+j=k-1$ for all $i\geq s+1$: by iterations, we get

$$
\begin{aligned}
a_{s+2,k-s-2} &= -\frac{(k-s-1)(k-s-2+2\upsilon)}{(s+2)(s+1+2\lambda)}a_{s+1,k-s-1},\\
a_{s+3,k-s-3} &= -\frac{(k-s-2)(k-s-1)(k-s-3+2\upsilon)(k-s-2+2\upsilon)}{(s+3)(s+2)(s+2+2\lambda)(s+1+2\lambda)}a_{s+1,k-s-1},\\
a_{2,k-2} &= C_k^2\frac{(k-1+2\upsilon)(k-2+2\upsilon)}{2\lambda(1+2\lambda)}a_{0,k},\\
&\vdots\\
a_{i,k-i} &= (-1)^{i-s+1}\\
&\quad\times\frac{(k-i+1)(k-i+2)\cdots(k-s-1)(k-i+2\upsilon)(k-i+1+2\upsilon)\cdots(k-s-2+2\upsilon)}{i(i-1)\cdots(s+2)(i-1+2\lambda)(i-2+2\lambda)\cdots(s+1+2\lambda)}\\
&\quad\times a_{s+1,k-s-1}.
\end{aligned}
$$

Now we will explain how the constants $b_{i,j}$ and $c_{i,j}$ can be eliminated except constants $b_{s,k-s-1}$ and $c_{s,k-s-1}$ because the component in (3.3) is zero.
The $\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\lambda+\nu})$ is generated by a family of cocycles depending on four free parameters: $a_{0,k}$, $a_{s+1,k-s-1}$, $b_{s,k-s-1}$, and $c_{s,k-s-1}$. Thus, the cohomology group in question is four-dimensional, generated by the 2-cocycle

$$
\begin{aligned}
C\Big(X\frac{d}{dx},Y\frac{d}{dx}\Big)(\phi,\psi)
&= b_{s,k-s-1}(XY''-X''Y)\phi^{(s)}\psi^{(k-s-1)}\\
&\quad+c_{s,k-s-1}(X'Y''-X''Y')\phi^{(s)}\psi^{(k-s-1)}\\
&\quad+\Big(a_{0,k}\phi\psi^{(k)}+a_{s+1,k-s-1}\phi^{(s+1)}\psi^{(k-s-1)}\\
&\quad+\sum_{\substack{i+j=k\\ i\neq(0,s+1)}}a_{i,j}\phi^{(i)}\psi^{(j)}\Big)(XY'-X'Y),
\end{aligned}
$$

where $a_{i,j}$ equals

$$
(-1)^sC_k^s\frac{(k-s+2\upsilon)(k-s+1+2\upsilon)(k-s+2+2\upsilon)\cdots(k-1+2\upsilon)}{(s-1+2\lambda)(s-2+2\lambda)\cdots2\lambda}a_{0,k},
$$

if $i\leq s$, and equals

$$
\begin{aligned}
&(-1)^{i-s+1}\\
&\times\frac{(k-i+1)(k-i+2)\cdots(k-s-1)(k-i+2\upsilon)(k-i+1+2\upsilon)\cdots(k-s-2+2\upsilon)}{i(i-1)\cdots(s+2)(i-1+2\lambda)(i-2+2\lambda)\cdots(s+1+2\lambda)}\\
&\qquad\times a_{s+1,k-s-1},
\end{aligned}
$$

if $i\geq s+1$.

(5) If $\lambda=\frac{-s}{2}$ and $\upsilon=\frac{-t}{2}$, where $s,t\in\{0,\dots,k-1\}$ and $i+j=k-1$, we distinguish many cases:

(a) For $t \leq k-s-2$, the space of solutions of the system (3.5) is one-dimensional, generated by $a_{s+1,k-s-1}$. In fact, there are six cases:
  (i) If $i = s$, we have $(k-s)(k-s+2\upsilon)a_{s,k-s} = 0$; then, $a_{s,k-s} = 0$.
  (ii) If $i < s$, we have

$$a_{i,j} = (-1)^i C_k^i \frac{(j+2\upsilon)(j+1+2\upsilon)\cdots(k-1+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots 2\lambda} a_{0,k},$$

  since $a_{s,k-s} = 0$.
  (iii) If $i = k-t-1$ and $j = t$, then we have

$$(k-t)(k-1-t+2\lambda)a_{k-t,t} = 0,$$

  and and as the condition $t \leq k-s-2$ involves $s < k-t-1$ and $(i+2\lambda)$ does not vanish only if $i = s$, so (3.5) implies $(k-t)(k-t-1+2\upsilon)a_{k-t,t} = 0$; so $a_{k-t,t} = 0$.
  (iv) If $i \neq s$ and $j \neq t$, the system (3.5) implies

$$a_{i+1,j} = -\frac{(j+1)(j+2\upsilon)}{(i+1)(i+2\lambda)} a_{i,j+1},$$

  and this last equality allows us to obtain

$$a_{i,j} = (-1)^{i-s+1} \frac{(j+1)(j+2)\cdots(k-s-1)}{i(i-1)\cdots(i+2)} \times \frac{(j+2\upsilon)(j+1+2\upsilon)\cdots(k-s-2+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots(s+1+2\lambda)} a_{s+1,k-s-1}.$$

  Since $a_{s,k-s} = 0$, we obtain

$$a_{0,k} = a_{1,k-1} = \cdots = a_{s,k-s} = 0.$$

  (v) If $s+1 \leq i < k-t-1$, we obtain

$$a_{i,j} = (-1)^{i-s+1} \frac{(j+1)(j+2)\cdots(k-s-1)}{i(i-1)\cdots(i+2)} \times \frac{(j+2\upsilon)(j+1+2\upsilon)\cdots(k-s-2+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots(s+1+2\lambda)} a_{s+1,k-s-1}.$$

  (vi) If $i > k-t-1$, we have $a_{i,j} = 0$, since $a_{k-t,t} = 0$.

We conclude that

$$a_{i,j} \simeq \begin{cases} 0, & \text{if } i \leq s; \\ 0, & \text{if } j \leq t; \\ (-1)^{i-s+1} \frac{(j+1)(j+2)\cdots(k-s-1)}{i(i-1)\cdots(i+2)} \\ \frac{(j+2\upsilon)(j+1+2\upsilon)\cdots(k-s-2+2\upsilon)}{(i-1+2\lambda)(i-2+2\lambda)\cdots(s+1+2\lambda)} a_{s+1,k-s-1}, & \text{otherwise.} \end{cases}$$

The constants $b_{i,j}$ and $c_{i,j}$ are eliminated as explained in the other cases.

Thus, the cohomology group in question is one-dimensional, generated by the 2-cocycles

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi)$$
$$=(XY'-X'Y)\left(\phi^{(s+1)}\psi^{(k-s-1)}+\sum_{\substack{i+j=k\\ i\neq s+1}}a_{i,j}\phi^{(i)}\psi^{(j)}\right).$$

(b) If $t>k-s-2$, then the space of solutions of the system (3.5) is two-dimensional, generated by $a_{s+1,k-s-1}$ and $a_{k-t-1,t+1}$. Secondly, the constants $b_{i,j}$ and $c_{i,j}$ are eliminated as explained in the other cases, except $b_{k-t-1,t}$ and $c_{k-t-1,t}$.
The $\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\lambda+\nu})$ is generated by a family of cocycles depending on four free parameters $a_{0,k}$, $a_{s+1,k-s-1}$, $b_{s,k-s-1}$, and $c_{s,k-s-1}$. Thus, the cohomology group in question is four-dimensional, generated by the 2-cocycle

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi,\psi)$$
$$\begin{aligned}&=b_{k-t-1,t}(XY''-X''Y)\phi^{(k-t-1)}\psi^{t}\\&\quad+c_{k-t-1,t}(X'Y''-X''Y')\phi^{(k-t-1)}\psi^{t}\\&\quad+\Bigg(a_{0,k}\phi\psi^{(k)}+a_{k,0}\phi^{(s+1)}\psi^{(k-s-1)}\\&\quad+\sum_{\substack{i+j=k\\ i,j\neq 0}}a_{i,j}\phi^{(i)}\psi^{(j)}\Bigg)(XY'-X'Y),\end{aligned}$$

where

$$a_{i,j}\simeq\begin{cases}(-1)^{k-j}C_k^j\frac{(j+2v)(j+1+2v)(j+2+2v)\cdots(k-1+2v)}{(i-1+2\lambda)(i-2+2\lambda)\cdots2\lambda}a_{0,k}, & \text{if } j\geq t+1;\\ (-1)^{k-i}C_k^i\frac{(i+2\lambda)(i+1+2\lambda)\cdots(k-1+2\lambda)}{(j-1+2v)(j-2+2v)\cdots2v}a_{k,0} & ,\text{if } i\geq s+1.\end{cases}$$

- For $\mu-\lambda-\upsilon=k$, where k is not a positive integer, every 2-cocycle on $\mathfrak{sl}(2)$ retains the following general from:

$$C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)(\phi\otimes\psi)=\sum_{0\leq n,m\leq 2}\sum_{i,j}a_{i,j,n,m}X^{(n)}Y^{(m)}\phi^{(i)}\psi^{(j)}.$$

The 2-cocycle condition is equivalent to $a_{i,j,n,m}=0$, $\forall\, i,j,n,m\in\mathbb{N}$. So the operator $C\left(X\frac{d}{dx},Y\frac{d}{dx}\right)$ is identically the zero map.

Thus,

$$\mathrm{H}^2(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\lambda+\nu})\simeq 0.$$

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