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COFINITENESS OF LOCAL COHOMOLOGY MODULES IN THE CLASS OF MODULES IN DIMENSION LESS THAN A FIXED INTEGER

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Abstract. Let n be a non-negative integer, R a commutative Noetherian ring with $\dim(R) \leq n+2$, $\mathfrak{a}$ an ideal of R, and X an arbitrary R-module. In this paper, we first prove that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module if X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules. Then, we show that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) : \dim\left(\frac{R}{\mathfrak{p}}\right) \geq n\}$ is a finite set for all i when $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq n+2$. As a consequence, it follows that $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all i whenever R is a semi-local ring with $\dim(R) \leq 3$ and X is an $\mathrm{FD}_{<1}$ R-module. Finally, we observe that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules forms an Abelian subcategory of the category of R-modules.

1. Introduction

We adopt throughout the following notation: let R denote a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R, M a finite (i.e., finitely generated) R-module, X an arbitrary R-module which is not necessarily finite, and n a non-negative integer. We refer the reader to [7, 8, 23] for basic results, notations, and terminology not given in this paper.

Hartshorne, in [14], defined an $\mathfrak{a}$-torsion R-module X to be $\mathfrak{a}$-*cofinite* if the R-module $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is finite for all i, and asked the following questions.

Question 1.1. Does the category of $\mathfrak{a}$-cofinite R-modules form an Abelian subcategory of the category of R-modules?

Question 1.2. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $\mathfrak{a}$-cofinite R-module for all i?

The following question is also an important problem in local cohomology [16, Problem 4].

Question 1.3. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ a finite set for all i?

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There have been many attempts in the literature to study the above questions. Hartshorne in [14, Proposition 7.6 and Corollary 7.7] showed that the answer to these questions is yes if R is a complete regular local ring and $\mathfrak{a}$ is a prime ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$. Huneke and Koh in [17, Theorem 4.1] and Delfino in [10, Theorem 3] extended Hartshorne's result [14, Corollary 7.7] and provided affirmative answers to Questions 1.2 and 1.3 in more general local rings R and one-dimensional ideals $\mathfrak{a}$. Delfino and Marley in [11, Theorems 1 and 2], Yoshida in [25, Theorem 1.1], Chiriacescu in [9, Theorem 1.4], and Kawasaki in [18, Theorems 1 and 8] showed that the answer to Questions 1.1–1.3 is yes if R is an arbitrary local ring and $\mathfrak{a}$ is an arbitrary ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$. Finally, in [21, Theorems 7.4 and 7.10] and [22, Theorems 2.6 and 2.10], Melkersson provided affirmative answers to these questions for the case that R is an arbitrary ring and either $\dim(R) \leq 2$ or $\mathfrak{a}$ is an arbitrary ideal of R with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 1$.

Recall that X is said to be an $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) *R-module* if there is a finite submodule Y of X such that $\dim_R\left(\frac{X}{Y}\right) < n$ [2, 4]. From [26, Theorem 2.3], the class of $\mathrm{FD}_{<n}$ R-modules is closed under taking submodules, quotients, and extensions. We say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i [3, Definition 4.1]. Note that X is an $\mathfrak{a}$-cofinite R-module if and only if X is an $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite R-module. Thus, as generalizations of Questions 1.1–1.3, we have the following questions (see [1, Question] and [24, Questions 1.5, 1.6, and 1.8]). Here, the set $\{\mathfrak{p} \in \mathrm{Ass}_R(X) : \dim\left(\frac{R}{\mathfrak{p}}\right) \geq n\}$ is denoted by $\mathrm{Ass}_R(X)_{\geq n}$.

Question 1.4. Does the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules form an Abelian subcategory of the category of R-modules?

Question 1.5. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i?

Question 1.6. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ a finite set for all i?

If R is a complete local ring with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq n+1$, then the answer to Questions 1.5 and 1.6 is yes from [1, Theorems 2.5 and 2.10]. In [24, Corollaries 3.3 and 4.5], the first author and Morsali removed the complete local assumption on R and provided affirmative answers to Questions 1.4–1.6 for the case that $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq n+1$, which are generalizations of Melkersson's results [22, Theorems 2.6 and 2.10]. In this paper, as generalizations of Melkersson's results [21, Theorems 7.4 and 7.10], we show that the answer to Questions 1.4–1.6 is also yes if $\dim(R) \leq n+2$. As a consequence, we provide an affirmative answer to Question 1.3 for the case that R is a semi-local ring with $\dim(R) \leq 3$. This result is a generalization of Marley's result in [19] where he showed that the answer to Question 1.3 is yes if R is a local ring with $\dim(R) \leq 3$ (see [19, Proposition 1.1 and Corollary 2.5]).

In the main result of Section 2, we observe that if $\dim(R) \leq n+2$ and X is an $\mathfrak{a}$-torsion R-module such that $\mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\mathrm{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules, then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Section 3 is devoted to the study of Questions 1.5 and 1.6. We show that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all i whenever $\dim(R) \leq n+2$ and $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq n+2$ (e.g., X is an $\mathrm{FD}_{<n}$ R-module).

It follows that if R is a semi-local ring with $\dim(R) \leq 3$ and $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq 3$ (e.g., X is an $\mathrm{FD}_{<1}$ R-module), then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-weakly cofinite R-module and $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all i. Recall that X is said to be an $\mathfrak{a}$-*weakly cofinite R-module* if X is an $\mathfrak{a}$-torsion R-module and the set of associated prime ideals of any quotient module of $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is finite for all i (see [12, Definition 2.1] and [13, Definition 2.4]). In Section 4, with respect to Question 1.4, we prove that when $\dim(R) \leq n+2$, the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules forms an Abelian subcategory of the category of R-modules.

2. A criterion for cofiniteness

The following two lemmas will be useful in the proof of the main result of this section. Note that when $\mathfrak{b}X = 0$, X is an $\mathrm{FD}_{<n}$ R-module if and only if X is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module.

Lemma 2.1. *Let t be a non-negative integer and let X be an R-module such that $\mathfrak{b}X = 0$ and $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$. Then $\operatorname{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t$.*

Proof. We prove this by using induction on t. The case $t = 0$ is clear from the isomorphisms

$$\operatorname{Hom}_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right) \cong \left(0 :_X \frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{b}}\right) \cong (0 :_X \mathfrak{a}+\mathfrak{b}) \cong \operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right).$$

Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\operatorname{Ext}^t_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module, since $\operatorname{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t-1$ by the induction hypothesis on $t-1$. From [23, Theorem 11.65], there is a spectral sequence

$$\mathrm{E}_2^{p,q} := \operatorname{Ext}^p_{\frac{R}{\mathfrak{b}}}\left(\operatorname{Tor}^R_q\left(\frac{R}{\mathfrak{b}}, \frac{R}{\mathfrak{a}+\mathfrak{b}}\right), X\right) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right).$$

Let $r \geq 2$ and set $B_r^{t,0} := \operatorname{Im}(\mathrm{E}_r^{t-r,r-1} \longrightarrow \mathrm{E}_r^{t,0})$. Then $B_r^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module because $\mathrm{E}_r^{t-r,r-1}$ is a subquotient of $\mathrm{E}_2^{t-r,r-1}$ that is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module by the induction hypothesis and [15, Proposition 3.4]. Thus, from the short exact sequence

$$0 \longrightarrow B_r^{t,0} \longrightarrow \mathrm{E}_r^{t,0} \longrightarrow \mathrm{E}_{r+1}^{t,0} \longrightarrow 0,$$

$\mathrm{E}_r^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module whenever $\mathrm{E}_{r+1}^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. There exists a finite filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = \operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$$

such that $\mathrm{E}_\infty^{t-i,i} \cong \frac{\phi^{t-i}H^t}{\phi^{t-i+1}H^t}$ for all i, $0 \leq i \leq t$. By assumption, $\operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an R-module. Thus, as we noted at the beginning of this section, $\operatorname{Ext}^t_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module and hence $\phi^t H^t$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Therefore $\mathrm{E}_\infty^{t,0} \cong \frac{\phi^t H^t}{\phi^{t+1}H^t}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module and so $\mathrm{E}_{t+2}^{t,0}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module, because $\mathrm{E}_\infty^{t,0} = \mathrm{E}_{t+2}^{t,0}$ as

$E_j^{t-j,j-1} = 0 = E_j^{t+j,1-j}$ for all $j \geq t+2$. Thus $E_2^{t,0} = \operatorname{Ext}^t_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. □

Lemma 2.2. *Let t be a non-negative integer and let X be an R-module such that $\mathfrak{b}X = 0$ and $\operatorname{Ext}^i_{\frac{R}{\mathfrak{b}}}\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module for all $i \leq t$. Then $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$.*

Proof. From [23, Theorem 11.65], there is a spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_{\frac{R}{\mathfrak{b}}}\left(\operatorname{Tor}^R_q\left(\frac{R}{\mathfrak{b}}, \frac{R}{\mathfrak{a}}\right), X\right) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R\left(\frac{R}{\mathfrak{a}}, X\right).$$

Let $0 \leq j \leq i \leq t$. By [15, Proposition 3.4], $E_2^{i-j,j}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Hence $E_\infty^{i-j,j}$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module as $E_\infty^{i-j,j} = E_{i+2}^{i-j,j}$ and $E_{i+2}^{i-j,j}$ is a subquotient of $E_2^{i-j,j}$. There exists a finite filtration

$$0 = \phi^{i+1}H^i \subseteq \phi^i H^i \subseteq \cdots \subseteq \phi^1 H^i \subseteq \phi^0 H^i = \operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$$

such that $E_\infty^{i-j,j} \cong \frac{\phi^{i-j}H^i}{\phi^{i-j+1}H^i}$ for all j, $0 \leq j \leq i$. Now, from the short exact sequences

$$0 \longrightarrow \phi^{i-j+1}H^i \longrightarrow \phi^{i-j}H^i \longrightarrow E_\infty^{i-j,j} \longrightarrow 0,$$

for all j, $0 \leq j \leq i$, $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ $\frac{R}{\mathfrak{b}}$-module. Therefore $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module. □

We are now ready to state and prove the main result of this section, which plays an important role in Sections 3 and 4 to study Questions 1.4–1.6.

Theorem 2.3. *Suppose that $\dim(R) \leq n+2$ and X is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules. Then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.*

Proof. Assume that $\mathfrak{a}$ is nilpotent. Then $\mathfrak{a}^t = 0$ for some integer t. By [15, Proposition 3.4], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}^t}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module and hence $X = (0 :_X \mathfrak{a}^t)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Now, assume that $\mathfrak{a}$ is not nilpotent. Since $\Gamma_{\mathfrak{a}}(R)$ is finite, there is an integer t such that $(0 :_R \mathfrak{a}^t) = \Gamma_{\mathfrak{a}}(R)$. Set $\mathfrak{b} := (0 :_R \mathfrak{a}^t)$ and $Y := \frac{X}{(0:_X \mathfrak{a}^t)}$. It is easy to see that $\mathfrak{b}Y = 0$, Y is an $(\mathfrak{a}+\mathfrak{b})$-torsion R-module, and $\dim\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}\right) \leq n+1$. Since $(0 :_X \mathfrak{a}^t)$, $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$, and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, X\right)$ are $\mathrm{FD}_{<n}$ R-modules from [15, Proposition 3.4], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ are $\mathrm{FD}_{<n}$ R-modules by the short exact sequence

$$0 \longrightarrow (0 :_X \mathfrak{a}^t) \longrightarrow X \longrightarrow Y \longrightarrow 0.$$

Thus, from [24, Corollary 2.3], $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}+\mathfrak{b}}, Y\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i. Hence $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, Y\right)$ is an $\mathrm{FD}_{<n}$ R-module for all i by Lemmas 2.1 and 2.2. Therefore X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from the above short exact sequence. □

The following corollary is an immediate application of the above theorem.

Corollary 2.4. *Suppose that* $\dim(R) \leq n+2$ *and* X *is an arbitrary* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are* $\mathrm{FD}_{<n}$ R*-modules. Then* $\Gamma_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module.*

Proof. By the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow \frac{X}{\Gamma_{\mathfrak{a}}(X)} \longrightarrow 0,$$

$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(X)\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(X)\right)$ are $\mathrm{FD}_{<n}$ R-modules. Thus the assertion follows from Theorem 2.3. □

By putting $n = 0$ in Theorem 2.3 and Corollary 2.4, we have the following results.

Corollary 2.5. *Suppose that* $\dim(R) \leq 2$ *and* X *is an* $\mathfrak{a}$*-torsion* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are finite* R*-modules. Then* X *is an* $\mathfrak{a}$*-cofinite* R*-module.*

Corollary 2.6. *Suppose that* $\dim(R) \leq 2$ *and* X *is an arbitrary* R*-module such that* $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *and* $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *are finite* R*-modules. Then* $\Gamma_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module.*

3. Cofiniteness and associated primes of local cohomology modules

The following is the main result of this section; it shows that the answer to Questions 1.5 and 1.6 is yes if $\dim(R) \leq n+2$.

Theorem 3.1. *Suppose that* $\dim(R) \leq n+2$ *and* X *is an arbitrary* R*-module. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module for all* i*;*
(ii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* i*;*
(iii) $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* $i \leq n+2$*.*

Proof. (i)⇒(ii). This follows by [3, Theorem 2.1].

(iii)⇒(i). We first show that if t is a non-negative integer such that $\operatorname{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t+1$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t$. We prove this by using induction on t. The case $t = 0$ follows from Corollary 2.4. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^t_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module, because $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq t-1$ from the induction hypothesis on $t-1$. By [3, Theorem 2.3], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^t_{\mathfrak{a}}(X)\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^t_{\mathfrak{a}}(X)\right)$ are $\mathrm{FD}_{<n}$ R-modules. Therefore $\mathrm{H}^t_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Theorem 2.3. This terminates the induction argument. Thus $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \neq n+2$ from [7, Theorem 6.1.2]. By [3, Theorem 2.3], $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \mathrm{H}^{n+2}_{\mathfrak{a}}(X)\right)$ is an $\mathrm{FD}_{<n}$ R-module. Also, from [7, Exercise 7.1.7], $\operatorname{Supp}_R(\mathrm{H}^{n+2}_{\mathfrak{a}}(X)) \subseteq \operatorname{Max}(R)$, because each R-module can be viewed as the direct limit of its finite submodules. Thus $\mathrm{H}^{n+2}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by [24, Lemma 2.1]. □

Corollary 3.2. *Suppose that* $\dim(R) \leq n+2$*,* X *is an arbitrary* R*-module, and* t *is a non-negative integer such that* $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<n}$ R*-module for all* $i \leq t+1$ *(resp. for all* $i \leq n+2$*). Then* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module for all* $i \leq t$ *(resp. for all* i*). In particular,* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ *is a finite set for all* $i \leq t$ *(resp. for all* i*).*

Proof. The first assertion follows from the proof of Theorem 3.1. The last assertion follows by the first one and [8, Exercise 1.2.28]. ☐

We have the following corollaries by taking $n = 0$ in Theorem 3.1 and Corollary 3.2.

Corollary 3.3 (see [21, Theorem 7.10])**.** *Suppose that* $\dim(R) \leq 2$ *and* X *is an arbitrary* R*-module. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* R*-module for all* i*;*
(ii) $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite* R*-module for all* i*;*
(iii) $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite* R*-module for all* $i \leq 2$*.*

Corollary 3.4. *Suppose that* $\dim(R) \leq 2$ *and* X *is an arbitrary* R*-module such that* $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is a finite* R*-module for all* $i \leq 2$*. Then* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ *is a finite set for all* i*.*

If R is a local ring with $\dim\left(\frac{R}{\mathfrak{a}}\right) \leq 2$, then the answer to Question 1.3 is yes by Bahmanpour–Naghipour's result [6, Theorem 3.1] (see also [20, Theorem 3.3(c)]). In [24, Corollary 5.6], the first author and Morsali generalized this result to arbitrary semi-local rings. In the next result, by putting $n = 1$ in Corollary 3.2, we provide an affirmative answer to Question 1.3 for the case that R is a semi-local ring with $\dim(R) \leq 3$. Note that our result is a generalization of Marley's result in [19], where he showed that if R is a local ring with $\dim(R) \leq 3$ and M is a finite R-module, then $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ is a finite set for all i (see [19, Proposition 1.1 and Corollary 2.5]). Note also that, if R is a semi-local ring and X is an $(\mathrm{FD}_{<1}, \mathfrak{a})$-cofinite R-module, then X is an $\mathfrak{a}$-weakly cofinite R-module by [5, Theorem 3.3].

Corollary 3.5. *Suppose that* R *is a semi-local ring with* $\dim(R) \leq 3$*,* X *is an arbitrary* R*-module, and* t *is a non-negative integer such that* $\mathrm{Ext}^i_R\left(\frac{R}{\mathfrak{a}}, X\right)$ *is an* $\mathrm{FD}_{<1}$ R*-module for all* $i \leq t+1$ *(resp. for all* $i \leq 3$*). Then* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-weakly cofinite* R*-module for all* $i \leq t$ *(resp. for all* i*). In particular,* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ *is a finite set for all* $i \leq t$ *(resp. for all* i*).*

4. Abelianness of the category of cofinite modules

The following theorem is the main result of this section; it shows that the answer to Question 1.4 is also yes if $\dim(R) \leq n+2$.

Theorem 4.1. *If* $\dim(R) \leq n+2$*, then the category of* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-modules forms an Abelian subcategory of the category of* R*-modules.*

Proof. The proof is similar to that of [24, Theorem 3.1]. We bring it here for the sake of completeness. Assume that X and Y are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules and $f : X \to Y$ is an R-homomorphism. We show that $\ker f$, $\operatorname{im} f$, and $\operatorname{coker} f$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules. From the short exact sequence

$$0 \longrightarrow \operatorname{im} f \longrightarrow Y \longrightarrow \operatorname{coker} f \longrightarrow 0,$$

$\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \operatorname{im} f\right)$ is an $\mathrm{FD}_{<n}$ R-module. Hence $\operatorname{Hom}_R\left(\frac{R}{\mathfrak{a}}, \ker f\right)$ and $\operatorname{Ext}^1_R\left(\frac{R}{\mathfrak{a}}, \ker f\right)$ are $\mathrm{FD}_{<n}$ R-modules by the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow X \longrightarrow \operatorname{im} f \longrightarrow 0.$$

Therefore $\ker f$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Theorem 2.3. Thus $\operatorname{im} f$ and $\operatorname{coker} f$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules from the above short exact sequences. ☐

As an immediate application of the above theorem, we have the following corollary.

Corollary 4.2. *Suppose that* $\dim(R) \leq n + 2$*,* N *is a finite* R*-module, and* X *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module. Then* $\operatorname{Ext}^j_R(N, X)$ *and* $\operatorname{Tor}^R_j(N, X)$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-modules for all* j*.*

We have the following results by taking $n = 0$ in Theorem 4.1 and Corollary 4.2.

Corollary 4.3 (see [21, Theorem 7.4])**.** *If* $\dim(R) \leq 2$*, then the category of* $\mathfrak{a}$*-cofinite* R*-modules forms an Abelian subcategory of the category of* R*-modules.*

Corollary 4.4. *Suppose that* $\dim(R) \leq 2$*,* N *is a finite* R*-module, and* X *is an* $\mathfrak{a}$*-cofinite* R*-module. Then* $\operatorname{Ext}^j_R(N, X)$ *and* $\operatorname{Tor}^R_j(N, X)$ *are* $\mathfrak{a}$*-cofinite* R*-modules for all* j*.*

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