REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 61, No. 1, 2020, Pages 161–168
Published online: June 30, 2020
https://doi.org/10.33044/revuma.v61n1a10

LIFTING VECTOR FIELDS FROM MANIFOLDS TO THE r-JET PROLONGATION OF THE TANGENT BUNDLE

JAN KUREK AND WŁODZIMIERZ M. MIKULSKI

Abstract. If $m \geq 3$ and $r \geq 0$, we deduce that any natural linear operator lifting vector fields from an m-manifold M to the r-jet prolongation J^rTM of the tangent bundle TM is the composition of the flow lifting $\mathcal{J}^r$ corresponding to the r-jet prolongation functor J^r with a natural linear operator lifting vector fields from M to TM. If $0 \leq s \leq r$ and $m \geq 3$, we find all natural linear operators transforming vector fields on M into base-preserving fibred maps $J^rTM \to J^sTM$.

1. Introduction

All manifolds considered in this paper are assumed to be finite dimensional, without boundary, and smooth. Maps between manifolds are assumed to be smooth (of class C^∞).

The general concept of bundle functors and natural operators can be found in the fundamental monograph [4].

In [1], J. Gancarzewicz proved that any natural linear operator A lifting vector fields $X \in \mathcal{X}(M)$ on an m-manifold M into vector fields $A(X) \in \mathcal{X}(TM)$ on the tangent bundle TM of M is of the form $A(X) = aX^C + bX^V$ for real numbers a and b, where $X^C = \mathcal{T}X \in \mathcal{X}(TM)$ is the complete (flow) lift of X to TM and $X^V \in \mathcal{X}(TM)$ is the vertical lift of X to TM.

In this paper, we prove that if $m \geq 3$ then any natural linear operator A lifting vector fields $X \in \mathcal{X}(M)$ on an m-manifold M into vector fields $A(X) \in \mathcal{X}(J^rTM)$ on the r-jet prolongation J^rTM of TM is of the form

$$A(X) = a\mathcal{J}^rX^C + b\mathcal{J}^rX^V \tag{1.1}$$

for (uniquely determined) real numbers a and b.

Moreover, if $0 \leq s \leq r$ and $m \geq 3$, we find all natural linear operators A transforming vector fields $X \in \mathcal{X}(M)$ on an m-manifold M into base-preserving fibred maps $A(X) : J^rTM \to J^sTM$.

2010 *Mathematics Subject Classification.* Primary: 58A20, 53A55.

Key words and phrases. Natural bundle; natural operator; vector field; jet prolongation functor; tangent bundle.

Natural operators lifting functions and vector fields are applied in almost all investigations of prolongation of geometric structures, see e.g. [8, 9]. That is why such natural operators are studied in many papers, see e.g. [1, 2, 3, 4, 5, 6, 7].

From now on, let $x^1, \dots, x^m$ denote the usual coordinates on $\mathbf{R}^m$ and $\partial_1, \dots, \partial_m$ be the canonical vector fields on $\mathbf{R}^m$.

2. Preliminaries

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms; let $\mathcal{FM}$ be the category of fibred manifolds (i.e. surjective submersions between manifolds) and their fibred maps; let $\mathcal{FM}_m$ be the category of fibred manifolds with m-dimensional bases and their fibred maps with local diffeomorphisms as base maps; and let $\mathcal{VB}$ be the category of vector bundles and their vector bundle homomorphisms.

The r-jet prolongation J^rY of an $\mathcal{FM}_m$-object $Y = (Y \to M)$ is the space of r-jets $j^r_x\sigma$ at points $x \in M$ of local sections σ of Y. It is a fibre bundle over Y with projection $j^r_x\sigma \mapsto \sigma(x)$. Every $\mathcal{FM}_m$-map $f : Y \to Y_1$ with the base map $\underline{f} : M \to M_1$ induces the fibred map $J^rf : J^rY \to J^rY_1$ by $j^r_x\sigma \mapsto j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$. The resulting functor $J^r : \mathcal{FM}_m \to \mathcal{FM}$ is a bundle functor in the sense of [4].

Let $Y = (Y \to M)$ be an $\mathcal{FM}_m$-object. A vector field $Z \in \mathcal{X}(Y)$ is called projectable if there is a vector field $\underline{Z} \in \mathcal{X}(M)$ on M being related with Z with respect to the projection $Y \to M$. We denote by $\mathcal{X}_{\text{proj}}(Y)$ the space of projectable vector fields on Y. Equivalently, $Z \in \mathcal{X}(Y)$ is projectable if and only if the flow $\{\text{Fl}^Z_t\}$ of Z is formed by $\mathcal{FM}_m$-maps. Thus for any $Z \in \mathcal{X}_{\text{proj}}(Y)$ we have $\mathcal{J}^rZ \in \mathcal{X}(J^rY)$ given by $\mathcal{J}^rZ = \frac{\partial}{\partial t}_{|t=0} J^r\text{Fl}^Z_t$.

Let $T : \mathcal{M}f_m \to \mathcal{FM}_m$ be the (usual) tangent functor sending any m-manifold M into the tangent bundle TM of M and any $\mathcal{M}f_m$-map $\varphi : M \to M_1$ into the tangent map $T\varphi : TM \to TM_1$ of φ. Composing T with J^r we obtain the bundle functor $J^rT : \mathcal{M}f_m \to \mathcal{FM}$ sending any m-manifold M into the space J^rTM of r-jets j^r_xX at points $x \in M$ of vector fields X on M and every $\mathcal{M}f_m$-map $\varphi : M \to N$ of two m-manifolds into $J^rT\varphi : J^rTM \to J^rTN$ given by $J^rT\varphi(j^r_xX) = j^r_{\varphi(x)}(T\varphi \circ X \circ \varphi^{-1})$. We see that J^rTM is (in the obvious way) a vector bundle over M and $J^rT\varphi : J^rTM \to J^rTN$ is a vector bundle map. So, $J^rT : \mathcal{M}f_m \to \mathcal{VB}$.

3. Natural operators

An $\mathcal{M}f_m$-natural linear operator $A : T_{|\mathcal{M}f_m} \rightsquigarrow T(J^rT)$ (lifting vector fields from m-manifolds to the r-jet prolongation of the tangent bundle) is an $\mathcal{M}f_m$-invariant family of $\mathbf{R}$-linear operators ($\mathbf{R}$-linear functions)

$$A : \mathcal{X}(M) \to \mathcal{X}(J^rTM)$$

for all m-manifolds M, where $\mathcal{X}(M)$ is the vector space of vector fields on M. The invariance of A means that if $X \in \mathcal{X}(M)$ and $X_1 \in \mathcal{X}(M_1)$ are φ-related (i.e. $T\varphi \circ X = X_1 \circ \varphi$) for a $\mathcal{M}f_m$-map $\varphi : M \to M_1$, then $A(X)$ and $A(X_1)$ are $J^rT\varphi$-related.

Example 3.1. Let $X \in \mathcal{X}(M)$ be a vector field on an m-manifold M. We have the (complete) flow lift $X^C = \mathcal{T}X \in \mathcal{X}_{\text{proj}}(TM)$ of X to TM. So, we have $\mathcal{J}^r X^C \in \mathcal{X}(J^r TM)$. Alternatively, $\mathcal{J}^r X^C$ is the flow lift of X to $J^r TM$ via the bundle functor $J^r T$. The function $\mathcal{X}(M) \to \mathcal{X}(J^r TM)$ given by $X \mapsto \mathcal{J}^r X^C$ is $\mathbf{R}$-linear. The resulting family $T_{|\mathcal{M}f_m} \rightsquigarrow T(J^r T)$ is an $\mathcal{M}f_m$-natural linear operator.

Example 3.2. Let $X \in \mathcal{X}(M)$ be as above. We have the vertical lift $X^V \in \mathcal{X}_{\text{proj}}(TM)$ of X to TM. So, we have $\mathcal{J}^r X^V \in \mathcal{X}(J^r TM)$. Clearly, $\mathcal{J}^r X^V_{|j^r_x Y} = \frac{d}{dt}_{|t=0}(j^r_x Y + t j^r_x X)$. The function $\mathcal{X}(M) \to \mathcal{X}(J^r TM)$ given by $X \mapsto \mathcal{J}^r X^V$ is $\mathbf{R}$-linear. The resulting family $T_{|\mathcal{M}f_m} \rightsquigarrow T(J^r T)$ is an $\mathcal{M}f_m$-natural linear operator.

Similarly, an $\mathcal{M}f_m$-natural linear operator $T_{|\mathcal{M}f_m} \rightsquigarrow (J^r T, J^s T)$ (transforming vector fields on m-manifolds into fibred base-preserving maps from the r-jet prolongation of the tangent bundle into the s-jet prolongation of the tangent bundle) is an $\mathcal{M}f_m$-invariant family of $\mathbf{R}$-linear operators ($\mathbf{R}$-linear functions)

$$A : \mathcal{X}(M) \to C^\infty_M(J^r TM, J^s TM)$$

for all m-manifolds M, where $\mathcal{X}(M)$ is the vector space of vector fields on M and $C^\infty_M(J^r TM, J^s TM)$ is the vector space of base-preserving fibred maps $J^r TM \to J^s TM$. The invariance of A means that if $X \in \mathcal{X}(M)$ and $X_1 \in \mathcal{X}(M_1)$ are φ-related vector fields for an $\mathcal{M}f_m$-map $\varphi : M \to M_1$, then so are $A(X) : J^r TM \to J^s TM$ and $A(X_1) : J^r TM_1 \to J^s TM_1$ (i.e. $J^s T\varphi \circ A(X) = A(X_1) \circ J^r T\varphi$).

Example 3.3. Let k be an integer such that $0 \le k \le r - s$. Given a vector field $X \in \mathcal{X}(M)$ on an m-manifold M we have a base-preserving fibred map

$$A^{\langle k \rangle}(X) : J^r TM \to J^s TM, \quad A^{\langle k \rangle}(X)(j^r_x Y) = j^s_x(\mathrm{ad}^k_Y(X)),$$

where $\mathrm{ad}_Y : \mathcal{X}(M) \to \mathcal{X}(M)$ is the adjoint map given by $\mathrm{ad}_Y(X) = [Y, X]$ and $\mathrm{ad}^k_Y = \mathrm{ad}_Y \circ \cdots \circ \mathrm{ad}_Y$ (k times). Thus we have the resulting $\mathcal{M}f_m$-natural linear operator $A^{\langle k \rangle} : T_{|\mathcal{M}f_m} \rightsquigarrow (J^r T, J^s T)$.

4. Preparatory lemmas

Lemma 4.1. *Let $A : T_{|\mathcal{M}f_m} \rightsquigarrow (J^r T, J^s T)$ be an $\mathcal{M}f_m$-natural linear operator with $A((x^1)^q \partial_2)(j^r_0 \partial_1) = 0$ for $q = 0, \dots, r$. If $0 \le s \le r$ and $m \ge 2$, then $A = 0$.*

Proof. First, prove that

$$A(x^\alpha \partial_j)(j^r_0 \partial_1) = 0 \tag{4.1}$$

for any $\alpha \in (\mathbf{N} \cup \{0\})^m$ and any $j = 1, \dots, m$. Let us consider three cases.

(I) Let $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbf{N} \cup \{0\})^m$ be such that $|\alpha| \le r$ and let $j \in \{2, \dots, m\}$. By the Frobenius theorem there exists a local embedding $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ of the form $\mathrm{id}_{\mathbf{R}} \times \psi$ such that $\varphi_* \partial_2 = \partial_2 + (x^2)^{\alpha_2} \dots (x^m)^{\alpha_m} \partial_j$ on some neighborhood of 0. Then $\varphi_* \partial_1 = \partial_1$ and $\varphi_*((x^1)^{\alpha_1} \partial_2) = (x^1)^{\alpha_1} \partial_2 + x^\alpha \partial_j$ in some neighborhood of 0. On the other hand, since $\alpha_1 \leq r$, by the assumption of the lemma we have

$$A((x^1)^{\alpha_1}\partial_2)(j^r_0\partial_1) = 0.$$

Then, using the invariance of A with respect to φ, we obtain

$$A((x^1)^{\alpha_1}\partial_2 + x^\alpha\partial_j)(j^r_0\partial_1) = 0.$$

Hence, we have (4.1) for any $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| \leq r$ and any $j \in \{2, \ldots, m\}$.

(II) Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbf{N} \cup \{0\})^m$ be such that $|\alpha| \leq r$ and let $j = 1$. For any $\tau \in \mathbf{R}$, the linear isomorphism $(x^1 + \tau x^2, x^2, \ldots, x^m)$ preserves ∂_1 and sends $x^\alpha\partial_2$ into $(x^1 - \tau x^2)^{\alpha_1}(x^2)^{\alpha_2} \ldots (x^m)^{\alpha_m}(\partial_2 + \tau\partial_1)$. Further, from the case (I) we have $A(x^\alpha\partial_2)(j^r_0\partial_1) = 0$. So, using the invariance of A with respect to $(x^1 + \tau x^2, x^2, \ldots, x^m)$, we obtain

$$A((x^1 - \tau x^2)^{\alpha_1}(x^2)^{\alpha_2} \ldots (x^m)^{\alpha_m}(\partial_2 + \tau\partial_1))(j^r_0\partial_1) = 0.$$

Both sides of the last equality are polynomials in τ. Considering the coefficients of the polynomials on τ, we obtain

$$A(x^\alpha\partial_1)(j^r_0\partial_1) - \alpha_1 A((x^1)^{\alpha_1-1}(x^2)^{\alpha_2+1} \ldots (x^m)^{\alpha_m}\partial_2)(j^r_0\partial_1) = 0.$$

(If $\alpha_1 = 0$ the term $\alpha_1 A(\ldots)(j^r_0\partial_1)$ does not occur.) Further, from the case (I) we have $\alpha_1 A((x^1)^{\alpha_1-1}(x^2)^{\alpha_2+1} \ldots (x^m)^{\alpha_m}\partial_2)(j^r_0\partial_1) = 0$. Hence we have (4.1) for any $\alpha \in (\mathbf{N} \cup \{0\})^m$ with $|\alpha| \leq r$ and $j = 1$.

(III) Now, let $\alpha \in (\mathbf{N} \cup \{0\})^m$ be such that $|\alpha| \geq r + 1$ and $j = 1, \ldots, m$. Then $j^r_0(\partial_2 + x^\alpha\partial_j) = j^r_0\partial_2$. So, by Lemma 42.4 in [4], there exists a local diffeomorphism $\varphi : \mathbf{R}^m \to \mathbf{R}^m$ such that $j^{r+1}_0\varphi = j^{r+1}_0\mathrm{id}$ and $\varphi_*\partial_2 = \partial_2 + x^\alpha\partial_j$ on some neighborhood of 0. Clearly, φ preserves $j^r_0\partial_1$. Further, from the case (I) for $j = 2$ and $\alpha = (0, \ldots, 0)$, we have $A(\partial_2)(j^r_0\partial_1) = 0$. Then by the invariance of A with respect to φ we obtain $A(\partial_2)(j^r_0\partial_1) = A(\partial_2 + x^\alpha\partial_j)(j^r_0\partial_1)$. Then we have (4.1) for any $\alpha \in (\mathbf{N} \cup \{0\})^m$ such that $|\alpha| \geq r + 1$ and $j = 1, \ldots, m$.

We are now in a position to complete the proof. From the cases (I)–(III) we get (4.1) for any $\alpha \in (\mathbf{N} \cup \{0\})^m$ and any $j = 1, \ldots, m$. Then from the linearity of A and the Peetre theorem it follows that $A(X)(j^r_0\partial_1) = 0$ for any $X \in \mathcal{X}(\mathbf{R}^m)$. Now, since the $\mathcal{M}f_m$-orbit of $j^r_0\partial_1$ is dense in J^rTM and A is $\mathcal{M}f_m$-invariant, we get that $A(X) = 0$ for any $X \in \mathcal{X}(M)$, i.e. $A = 0$. □

Lemma 4.2. *Let $0 \leq s \leq r$ and $m \geq 2$. Let $A : T_{|\mathcal{M}f_m} \rightsquigarrow (J^rT, J^sT)$ be an $\mathcal{M}f_m$-natural linear operator. Given $k = 0, \ldots, r$ we have*

$$A((x^1)^k\partial_2)(j^r_0\partial_1) = \sum_{l=0}^{\min(k,s)} \mu^k_l j^s_0((x^1)^l\partial_2) \tag{4.2}$$

for some (uniquely determined) real numbers μ^k_l for $k = 0, \ldots, r$ and $l = 0, \ldots, \min(k, s)$.

Proof. We can write

$$A(a(x^1)^k\partial_2)(bj^r_0\partial_1) = \sum_{j=1}^{m}\sum_{|\alpha|\le s}\lambda^{j,k}_\alpha(a,b)j^s_0(x^\alpha\partial_j),$$

where $\lambda^{j,k}_\alpha$ are some (uniquely determined) smooth maps. Using the invariance of A with respect to $(\tau_1x^1,\dots,\tau_mx^m)$ for $\tau_1=1$, $\tau_2\neq 0$, …, $\tau_m\neq 0$, we get the homogeneity condition

$$\tau_2\lambda^{j,k}_\alpha(a,b) = \frac{\tau_j}{\tau^\alpha}\lambda^{j,k}_\alpha(a,b).$$

Then $\lambda^{j,k}_\alpha(a,b)=0$ if $\tau_2\neq\frac{\tau_j}{\tau^\alpha}$. Hence

$$A(a(x^1)^k\partial_2)(bj^r_0\partial_1) = \sum_{l=0}^{s}\mu^k_l(a,b)j^s_0((x^1)^l\partial_2),$$

where μ^k_l are (uniquely determined) smooth maps. Now, using the invariance of A with respect to $(\tau x^1,x^2,\dots,x^m)$ for $\tau\neq 0$, we obtain the homogeneity condition

$$\frac{1}{\tau^k}\mu^k_l(a,\tau b) = \frac{1}{\tau^l}\mu^k_l(a,b).$$

Consequently, $\mu^k_l(a,b)=0$ if $l>k$. The proof of the lemma is complete. □

Lemma 4.3. *Let $0\le s\le r$ and $m\ge 3$. The vector space of all $\mathcal{M}f_m$-natural linear operators $A: T_{|\mathcal{M}f_m}\rightsquigarrow(J^rT,J^sT)$ has dimension $\le r-s+1$.*

Proof. Let $A: T_{|\mathcal{M}f_m}\rightsquigarrow(J^rT,J^sT)$ be an $\mathcal{M}f_m$-natural linear operator. Let μ^k_l for $k=0,\dots,r$ and $l=0,\dots,\min(k,s)$ be the real numbers from Lemma 4.2. By Lemma 4.1, A is uniquely determined by this system (μ^k_l) of real numbers. So, it remains to show that the system (μ^k_l) is uniquely determined by the subsystem (μ^k_0) of real numbers μ^k_0 for $k=0,\dots,r-s$. Let us consider two cases.

(I) $s=0$. Then $(\mu^k_l)=(\mu^k_0)$. So, this case is trivial.

(II) $s\ge 1$. We have $\mu^k_l=\mu^0_0$ for $k=0$ and $l=0,\dots,\min(k,s)=0$. So, we can assume $k\ge 1$. For a real number τ, let $\psi_\tau:\mathbf{R}^{m-1}\to\mathbf{R}^{m-1}$ be a local diffeomorphism such that $(\psi_\tau)_*\partial_2=\partial_2+\tau x^2\partial_2$ on some neighborhood of 0. Then from the invariance of A with respect to $\mathrm{id}_\mathbf{R}\times\psi_\tau$ and (4.2) for $k-1$ instead of k it follows that

$$A((x^1)^{k-1}(\partial_2+\tau x^2\partial_2))(j^r_0\partial_1) = \sum_{l=0}^{\min(k-1,s)}\mu^{k-1}_l j^s_0((x^1)^l(\partial_2+\tau x^2\partial_2)).$$

Consequently, if we consider the coefficients on τ of both sides, we get

$$A((x^1)^{k-1}x^2\partial_2)(j^r_0\partial_1) = \sum_{l=0}^{\min(k-1,s)}\mu^{k-1}_l j^s_0((x^1)^lx^2\partial_2). \tag{4.3}$$

Similarly, from the invariance of A with respect to $(x^1+\tau x^2, x^2, \dots, x^m)$ and (4.2) it follows that

$$A((x^1-\tau x^2)^k(\partial_2+\tau\partial_1))(j^r_0\partial_1) = \sum_{l=0}^{\min(k,s)} \mu^k_l j^s_0((x^1-\tau x^2)^l(\partial_2+\tau\partial_1)).$$

So, we have

$$\begin{aligned} &-kA((x^1)^{k-1}x^2\partial_2)(j^r_0\partial_1) + A((x^1)^k\partial_1)(j^r_0\partial_1) \\ &\qquad = -\sum_{l=0}^{\min(k,s)} l\mu^k_l j^s_0((x^1)^{l-1}x^2\partial_2) + \sum_{l=0}^{\min(k,s)} \mu^k_l j^s_0((x^1)^l\partial_1). \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we get

$$\begin{aligned} A((x^1)^k\partial_1)(j^r_0\partial_1) &= k\sum_{l=0}^{\min(k-1,s)} \mu^{k-1}_l j^s_0((x^1)^l x^2\partial_2) \\ &\quad - \sum_{l=0}^{\min(k,s)} l\mu^k_l j^s_0((x^1)^{l-1}x^2\partial_2) + \sum_{l=0}^{\min(k,s)} \mu^k_l j^s_0((x^1)^l\partial_1). \end{aligned} \tag{4.5}$$

(If $l=s$ then $j^s_0((x^1)^l x^2\partial_2)=0$. If $l=0$, then $l\mu^k_l j^s_0((x^1)^{l-1}x^2\partial_2)$ does not occur.) Using the invariance of A with respect to the embedding switching x^2 and x^3 (we use the assumption $m\geq 3$) and preserving the other coordinates, from (4.5) we get

$$\begin{aligned} A((x^1)^k\partial_1)(j^r_0\partial_1) &= k\sum_{l=0}^{\min(k-1,s)} \mu^{k-1}_l j^s_0((x^1)^l x^3\partial_3) \\ &\quad - \sum_{l=0}^{\min(k,s)} l\mu^k_l j^s_0((x^1)^{l-1}x^3\partial_3) + \sum_{l=0}^{\min(k,s)} \mu^k_l j^s_0((x^1)^l\partial_1). \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we see that the coefficients on $j^s_0((x^1)^{l-1}x^2\partial_2)$ (on the right hand side of (4.5)) must be 0, i.e.

$$-l\mu^k_l + k\mu^{k-1}_{l-1} = 0$$

for $l=1,\dots,\min(k,s)$. So, by induction, the system (μ^k_l) is uniquely determined by $\mu^0_0,\dots,\mu^{r-s}_0$. The proof of the lemma is complete. □

Lemma 4.4. *Let $0\leq s\leq r$ and $m\geq 1$. The system of $\mathcal{M}f_m$-natural linear operators $A^{\langle k\rangle}$ from Example 3.3 for $k=0,\dots,r-s$ is linearly independent.*

Proof. Suppose $\sum_{k=0}^{r-s}\lambda_k A^{\langle k\rangle}=0$. We prove that $\lambda_0=\dots=\lambda_q=0$ for $q=0,\dots,r-s$. We proceed by induction with respect to q.

(i) We start with $q=0$. Since $A^{\langle 0\rangle}(\partial_1)(j^r_0\partial_1)=j^s_0\partial_1$ and $A^{\langle k\rangle}(\partial_1)(j^r_0\partial_1)=0$ for $k=1,\dots,r-s$, then $0=\sum_{k=0}^{r-s}\lambda_k A^{\langle k\rangle}(\partial_1)(j^r_0\partial_1)=\lambda_0 j^s_0\partial_1$. Then $\lambda_0=0$.

(ii) Now, we make the inductive step. Let $r-s-1\geq q\geq 0$ and assume that $\lambda_0=\dots=\lambda_q=0$. Then $0=\sum_{k=0}^{r-s}\lambda_k A^{\langle k\rangle}\left(\frac{1}{(q+1)!}(x^1)^{q+1}\partial_1\right)(j^r_0\partial_1)=\lambda_{q+1}j^s_0\partial_1$, because $A^{\langle q+1\rangle}\Big(\frac{1}{(q+1)!}(x^1)^{q+1}\partial_1\Big)(j^r_0\partial_1) = j^s_0\partial_1$ and $A^{\langle k\rangle}((x^1)^{q+1}\partial_1)(j^r_0\partial_1) = 0$ for $k = q+2,\dots,r-s$. Then $\lambda_{q+1} = 0$, i.e. $\lambda_0 = \dots = \lambda_{q+1} = 0$, as well.

Thus we have proved that $\lambda_0 = \dots = \lambda_q = 0$ for $q = 0,\dots,r-s$. For $q = r-s$ we get $\lambda_0 = \dots = \lambda_{r-s} = 0$. The proof of the lemma is complete. □

5. Main results

Theorem 5.1. *Let $0 \le s \le r$ and $m \ge 3$. Any $\mathcal{M}f_m$-natural linear operator $A : T_{|\mathcal{M}f_m} \rightsquigarrow (J^rT, J^sT)$ is the linear combination of $A^{\langle k\rangle}$ for $k = 0,\dots,r-s$ with (uniquely determined) real coefficients.*

Proof. It is an immediate consequence of Lemmas 4.3 and 4.4. □

Theorem 5.2. *Let $m \ge 3$ and $r \ge 0$ be integers. Any $\mathcal{M}f_m$-natural linear operator $A : T_{|\mathcal{M}f_m} \rightsquigarrow T(J^rT)$ is of the form* (1.1) *for (uniquely determined) reals a and b.*

Proof. Let $A : T_{|\mathcal{M}f_m} \rightsquigarrow T(J^rT)$ be an $\mathcal{M}f_m$-natural linear operator.

Using the source projection $\pi^r : J^rTM \to M$ we produce the $\mathcal{M}f_m$-natural linear operator $T\pi^r \circ A : T_{|\mathcal{M}f_m} \rightsquigarrow (J^rT, J^0T)$. By Theorem 5.1 for $s = 0$,

$$T\pi^r \circ A = \sum_{k=0}^{r} \lambda_k A^{\langle k\rangle},$$

where λ_k are the real numbers. First, we are going to prove that $\lambda_1 = \dots = \lambda_r = 0$.

It is easy to see that $A^{\langle k\rangle}\Big(\frac{1}{q!}(x^1)^q\partial_1\Big)(j^r_0\partial_1) = \delta_{k,q}\partial_{1|0}$ (the Kronecker delta). So, $T\pi^r \circ A\left(\frac{1}{k!}(x^1)^k\partial_1\right)(j^r_0\partial_1) = \lambda_k\partial_{1|0}$. Then

$$A\left(\frac{1}{k!}(x^1)^k\partial_1\right)(j^r_0\partial_1) = \lambda_k\mathcal{J}^r\partial_1^C(j^r_0\partial_1) + v \tag{5.1}$$

for some (depending on k) π^r-vertical vector v over $j^r_0\partial_1$.

Since $j^r_0\partial_1 = j^r_0\Big(\partial_1 + \frac{1}{(r+1)!}(x^1)^{r+1}\partial_1\Big)$, there exists a local diffeomorphism φ with $j^{r+1}_0\varphi = \mathrm{id}$ sending the germ at 0 of ∂_1 into the germ at 0 of $\partial_1 + \frac{1}{(r+1)!}(x^1)^{r+1}\partial_1$. Such φ preserves $j^r_0\partial_1$ and preserves $j^{r+1}_0\big(\frac{1}{k!}(x^1)^k\partial_1\big)$ if $k \ge 1$. So, if $k \ge 1$, φ preserves the left-hand side of (5.1) because of the order argument. Indeed, by Lemma 42.5 in [4], A is of order $\le r+1$ because J^rT is of order $\le r+1$. Moreover, φ preserves v. Indeed, the vertical bundle VJ^rT of J^rT is of order $r+1$ because J^rT is of order $r+1$.

On the other hand, φ does not preserve $\mathcal{J}^r\partial_1^C(j^r_0\partial_1)$, because

$$\mathcal{J}^r\left(\frac{1}{(r+1)!}(x^1)^{r+1}\partial_1\right)^C(j^r_0\partial_1) = j^r_0\left(\frac{1}{r!}(x^1)^r\partial_1\right) \ne 0,$$

where we identify E_x with V_vE in the obvious way, for any vector bundle $E \to M$, $v \in E_x$, and $x \in M$. Indeed, if φ_t is the flow of $\frac{1}{(r+1)!}(x^1)^{r+1}\partial_1$, then

$$\mathcal{J}^r\left(\frac{1}{(r+1)!}(x^1)^{r+1}\partial_1\right)^C(j^r_0\partial_1) = \frac{d}{dt}_{|t=0}J^rT\varphi_t(j^r_0\partial_1) = \frac{d}{dt}_{|t=0}j^r_0((\varphi_t)_*\partial_1)$$
$$= j^r_0\left(\frac{d}{dt}_{|t=0}(\varphi_t)_*\partial_1\right) = j^r_0\left(\left[\partial_1, \frac{1}{(r+1)!}(x^1)^{r+1}\partial_1\right]\right) = j^r_0\left(\frac{1}{r!}(x^1)^r\partial_1\right).$$

Consequently, $\lambda_k = 0$ for $k \in \{1, \dots, r\}$, as well. Then $T\pi^r \circ A(X)(j^r_xY) = \lambda_0 X(x)$ for any $X \in \mathcal{X}(M)$ and any $j^r_xY \in J^rTM$. Then replacing $A(X)$ by $A(X) - \lambda_0\mathcal{J}^rX^C$, we may assume that $A(X)$ is vertical for any $X \in \mathcal{X}(M)$ and any m-manifold M. Let $pr : VJ^rTM \to J^rTM$ be the projection given by $\frac{d}{dt}_{|t=0}(j^r_xY + tj^r_xY_1) \mapsto j^r_xY_1$. Then the composition $pr \circ A : T_{|\mathcal{M}f_m} \rightsquigarrow (J^rT, J^rT)$ is an $\mathcal{M}f_m$-natural linear operator. So, by Theorem 5.1, $pr \circ A$ is a constant multiple of $A^{\langle 0\rangle}$. Then $A(X)$ is a constant multiple of $\mathcal{J}^rX^V$.

The proof of the theorem is thus complete. □

References

[1] J. Gancarzewicz, Liftings of functions and vector fields to natural bundles, *Dissertationes Math. (Rozprawy Mat.)* **212** (1983), 55 pp. MR 0697471.

[2] J. Gancarzewicz, Relèvements des champs de vecteurs aux fibrés naturels, *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 1, 59–61. MR 0691028.

[3] I. Kolář, On the natural operators on vector fields, *Ann. Global Anal. Geom.* **6** (1988), no. 2, 109–117. MR 0982760.

[4] I. Kolář, P. W. Michor and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993. MR 1202431.

[5] J. Kurek and W. M. Mikulski, Lifting vector fields to the rth order frame bundle, *Colloq. Math.* **111** (2008), no. 1, 51–58. MR 2353930.

[6] M. Kureš and W. M. Mikulski, Natural operators lifting vector fields to bundles of Weil contact elements, *Czechoslovak Math. J.* **54(129)** (2004), no. 4, 855–867. MR 2099999.

[7] W. M. Mikulski, Some natural operations on vector fields, *Rend. Mat. Appl. (7)* **12** (1992), no. 3, 783–803. MR 1205977.

[8] A. Morimoto, Prolongation of connections to bundles of infinitely near points, *J. Differential Geom.* **11** (1976), no. 4, 479–498. MR 0445422.

[9] K. Yano and S. Ishihara, *Tangent and cotangent bundles: differential geometry*, Marcel Dekker, New York, 1973. MR 0350650.

Jan Kurek
Institute of Mathematics UMCS, M. Curie-Sklodowskiej 1, 20-031 Lublin, Poland
kurek@hektor.umcs.lublin.pl

Włodzimierz M. Mikulski✉
Faculty of Mathematics and Computer Science UJ, ul. Łojasiewicza 6, 30-348 Krakow, Poland
Wlodzimierz.Mikulski@im.uj.edu.pl

Received: January 25, 2019
Accepted: May 7, 2019