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# SELBERG ZETA-FUNCTION ASSOCIATED TO COMPACT RIEMANN SURFACE IS PRIME

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ABSTRACT. Let $Z(s)$ be the Selberg zeta-function associated to a compact Riemann surface. We consider decompositions $Z(s) = f(h(s))$, where $f$ and $h$ are meromorphic functions, and show that such decompositions can only be trivial.

## 1. Introduction

We continue the investigation of decompositions of the Selberg zeta-function which was started in Garunkštis and Steuding [6]. First we reproduce required definitions. Let $s = \sigma + it$ be a complex variable and $X$ a compact Riemann surface of genus $g \geq 2$ with constant negative curvature $-1$. The surface $X$ can be written as a quotient $\Gamma\backslash H$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a strictly hyperbolic Fuchsian group and $H$ is the upper half-plane of $\mathbb{C}$. Then the Selberg zeta-function associated with $X = \Gamma\backslash H$ is defined by (see Hejhal [8, § 2.4, Definition 4.1])

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}). \tag{1.1}$$

Here $\{P_0\}$ is the conjugacy class of a primitive hyperbolic element $P_0$ of $\Gamma$ and $N(P_0) = \alpha^2$ if the eigenvalues of $P_0$ are $\alpha$ and $\alpha^{-1}$ with $|\alpha| > 1$. Equation (1.1) defines the Selberg zeta-function in the half-plane $\sigma > 1$. The function $Z(s)$ can be extended to an entire function (see [8, § 2.4, Theorem 4.25]).

**Definition 1.1** (Gross [7], Chuang and Yang [1, Section 3.2], [6])**.** Let $F$ be a meromorphic function. Then an expression

$$F(z) = f(h(z)), \tag{1.2}$$

where $f$ is meromorphic and $h$ is entire ($h$ may be meromorphic when $f$ is a rational function), is called a *decomposition* of $F$, with $f$ and $h$ as its left and right components, respectively. $F$ is said to be *prime* in the sense of a decomposition

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if for every representation of $F$ of the form (1.2) we have that either $f$ or $h$ is linear. If every representation of $F$ of the form (1.2) implies that $f$ is rational or $h$ is a polynomial, we say that $F$ is *pseudo-prime* in the sense of a decomposition. Furthermore, $F$ is said to be *left-prime* (*right-prime*) if every factorization (1.2) implies that $f$ is linear whenever $h$ is transcendental ($h$ is linear whenever $f$ is transcendental).

Liao and Yang [10] showed that the Riemann zeta-function is prime. In [6] the following theorem is proved.

**Theorem A.** *The Selberg zeta-function $Z$ associated with a compact Riemann surface of genus $g$ is pseudo-prime and right-prime. Moreover, if $Z(s) = f(h(s))$, where $f$ is rational and $h$ is meromorphic, then $f$ is a polynomial of degree $k$, where $k$ divides $2g-2$, and $h$ is an entire function.*

Here we complete Theorem A.

**Theorem 1.2.** *The Selberg zeta-function $Z$ associated with a compact Riemann surface of genus $g \geq 2$ is prime.*

Theorem 1.2 follows from Theorem A, the property that $Z(s)$ has a simple zero at $s = 1$ ([8, § 2.4, Theorem 4.11]), and the following lemma.

**Lemma 1.3.** *If there exist a polynomial $P$ and an entire function $h$ such that $Z(s) = P(h(s))$ then the polynomial $P$ has only one root in the complex plane (counting without multiplicities).*

The proof of Lemma 1.3 is based on the distribution of zeros of $Z(s)-a$, $a \in \mathbb{C}$, (such zeros are called $a$-points of $Z(s)$) and of zeros of $Z'(s)$ in the left half-plane of $\mathbb{C}$. These zeros are described below.

The Selberg zeta-function $Z(s)$ has trivial zeros at integers $s = -n$, $n \geq 1$, of multiplicity $(2g-2)(2n+1)$; at $s = 0$ with multiplicity $2g-1$; and an already mentioned zero at $s = 1$ with multiplicity 1 (see [8, § 2.4, Theorem 4.11], also for nontrivial zeros).

For the trivial zeros of $Z'(s)$, Theorem 1 from [5] together with the equality $\overline{Z(s)} = Z(\overline{s})$ give the following proposition.

**Proposition 1.4.**

*(i) There is $\sigma_0 \geq 1$ such that $Z'(s) \neq 0$ in $\sigma \geq \sigma_0$;*
*(ii) the function $Z'(s)$ has zeros at $s = n$ of multiplicity $(2g-2)(1-2n)-1$ for any $n \leq -1$, and at $s = 0$ of multiplicity $2g-2$.*

*Moreover, for any $0 < \varepsilon < 1/2$, there is a constant $n_0 = n_0(\varepsilon) \leq -1$ such that*

*(iii) $Z'(s)$ has a simple real zero in the disc $|s + 1/2 - n| \leq \varepsilon$ for any $n \leq n_0$;*
*(iv) $Z'(s)$ has no other zeros in $\sigma \leq n_0$ except those mentioned in (ii) and (iii).*

For more about the zeros of the derivative of the Selberg zeta-function see [4, 11, 12].

For the $a$-points of $Z(s)$ we will prove the following two statements.

**Proposition 1.5.** *Let $b > 0$ and $1/6 < r < 1/2$. Then there exists a negative number $N = N(Z, b, r)$ such that, for $a \in \mathbb{C}$, $0 < |a| \leq b$, the function $Z(s) - a$ has $(2g-2)(1-2n)$ simple zeros in $|s-n| < r$, where $n < N$ are integers. Furthermore, $Z(s) - a$ has no other zeros in $\sigma < N$.*

On the other hand, Proposition 1.4 implies that, for sufficiently large negative $n$, a neighborhood of $n + 1/2$ contains a double zero of $Z(s) - Z(n + 1/2)$.

Using Proposition 1.5 and the particular kind of polynomials $P(z) = z^k + C$ we can easily demonstrate the main idea of the proof of Lemma 1.3. Indeed, let

$$Z(s) = h(s)^k + C,$$

where $C \neq 0$ and $h(s)$ is an entire function. Then all zeros of $Z(s) - C$ are at least of order $k$. By Proposition 1.5 we see that $k = 1$ and Lemma 1.3 is true for this particular kind of polynomials. To consider the general case we will need the following consequence of Proposition 1.5.

**Corollary 1.6.** *Let $a : [0, 1] \to \mathbb{C} \setminus 0$ be a continuous function. Then for any sufficiently large negative $n$ there are $(2g - 2)(2n + 1)$ continuous functions $s_j : [0, 1] \to \mathbb{C}$ such that, for each $j$, we have $Z(s_j(x)) = a(x)$, $|s_j(x) - n| < 1/3$, and $s_j(x) \neq s_m(x)$ if $j \neq m$ and $x \in [0, 1]$.*

In the last corollary, $1/3$ can be replaced by any number $r$, $1/6 < r < 1/2$. Various properties of $a$-points of Selberg zeta-functions were considered in [2, 3].

The next section contains the proofs of Proposition 1.5, Corollary 1.6, and Lemma 1.3.

## 2. Proofs

*Proof of Proposition 1.5.* We have (see [8, § 2.4, Theorem 4.12])

$$Z(s) = f(s)Z(1 - s), \tag{2.1}$$

where

$$f(s) = \exp\left(\operatorname{area}(X) \int_0^{s-1/2} v \tan(\pi v)\, dv\right).$$

It is known ([5, Lemma 6]) that, for $t \geq 0$ and $s$ not an integer,

$$\int_0^{s-1/2} v \tan(\pi v)\, dv$$
$$= \frac{i(s-1/2)^2}{2} - \frac{s-1/2}{\pi} \log\left(1 + e^{2\pi i(s-1/2)}\right) + \frac{i}{2\pi^2} \operatorname{Li}_2(-e^{2\pi i(s-1/2)}) + \frac{i}{24},$$

where the integration is along the straight line segment joining the origin to $s - 1/2$ if $s$ is not on the real line; if $s$ is on the real line, and not an integer, we define the integral by the requirement of continuity as $s$ is approached from the upper half-plane; furthermore, the branch of the logarithm is chosen such that

$$-\pi/2 \leq \Im \log\left(1 + e^{2\pi i(s-1/2)}\right) \leq \pi/2.$$

Then, for $\sigma \to -\infty$,

$$
\begin{aligned}
|f(s)|& \\
&= \exp\Big(\operatorname{area}(X)\Big(-(\sigma-1/2)t - \frac{\sigma-1/2}{\pi}\log|1+e^{2i\pi(s-1/2)}| + O(|t|+1)\Big)\Big)
\end{aligned}
\tag{2.2}
$$

uniformly in $t \geq 0$. Let

$$
g(\sigma,t) = t + \frac{1}{\pi}\log|1+e^{2i\pi(\sigma-1/2+it)}|.
$$

We will observe that there is $\delta_r > 0$ such that

$$
g(\sigma,t) > \delta_r, \tag{2.3}
$$

where $s=\sigma+it$ lies on the semicircle $|s-n|=r$, $t\geq 0$, $n\in\mathbb{Z}$, and $1/6<r<1/2$. Note that $g(x+n,t)=g(-x+n,t)$, $x\in\mathbb{R}$. Thus it is enough to prove (2.3) for the following quarter of the circle: $|s-n|=r$, $t\geq 0$, $0\leq \sigma-n\leq r$, which we parametrize by $t=x$, $\sigma=\sqrt{r^2-x^2}+n$, $x\in[0,r]$. Consequently we consider the function

$$
q(x) = g(\sqrt{r^2-x^2}+n, x).
$$

Straightforward calculations show that $q(0)>0$ and $q'(x)>0$ for $0\leq x\leq r$, $1/6<r\leq 1/2$. This establishes the inequality (2.3).

Hence, for any given real positive number $Y$ and $1/6<r<1/2$, there is a negative number $M=M(Y,r)$ such that

$$
|f(s)| = \exp\left(\operatorname{area}(X)(-(\sigma-1/2)g(\sigma,t)+O(|t|+1))\right) > Y,
$$

if $|s-n|=r$, $t\geq 0$, and $n<M$. The Dirichlet series expansion of $Z(s)$ yields

$$
|Z(s)| > 1/2 \tag{2.4}
$$

if $\sigma$ is sufficiently large. Note that

$$
\overline{Z(s)} = Z(\bar{s}). \tag{2.5}
$$

Then Rouché's theorem gives that for sufficiently large negative $n$ the functions $Z(s)$ and $Z(s)-a$ have the same number of zeros in the disc $|s-n|\leq r$. In this disc $Z(s)$ has only one distinct zero at $s=n$ and clearly $Z(n)\neq a$. This, $1/6<r<1/2$, and Proposition 1.4 give that $Z(s)-a$ and $(Z(s)-a)'=Z'(s)$ have no common zeros in $\sigma<N$. Accordingly, all zeros of $Z(s)-a$ located in $|s-n|\leq r$ are simple.

It remains to show that for any sufficiently large negative $n$ the area $\{s : |s-n| > r,\ n-1/2\leq\sigma\leq n+1/2\}$ is free from zeros of $Z(s)-a$. This follows by the inequalities $\partial g(\sigma,t)/\partial t>0$ if $t>0$, $\sigma\in\mathbb{R}$ and $g(\sigma,0)>0$ if $|\sigma-1/2-n|<1/3$, together with formulas (2.1)–(2.5). Proposition 1.5 is proved. □

**Lemma 2.1.** *If the polynomial $P(z)$ has at least two different roots, then there is a nonzero constant $c$ such that $P(z) - c$ has a multiple root.*

*Proof.* Let $\deg P = k \geq 2$. Conversely to the statement of the lemma, suppose that the roots of $P(z) - c$ are simple for all $c \neq 0$. Then $(P(z) - c)' = P'(z)$ has no common roots with $P(z) - c$ for any $c \neq 0$. Therefore, for any root $z_j'$, $j \in \{1, \ldots, k-1\}$, of $P'(z)$, we have $P(z_j') = 0$. This is possible only if $P(z) = a(z-z_1')^k$ and $z_j' = z_1'$, for all $j \in \{2, \ldots, k-1\}$. The contradiction obtained proves the lemma. □

*Proof of Corollary 1.6.* By Proposition 1.5, for any large negative $n$ and fixed $x \in [0,1]$, there are exactly $(2g-2)(n+1)$ simple zeros $s_j(x)$ of $Z(s) - a(x)$ in the disc $|s - n| < 1/3$. Then the corollary follows by the implicit function theorem ([9, Theorem 2.4.1]) from which we see that $Z(s)$ is a one-to-one function in some neighborhood of each $s_j(x)$, $j = 1, \ldots, (2g-2)(n+1)$, $x \in [0,1]$. □

*Proof of Lemma 1.3.* Note that $P$ cannot be a constant polynomial. To obtain a contradiction, assume that $Z(s) = P(h(s))$ and the polynomial $P$, $\deg P = k$, has at least two different roots. Then Lemma 2.1 implies the existence of $a_1$ such that $P'(a_1) = 0$ and $P(a_1) \neq 0$. Therefore we can write

$$P(z) - P(a_1) = d(z - a_1)^{k_1} \ldots (z - a_m)^{k_m}, \tag{2.6}$$

where $k_1 \geq 2$ and $k_1 + \cdots + k_m = k$. In view of Proposition 1.5 there are infinitely many zeros of $Z(s) - P(a_1)$ each of which lies at a distance smaller than $1/3$ from some negative integer. Thus there are an infinite subset $S$ of these zeros and $a_j$ defined by (2.6) such that $h(\rho) - a_j = 0$ for $\rho \in S$. If $k_j \geq 2$ then the zeros $\rho$ are multiple zeros of $Z(s) - P(a_1)$ and this contradicts Proposition 1.5. Hence $k_j = 1$, $P'(a_j) \neq 0$, and by (2.6) we see that $j \geq 2$. Therefore there is a continuous function $a : [0,1] \to \mathbb{C}$, such that $a(0) = a_j$, $a(1) = a_1$, and

$$P'(a(x)) \neq 0 \quad \text{for } x \in [0,1). \tag{2.7}$$

By Corollary 1.6 there is a continuous function $\psi : [0,1] \to \mathbb{C}$ such that $\psi(0) \in S$,

$$Z(\psi(x)) = P(a(x)), \tag{2.8}$$

and, for $x \in [0,1]$ and some large negative integer $n$,

$$|\psi(x) - n| < 1/3.$$

Note that $Z(\psi(x)) = P(h(\psi(x)))$. To get the contradiction we will show that $h(\psi(1)) = a_1$. By (2.8)

$$P(h(\psi(x))) = P(a(x)).$$

In view of (2.7) we have that $P(z)$ is a one-to-one function in a sufficiently small neighborhood of any $a(x)$, $x \in [0,1)$. Then $h(\psi(0)) = a(0)$ leads to the equality $h(\psi(x)) = a(x)$ for $x \in [0,1)$. Continuity gives $h(\psi(1)) = a(1) = a_1$ and thus $z = \psi(1)$ is a multiple zero of $Z(z) - Z(\psi(1))$. This contradicts Proposition 1.5 and proves Lemma 1.3. □

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