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HOMOGENEOUS EINSTEIN MANIFOLDS

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ABSTRACT. This survey builds on the two surveys by Wang and Lauret, written 10–15 years ago, to give the current state of affairs regarding homogeneous Einstein spaces.

1. Introduction

In Riemannian geometry, we have three standard notions of curvature which give a glimpse at the shape of a given metric and help us to discern how a space might be related to other familiar, model spaces. These three curvatures are the sectional, Ricci, and scalar curvatures.

The spaces of constant sectional curvature are well understood with their simply-connected covers being spheres, hyperbolic spaces, and Euclidean spaces. At the other extreme is scalar curvature. All manifolds, of dimension 3 or greater, admit metrics of constant negative scalar curvature [11, Theorem 4.35]; even further, compact manifolds are known to admit constant scalar curvature metrics in every conformal class; this is the so-called Yamabe problem [92]. In the homogeneous setting, scalar curvature is always constant and reveals some interesting, but limited, information about the underlying space [9]. Recall that a Riemannian manifold (M, g) is called *homogeneous* if its isometry group acts transitively.

Perhaps in the Goldilocks zone is Ricci curvature. If one were to hope to endow a given manifold with a special geometry against which other metrics could be weighed, this seems to be a reasonable invariant to hold constant and study [11]. A Riemannian metric g is called *Einstein* if it has constant Ricci curvature, i.e.,

$$\operatorname{ric}_g = cg \tag{1.1}$$

for some $c \in \mathbb{R}$.

The general setting of manifolds differs substantially from the homogeneous setting in both technique and results. For the reader interested in the general setting, we direct them to the survey [3]. Our interest is solely in the homogeneous setting and follows on two excellent surveys, written 10–15 years ago: see [93] for the compact setting and [66] for the non-compact setting.

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Question. *Do all homogeneous spaces admit Einstein metrics?*

Question. *Can the homogeneous Einstein spaces be classified?*

Question. *What special properties do homogeneous Einstein metrics have?*

In the homogeneous setting, Equation (1.1) is nothing more than a collection of quadratic equations, and Einstein metrics are nothing more than the (positive definite) real solutions. The simplicity of these algebraic equations is deceiving. Solutions to these equations have been elusive and there is now a mountain of serious work invested over the past 70 years aimed at understanding these natural model spaces.

As is to be expected, the first efforts at addressing the existence question start with simply asking about constraints, both topological and Lie theoretic, of signed Ricci curvature.

Consider a Riemannian manifold (M, g) with G a transitive group of isometries such that H is compact. As H is compact, G/H carries a manifold structure which is diffeomorphic to M; in this way, we have $M = G/H$. Certainly, $G = \mathrm{Isom}(M, g)$ is such a group, but in practice there are often other presentations of M as a homogeneous space. Once we have fixed a homogeneous presentation G/H of M, any reference to a metric will assume that the metric is G-invariant, unless said otherwise.

If G/H admits a metric of positive Ricci curvature, by homogeneity the Ricci curvature is bounded below by a positive constant, and so Myer's theorem yields that G/H is compact with finite fundamental group. Consequently, the semi-simple part of G must act transitively [55]. In Section 2, we discuss the current state of knowledge for the compact setting.

In the case of zero Ricci curvature, it turns out there is little variety. Here the Riemannian metric must be flat and G/H is isometric to $\mathbb{R}^k \times T^{n-k}$ [2].

When G/H admits a metric of negative Ricci curvature, we know that G and G/H must be non-compact [12]. This case has attracted substantial attention over the past 20 years. We discuss this setting in Sections 3, 4, and 5.

There are two main, driving questions in the study of homogeneous Einstein spaces at present. In the compact setting, Einstein metrics are not unique on a given homogeneous space. On a given manifold, there are not even a finite number of homogeneous Einstein metrics. However, if we fix the homogeneous presentation, then we have the following fundamental question.

Open Problem. *On a given compact homogeneous space G/H, are there only a finite number of G-invariant Einstein metrics (up to isometry and scaling)?*

In Section 2, we present several theorems on both existence and non-existence of G-invariant Einstein metrics on G/H. The classification of G/H admitting such metrics, and the metrics themselves, is a wide open problem of continued interest.

In the non-compact setting, the Alekseevsky Conjecture has been the primary driver of research on Einstein spaces.

Alekseevsky Conjecture. *If G/H is connected and admits an Einstein metric with negative scalar curvature, then it is diffeomorphic to $\mathbb{R}^n$.*

Since the late 1990s, there has been a flurry of activity towards resolving this conjecture. A solution has recently been put forward [21]. We give an overview of the progress towards this resolution below. Even with a resolution of the Alekseevsky Conjecture, a topological result, the algebraic problem of classifying the non-compact homogeneous Einstein spaces is also wide open and actively being pursued, see Section 3.

Finally, any discussion of Einstein metrics would be incomplete without also considering their closely related cousins, the Ricci solitons. Recall that a Riemannian metric g on M is a *Ricci soliton* if

$$\mathrm{ric}_g = cg + L_X g$$

for some $c \in \mathbb{R}$ and some smooth vector field X on M, where L_X is the Lie derivative. These are important for two reasons. First, both kinds of spaces arise naturally as fixed points/generalized fixed points of the (normalized) Ricci flow, see Section 6.1. Second, and most importantly for this article, in the non-compact homogeneous setting Ricci solitons are intimately coupled to Einstein metrics, see Section 3.

A note from the surveyor. It is a privilege and an honor to have been asked to write a survey on the topic of Einstein metrics on homogeneous spaces. To do so within 20 pages is quite a challenge, but it is also an opportunity for presenting a condensed narrative that should, hopefully, not overwhelm a newcomer to the field, giving them just enough to whet their appetite along with a sufficient amount of references to help them dig deeper into wherever their interests lie and the whims of the moment point them.

While many important contributions will not be mentioned, it is important to acknowledge that some of these played a crucial role in driving progress, even if their technical results were overshadowed by later works. These contributions served as necessary milestones along the way. Other omissions will have occurred either by accident or lack of awareness and knowledge on the author's part—for those, I apologize in advance.

Finally, I have tried to fill the narrative with open questions that still need resolution. Despite the significant progress made in recent years, these questions should demonstrate that the story is not fully written for homogeneous Einstein manifolds.

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2. Compact homogeneous Einstein spaces

An excellent and robust survey of compact homogeneous Einstein spaces can be found in [93]. There the interested reader will find numerous explicit examples and a full treatment of the compact setting up to 2012. Since the writing of that survey, there have been advances in the form of obtaining new and different kinds of examples of Einstein metrics on compact homogeneous spaces; however, the finiteness conjecture remains an open problem (see below). Our treatment of the compact setting, in this section, should be viewed as an attempt to stimulate and motivate the reader to go deeper by exploring the survey by Wang. The interested reader should also consult [16].

The main tool for approaching Einstein metrics on compact homogeneous spaces is the scalar curvature function restricted to the set of volume one metrics. Let $\mathcal{M}_1^G$ denote the volume one G-invariant metrics on G/H and

$$S : \mathcal{M}_1^G \to \mathbb{R},$$

the scalar curvature function. Einstein metrics are critical points of this function [56, 82].

Open Problem 2.1. *On a given homogeneous space G/H, are there a finite number of G-invariant Einstein metrics (up to isometry and scaling)?*

Recall that we must restrict to a fixed homogeneous structure G/H as there do exist examples of infinitely many, inequivalent homogeneous structures on the same underlying manifold which admit G-invariant Einstein metrics [95]. For example, $S^2 \times S^3$ can be presented as $(\mathrm{SU}(2) \times \mathrm{SU}(2))/U(1)_{pq}$ with p and q relatively prime.

Theorem 2.2 ([22])**.** *The moduli space of Einstein metrics on $\mathcal{M}_1^G$ has finitely many components, each of which is compact. Even further, this moduli space is an algebraic variety.*

Part of the challenge of the above is that the set of solutions to the Einstein equation $\mathrm{ric}(g) = cg$ is not necessarily discrete. For example, on a compact semi-simple Lie group, one can pull back any solution by an automorphism to have an isometric metric, which is of course also going to be Einstein. On homogeneous spaces more generally, one has to contend with $N_G(H)/H$, which acts on $\mathcal{M}_1^G$. (Note that we are not asserting that all isometries arise in this manner. That constitutes a distinct and intriguing problem in its own right. See, e.g., [35, 38, 39].) As a special case, the following finiteness conjecture has been made.

Open Problem 2.3 ([22])**.** *If G/H is a compact homogeneous space whose isotropy representation consists of pairwise inequivalent irreducible summands (e.g., when $\operatorname{rank} G = \operatorname{rank} H$), then the algebraic Einstein equations have only finitely many real solutions.*

The above results and questions sample from the leading edge of knowledge on the general compact setting. Before putting forward other general results, we give a few examples to demonstrate both existence and non-existence results for particular homogeneous spaces.

2.1. **Examples of compact homogeneous Einstein spaces.**

For the moment, let G to be a Lie group, not necessarily compact, and consider G/H where H is compact. Take an Ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. If one prefers, one can choose $\mathfrak{p}$ to be the orthogonal complement to $\mathfrak{h}$ under the Killing form of $\mathfrak{g}$. Then the set of G-invariant metrics on G/H is in one-to-one correspondence with the Ad_H-invariant inner products on $\mathfrak{p} \simeq T_{eH}G/H$.

The isotropy action of H on $T_{eH}G/H$ is equivalent to the Ad_H-representation on $\mathfrak{p}$ and the simplest examples of homogeneous Einstein spaces occur when this presentation is irreducible; that is, our homogeneous space is so-called isotropy irreducible. The isotropy irreducible spaces are classified, see [77, 100]. These spaces include the following examples.

Example 2.4. Irreducible symmetric spaces are Einstein manifolds.

In the positive scalar curvature setting (i.e., compact), this includes the following.

Example 2.5. Let G be a compact simple Lie group. If B is the Killing form of $\mathfrak{g} = \operatorname{Lie} G$, then we may consider the left-invariant g_B, which is $-B$ on $\mathfrak{g} \simeq T_eG$.

If G is compact semi-simple, then we can build an Einstein metric by scaling the above metric on each simple factor. Likewise, for symmetric spaces, or a product of isotropy irreducible spaces, whose factors are of the same type (i.e., all of the same scalar curvature sign), we can scale the factors appropriately to put an Einstein metric on such a product.

Theorem 2.6 ([56, 27])**.** *If G is a compact semi-simple group other than* $\mathrm{SU}(2)$*, then G admits at least two Einstein metrics which are not equivalent up to scaling and isometry. The group* $\mathrm{SU}(2)$ *admits a single Einstein metric.*

In the more general, homogeneous setting, we have full information up to dimension 7, including an affirmative answer to Open Problem 2.1, and partial information up to dimension 11.

Theorem 2.7. *The compact homogeneous Einstein spaces are classified in dimension 7 and less. For each homogeneous space G/H, there are a finite number of G-invariant Einstein metrics (up to scaling and isometry) on each G/H.*

Details of this classification can be found in [83], which builds on work in dimensions 6 and less in [1, 87]. Although as yet there is no classification for higher dimensions, one does know that Einstein metrics always exist up to dimension 11.

Theorem 2.8 ([15])**.** *Let G be compact semi-simple with closed subgroup H. If G/H is simply-connected with dimension less than or equal to 11, then it admits a homogeneous Einstein metric.*

We point out that the existence of homogeneous Einstein metrics up to dimension 11 is subtle. For some groups G, there is not always a G-invariant Einstein metric. Instead, the authors produce a canonical presentation of the underlying manifold as a homogeneous space by changing the group G acting on the manifold to a canonical group G_{can} so that G/H is diffeomorphic to $G_{\mathrm{can}}/H_{\mathrm{can}}$ and then prove the existence of a G_{can}-invariant Einstein metric. The existence of G-invariant Einstein metrics is well understood when G is simple, but the semisimple case is not fully understood.

Among the examples given in [94] which do not admit homogeneous Einstein metrics, for any transitive group G, there is an example of dimension 12, namely, $\mathrm{SU}(4)/\mathrm{Sp}(2)$, and so the above theorem is sharp.

Another case of note is homogeneous spaces G/H for which the isotropy representation has only two summands. The collection of such spaces admitting Einstein metrics has been classified for G simple [28, 42] with one recent addition to the list [60, Remark 6.1]. The classification of Einstein metrics on these spaces remains open.

Open Problem 2.9. *For compact homogeneous spaces whose isotropy representation has two summands and admits an Einstein metric, verify the classification when G is simple, classify the Einstein metrics on these spaces, and classify the case when G is semi-simple.*

2.2. **General existence and non-existence results in the compact setting.** As Einstein metrics are critical points of the scalar curvature function (restricted to the volume one metrics), one might naturally look for maxima and minima of this function.

Theorem 2.10 ([94])**.** *A global minimum of S exists on $\mathcal{M}_1^G$ if and only if G/H is the product of several isotropy irreducible homogeneous spaces. The minimum occurs at a unique metric which is a product of the Einstein metrics on each factor.*

Alternatively, one may consider global maxima of S. If H is a maximal proper subgroup of G, then G/H admits a G-invariant Einstein metric. Here the Einstein metric is a maximum of the scalar curvature function on $\mathcal{M}_1^G$ [94]. See below for a general statement on the existence of maxima.

Even further, one can ask about the boundedness of S on $\mathcal{M}_1^G$. Scalar curvature is rarely bounded below. This occurs only when G/H is a product of isotropy irreducible spaces and $\mathbb{R}^k$, with $k \geq 0$, see [94, Theorem 2.1]. For the case of S bounded above, we consider when H is not a maximal subgroup of G; i.e., we must concern ourselves with intermediate subgroups, $H \subset K \subset G$, and how these fit together. At this point, one of two things can happen: either K/H is a torus, and hence flat, or K/H admits a (K-invariant) metric of positive scalar curvature. Notation: we say the extension $H \subset K$ is toral if K/H is a torus and non-toral otherwise. At the Lie algebra level, toral just means $\mathfrak{k}/\mathfrak{h}$ is abelian.

Consider the fibration $K/H \to G/H \to G/K$. If $K \subset G$ is non-toral, e.g., when K is maximal, then G/K admits a metric of positive scalar curvature. By shrinking the fibers K/H and expanding the base G/K, one can create a sequence of metrics on G/H which have unit volume, but with scalar curvature going to infinity; see [11, Equation 9.71]. We have the following from [13, Theorem 5.22].

Theorem 2.11. *Let G/H be a compact homogeneous space. The scalar curvature S is bounded from above on $\mathcal{M}_1^G$ if and only if there exist no non-toral, intermediate subalgebras which are Ad_H-invariant. In this case, a global maximum is obtained.*

We note that in the case of H connected, the condition that our intermediate subalgebra be Ad_H-invariant is superfluous.

The problem now becomes one of finding non-extremal critical points of $S(g)$. Fortunately, we have a powerful tool at our disposal, namely that the Palais–Smale condition holds on super-level sets of S [22, Theorem A]. That is, on the set $(\mathcal{M}_1^G)_\epsilon = \{g \in \mathcal{M}_1^G \mid S(g) \geq \epsilon > 0\}$, if there exists a sequence $g_i \in (\mathcal{M}_1^G)_\epsilon$ with $-\operatorname{grad}(S)(g_i) \to 0$, then there exists a convergent subsequence of $\{g_i\}$.

Obviously, the limiting metric $g_i \to g$ will be a critical point of S on $\mathcal{M}_1^G$ and so Einstein. Now, if one can show that the super level set $(\mathcal{M}_1^G)_\epsilon$ has non-trivial topology, then one can argue for the existence of critical points via a mountain pass lemma.

There are a few approaches for ensuring $(\mathcal{M}_1^G)_\epsilon$ has non-trivial topology; we present two which rely on building a combinatorial object using intermediate groups. Given $H \subset G$, consider a maximal torus T in the normalizer $N(H)$ of H which is complementary to H. Then there are a finite number of intermediate subgroups K satisfying

$$TH \subset K \subset G.$$

Considering all flags of such subgroups $K_1 \subset K_2 \subset \cdots \subset K_r$, one can build a simplicial complex where flags of length 2 correspond to vertices, flags of length 3 to edges, and so on. This simplicial complex is denoted by $\Delta_{G/H}$.

Theorem 2.12 ([13]). *If $\Delta_{G/H}$ is non-contractible, then G/H admits a G-invariant Einstein metric.*

Other existence and non-existence results along these lines can be found in [13, 14]. Remarkably, one can use a finite amount of Lie algebra data to analyze infinitely different homogeneous spaces simultaneously; e.g., one can recover the existence of Einstein metrics on the Aloff–Wallach spaces this way.

Continuing in this vein, Graev [40] constructs a different combinatorial object from the inclusion of subalgebras of $\mathfrak{g}$ called the *nerve*, denoted by $X_{G/H}$. Here, though, one uses all flags at the Lie algebra level $\mathfrak{h} \subset \mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \cdots \subset \mathfrak{k}_r \subset \mathfrak{g}$ with the only condition being that $\mathfrak{h} \subset \mathfrak{k}_1$ is non-toral.

Theorem 2.13 ([40]). *Let G/H be a compact homogeneous space with G and H connected. If $X_{G/H}$ is non-contractible, then G/H admits a G-invariant Einstein metric.*

We do not attempt to further elaborate on these notions beyond the glimpse provided above; instead, we refer the reader to the recent work [16]. Here the reader will find numerous examples, open problems on the existence, non-existence, and classification of Einstein metrics in the compact setting.

2.3. **Results on compact homogeneous Einstein manifolds since 2012.** In the decade following Wang's extensive survey [93], the bulk of new results for compact homogeneous Einstein manifolds has been the construction of new types of Einstein metrics. In some cases, these have been non-naturally reductive Einstein metrics [101, 23, 102], the classification of Einstein metrics for some special homogeneous spaces [86, 24], and the generation of new examples of Einstein metrics on particular homogeneous spaces, sometimes with the aid of computer algebra systems, [7, 8, 26]. For a detailed treatment of Einstein metrics on flag manifolds, see [6].

3. Non-compact homogeneous Einstein spaces: solvmanifolds

As in the compact case, there are non-compact Lie groups which do admit Einstein metrics, namely solvable Lie groups. If a solvable Lie group is unimodular, then it is Einstein if and only if it is flat [29]. Thus, a solvable Lie group admitting an Einstein metric of negative scalar curvature must be non-unimodular. To date, all known examples of non-compact homogeneous Einstein metrics on Lie groups occur on solvable groups. As has recently been announced, this collection exhausts the set of non-compact homogeneous Einstein spaces [21] and so we begin by focusing our attention on these groups.

As we will see in what follows, the nilradicals of Einstein solvmanifolds have their own special properties. For now, we call a nilpotent Lie group an Einstein nilradical if it is the nilradical of a solvable Lie group admitting an Einstein metric. We call a nilpotent Lie algebra an Einstein nilradical if its simply-connected Lie group is an Einstein nilradical. (Simple connectivity turns out to be necessary.)

3.1. **Examples of non-compact homogeneous Einstein spaces.** As in the compact setting, symmetric spaces provide a familiar family of examples. If M is a non-compact irreducible symmetric space which is not flat, then $M = G/K$, where $G = \mathrm{Isom}(M)$ is simple and K is a maximal compact subgroup. Even further, we have an Iwasawa decomposition $G = KAN$ with $S = AN$ being a solvable simply-transitive group of isometries on M.

The group S above features several properties which, as we will see shortly, are the norm for solvable groups admitting Einstein metrics.

(1) S is simply-connected.
(2) The nilradical of S is N, and A is a complementary, abelian subgroup, i.e., $A \cap N = \{e\}$.
(3) The Lie algebra $\mathfrak{s} = \mathrm{Lie}\, S$ contains a special element $H \in \mathfrak{a} = \mathrm{Lie}\, A \subset \mathfrak{s}$ such that
$$\mathrm{tr}(ad_H \circ D) = \mathrm{tr}\, D \quad \text{for all derivations } D \in \mathrm{Der}(\mathfrak{s}).$$

For more details on the structure of Einstein solvmanifolds, see Section 3.2.

By left-invariance, to build examples and study metrics on Lie groups, it suffices to work with Lie algebras and inner products on the Lie algebra. We adopt this approach going forward.

Example 3.1. Let $\mathfrak{n} = \mathbb{R}^{n-1}$ be an abelian Lie algebra. Consider the derivation D which is the identity on $\mathfrak{n}$. Then $\mathfrak{s} = \mathbb{R}D \ltimes \mathfrak{n}$ is the Lie algebra of a solvable group S which admits an Einstein metric. This produces the rank 1 symmetric space which is hyperbolic n-space $\mathbb{H}^n$.

Example 3.2. Let $\mathfrak{n}$ be the Heisenberg Lie algebra. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and write $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ for some vector complement $\mathfrak{v}$ of $\mathfrak{z}$. Define a derivation D of $\mathfrak{n}$ which is the identity on $\mathfrak{v}$ and twice the identity on $\mathfrak{z}$. Then $\mathfrak{s} = \mathbb{R}D \ltimes \mathfrak{n}$ admits an inner product such that the corresponding metric on S is Einstein. This is the rank 1 symmetric space complex hyperbolic space.

In the two examples above, using any other derivation will produce a group that cannot admit an Einstein metric. Unlike in the compact setting, not all (non-unimodular) solvable groups admit left-invariant Einstein metrics in low dimensions due to the necessary algebraic structures above. However, in low dimensions, those are the only constraints.

Theorem 3.3 ([63, 96]). *Up to dimension 6, every nilpotent Lie algebra is an Einstein nilradical.*

In dimension 7, there are examples of non-Einstein nilradicals, but also continuous families of Einstein nilradicals; these are classified in dimension 7 [73, 33]. At this point, we start to see more divergence in the theories for compact and non-compact spaces. In the compact setting, in any given dimension, the number of compact semi-simple Lie groups is finite; in the solvable setting, beyond low dimensions you have large dimensional continuous families of solvable groups.

Another difference between the compact and non-compact settings is that, for non-unimodular solvable Lie groups, Einstein metrics are not critical points of the scalar curvature function. However, one can realize them as critical points of a modified scalar curvature function [43, 62].

As we will see in Section 3.2, Einstein metrics on solvable Lie groups are unique up to scaling and isometry, yet another divergence from the compact realm.

Theorem 3.4 ([31, 44]). *A generic 2-step nilpotent Lie algebra is an Einstein nilradical.*

The first examples of continuous families of non-Einstein nilradicals appeared in [97]. These were followed by other techniques for building (continuous) families of both Einstein and non-Einstein nilradicals [89, 4, 47, 90]. See [72, 88] for a deeper investigation of non-Einstein nilradicals among 2-step nilpotent algebras. In the presence of additional curvature conditions, one can classify the Einstein solvmanifolds in low dimensions. For example, if one requires negative sectional curvature, then these are classified in dimensions up to 7 [80, 79].

Unlike in the compact setting, we have a convenient reduction to irreducible algebras.

Theorem 3.5 ([45, 81]). *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a nilpotent Lie algebra which is a direct sum of ideals. If $\mathfrak{n}$ is an Einstein nilradical, so are the factors $\mathfrak{n}_1$ and $\mathfrak{n}_2$.*

Note that it is not assumed a priori that the Einstein metric makes $\mathfrak{n}_1$ and $\mathfrak{n}_2$ orthogonal. Instead, this is true after the fact.

3.2. **Structure results for Einstein solvmanifolds.** Presently, there are few techniques for precluding the existence of Einstein metrics. First, among Einstein metrics, zero scalar curvature ($c = 0$) corresponds to unimodular Lie groups, and negative scalar curvature ($c < 0$) corresponds to non-unimodular Lie groups [29]. As the Ricci flat homogeneous spaces are precisely the flat spaces [2], we focus our attention on the setting of negative scalar curvature.

To demonstrate the stark contrast between the compact and non-compact settings, we begin with the following.

Theorem 3.6 ([43]). *A left-invariant (standard) Einstein metric on a non-unimodular solvable Lie group is unique up to scaling and isometry.*

A solvable Lie group with left-invariant metric is called *standard* if the corresponding Lie algebra $\mathfrak{s}$ with inner product satisfies $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical and $\mathfrak{a} = \mathfrak{n}^\perp$ is an abelian Lie algebra. The standard condition turns out to always be satisfied (see below) and so we place it in parentheses.

Remark 3.7. In the theorem above, we actually know that any two Einstein metrics are equivalent up to scaling and pull-back by an automorphism. While pull-back by an automorphism will always give isometric metrics on a Lie group, in general not all isometries arise this way for all Lie groups, even among solvable Lie groups. See [39].

We have a recipe for building Einstein solvmanifolds which goes as follows. Take as ingredients:

(1) a nilpotent Lie algebra $\mathfrak{n}$ with a so-called Ricci soliton metric (see Equation 6.1);
(2) an abelian subalgebra $\mathfrak{a} \subset \mathrm{Der}(\mathfrak{n})$ of fully-reducible operators whose real parts are non-trivial;
(3) the existence of an element $H \in \mathfrak{a}$ such that $D = \mathrm{ad}\, H \in \mathrm{Der}(\mathfrak{n})$ satisfies
$$\mathrm{tr}(D\phi) = \mathrm{tr}(\phi) \quad \text{for all } \phi \in \mathrm{Der}(\mathfrak{n}) \tag{3.1}$$
with positive eigenvalues.

We can assume, by conjugating $\mathfrak{a}$ by an automorphism of $\mathfrak{n}$, that $\mathfrak{a}$ consists of normal operators relative to the Ricci soliton inner product on $\mathfrak{n}$. On $\mathfrak{a}$, use the metric $\langle A, B\rangle = \mathrm{tr}(S(ad_A) \circ S(ad_B))$, where $S(X) = \frac{1}{2}(X + X^t)$ denotes the symmetric part of the normal operator X. On $\mathfrak{n}$, use the Ricci soliton inner product. Extend these to an inner product on the Lie algebra $\mathfrak{s} = \mathfrak{a} \ltimes \mathfrak{n}$ so that $\mathfrak{a} \perp \mathfrak{n}$. We now have the metric Lie algebra of a solvable Lie group with left-invariant Einstein metric. Remarkably, this construction exhausts the class of Einstein solvmanifolds.

Theorem 3.8 ([67, 69]). *All Einstein solvmanifolds are standard and arise via the construction above.*

We note that partial progress on the problem of showing that Einstein solvmanifolds are standard was made by others (see, e.g., [85] and other references of [67]). This theorem is the root of all non-existence results for Einstein metrics on solvable Lie groups. We note a few important consequences.

First, the special derivation D defined in Equation (3.1) must have eigenvalues which are integers (up to scaling) and so $\mathfrak{n}$ must be $\mathbb{N}$-graded. Recall that a (nilpotent) Lie algebra is $\mathbb{N}$-graded if
$$\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k \text{ with } [\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$
Here some of the $\mathfrak{n}_i$ may be trivial. This is equivalent to the existence of a derivation with eigenvalues which are positive integers.

Example 3.9. Characteristically nilpotent Lie algebras cannot be Einstein nilradicals.

Recall that a nilpotent Lie algebra is called *characteristically nilpotent* if $\mathrm{Der}(\mathfrak{n})$ is nilpotent. Going beyond the necessity of the existence of a positive derivation, one must have a derivation satisfying Equation (3.1) and this has served as inspiration for the pre-Einstein derivation as defined in [81]. This pre-Einstein derivation turns out to play a deeper role in the geometry of solvmanifolds, is unique up to conjugation in $\mathrm{Der}(\mathfrak{n})$, and is a necessary ingredient in understanding maximal symmetry (see Section 6.3).

Theorem 3.10 ([81])**.** *If $\mathfrak{n}$ is an Einstein nilradical, then its pre-Einstein derivation must be positive, i.e., have positive eigenvalues.*

Open Problem 3.11. *Is there a list of algebraic invariants which completely determines when a solvable algebra admits an Einstein metric?*

Despite not having a list of algebraic invariants, we do know that the existence of an Einstein metric on a solvable Lie group is intrinsic to the Lie algebra and is a 'local problem' on the space of left-invariant metrics. We are careful to point out that finding the Einstein metric itself is a global problem on the space of metrics. In [46] it was shown that one can determine the existence of an Einstein metric by measuring (1) algebraic invariants, and (2) local deformations of any choice of left-invariant metric. Like most of the results on existence of Einstein metrics, this result uses the robust tool of geometric invariant theory described in Section 4.1

4. Non-compact homogeneous Einstein spaces: tools and structure results

A central tool in the study of non-compact homogeneous Einstein spaces of negative scalar curvature is geometric invariant theory. Its introduction to homogeneous Riemannian geometry was in the work [43]. We motivate its introduction with a close look at the Ricci tensor of a Lie group with left-invariant metric. Note that everything that follows can and has been extended to homogeneous spaces.

On a Lie group with left-invariant metric, one can consider the Ricci tensor and decompose it as follows:

$$\mathrm{Ric} = M - \frac{1}{2}B - S(ad_H),$$

where $B(X,X) = -\operatorname{tr}(ad_X)^2$ is the Killing form, H a mean curvature vector satisfying $\langle H, X\rangle = \operatorname{tr} ad_X$ for all $X \in \mathfrak{g}$, with $S(ad_H)$ the symmetric part of ad_H, and M a mysterious component [11]. This mysterious tensor M is the moment map for the natural, 'change of basis' action on the space of Lie brackets; to our knowledge, this crucial observation was first made by Lauret [64, 65].

4.1. Geometric invariant theory as a tool.

Here we describe a tool which has been exploited in the nilpotent, solvable, and general homogeneous setting for understanding Einstein and Ricci soliton geometries. We describe everything in the setting of Lie groups with left-invariant metrics for simplicity; however, the reader should note that the ideas all extend to the full homogeneous setting [59].

Consider a Lie group G with left-invariant metric g. If G is simply-connected, all the data for (G, g) is encoded in the metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. Here the inner product on $\mathfrak{g} \simeq T_eG$ is the restriction of g to T_eG. A metric Lie algebra comprises three elements: an underlying vector space $\mathbb{R}^n$, a Lie bracket $\mu = [\cdot,\cdot]$, and an inner product $\langle\cdot,\cdot\rangle$. We will let $G_{\mu,\langle\cdot,\cdot\rangle}$ represent the simply-connected Lie group with metric Lie algebra $(\mathbb{R}^n, \mu, \langle\cdot,\cdot\rangle)$.

On the one hand, we can vary the inner product to obtain all possible left-invariant geometries on G via

$$\phi \cdot \langle\cdot,\cdot\rangle = \langle\phi\,\cdot, \phi\,\cdot\rangle \quad \text{for } \phi \in \mathrm{GL}(n,\mathbb{R});$$

here $\mathrm{GL}(n,\mathbb{R})$ is acting (via a right action) on the space of inner products which is the open set $\mathcal{P}$ of positive definite, symmetric bilinear forms. On the other hand, we can vary the Lie structure via the 'change of basis' action as follows:

$$\phi \cdot \mu = \phi\mu(\phi^{-1}\cdot, \phi^{-1}\cdot) \quad \text{for } \phi \in \mathrm{GL}(n,\mathbb{R});$$

here $\mathrm{GL}(n,\mathbb{R})$ is acting (via a left action) on the space of $\mathbb{R}^n$-valued anti-symmetric, bilinear forms $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. The set of Lie brackets is an algebraic variety in this vector space as the Jacobi condition is polynomial. These two perspectives are equivalent and we have that the following Lie groups with left-invariant metrics are isometric:

$$G_{\mu,\phi\cdot\langle\cdot,\cdot\rangle} \simeq G_{\phi\cdot\mu,\langle\cdot,\cdot\rangle}.$$

The isometry between these Lie groups with left-invariant metrics arises from lifting the Lie algebra isomorphism $\phi : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \phi\cdot\mu)$ which is simultaneously an isometry of the vector spaces with inner products.

The slight shift in perspective to vary the Lie bracket turns out to be quite powerful. We demonstrate the usefulness of this shift in perspective for nilpotent Lie groups. For a simply-connected, nilpotent Lie group N, the set of isometry classes in $\mathcal{P}$ are precisely the orbits of $\mathrm{Aut}(N)$ [39]. However, in the space of Lie brackets the isometry classes become the orbit of $O(n)$.

Additionally, by working in the space of Lie brackets, we have a natural way of compactifying the set of left-invariant metrics. For a nilpotent Lie bracket μ, $sc(N_\mu) = -\frac{1}{4}|\mu|$ and so normalizing to left-invariant metrics of $sc = -1$ amounts to restricting oneself to a sphere in $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. The compactification of the set of left-invariant metrics on a nilpotent Lie group is then

$$\overline{\mathrm{GL}(n,\mathbb{R})\cdot\mu} \cap S,$$

where S is a sphere of radius 4 in $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. Obviously, in the boundary, we have non-isomorphic Lie groups that appear in the compactification. This turns out to be an important, and useful, fact.

Further, for nilpotent Lie groups $G_{\mu,\langle\cdot,\cdot\rangle}$, we have

$$\mathrm{Ric} = M_\mu,$$

where Ric is the $(1,1)$-Ricci tensor and M_μ is the moment map of the $\mathrm{GL}(n,\mathbb{R})$ action on $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. This follows as nilpotent groups being unimodular implies $H = 0$, and nilpotentcy implies the Killing form B vanishes.

From geometric invariant theory (GIT) [57], there is a natural stratification of $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ which plays a crucial role in the possible geometries a Lie group can admit. These stratifications for non-nilpotent groups play an essential role in the structure results for general, non-compact homogeneous Einstein manifolds [59]. For example, in the setting of solvmanifolds, the GIT stratification is the key to proving the converse of a Cauchy–Schwarz inequality, enabling Lauret to prove that Einstein solvmanifolds are standard [67].

We lay out the basics of the GIT stratification to motivate the interested reader to dig deeper. For the $\mathrm{GL}(n,\mathbb{R})$ action on $V = \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, we have an induced action of $\mathfrak{gl}(n,\mathbb{R})$ on V which will be denoted by π. We implicitly define the function

$$m : V = \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \mathfrak{gl}(n,\mathbb{R})$$

via

$$\langle m(\mu), X\rangle = \frac{1}{|\mu|^2}\langle \pi(X)\mu, \mu\rangle.$$

Note that $m(\mu) = \frac{1}{|\mu|^2} M_\mu$. The function $F(\mu) = |m(\mu)|^2$ has a finite number of critical values. Let $\mathcal{C}$ denote the critical points of F. There is a finite collection of diagonal metrics $\mathcal{B}$ such that if μ is a critical point, then $m(\mu) = k\beta k^{-1}$ for some $\beta \in \mathcal{B}$ and some $k \in O(n)$.

Let $\mathcal{C}_\beta$ be the critical points with $m(\mu)$ conjugate to β. We define $\mathcal{S}_\beta$ to be the points in V which flow under the negative gradient flow to $\mathcal{C}_\beta$. In this way, the critical points are precisely the minima of F on the stratum $\mathcal{S}_\beta$.

As to be expected, for the particular representation of $\mathrm{GL}(n,\mathbb{R})$ above, these strata enjoy extra properties since the points in our space correspond to algebras. For example:

(1) If $\mu \in \mathcal{S}_\beta$, $\langle [\beta, D], D\rangle \geq 0$ for all $D \in \mathrm{Der}(\mu)$.
(2) If μ is nilpotent, then the critical points of F are precisely the Ricci soliton metrics; see Equation (6.1).
(3) If μ is solvable, then $\mathrm{GL}(n,\mathbb{R}) \cdot \mu \cap \mathcal{C} \neq \emptyset$ if and only if G_μ admits a left-invariant Einstein metric. Here the critical point is not the Einstein metric, but one can obtain the Einstein metric by simply dilating the orthogonal complement to the nilradical.

The GIT stratification is a cornerstone to obtaining the general structure results for homogeneous, Einstein spaces.

Remark 4.1. The application of GIT here is an adaptation of the classical theory over $\mathbb{C}$ to real algebraic, reductive groups. Interestingly, one can generalize this theory to real groups which are not necessarily algebraic and merely ‘self-adjoint’ relative to some inner product (see [19]).

4.2. **Structure results for non-compact homogeneous Einstein spaces in general.** We now consider the full setting of non-compact homogeneous Einstein spaces and not just the solvmanifold setting.

If a homogeneous space has zero Ricci curvature, then we know it is flat [2]. If a homogeneous space has positive Ricci curvature, then it must be compact [12]. Thus, in the non-compact setting, we focus our attention on the case of Einstein spaces with negative scalar curvature.

Using geometric invariant theory in the full homogeneous setting and exploiting the GIT stratification for our particular case, we have the following.

Theorem 4.2 ([59, 52, 5]). *Let M be a non-compact homogeneous Einstein space of negative scalar curvature. Let G denote the connected component of the identity of the isometry group. Here $M = G/H$ with H compact. We have a Levi decomposition $G = G_1G_2$, where G_1 is a maximal semi-simple and G_2 the radical (i.e., maximal, normal solvable Lie subgroup) such that*

(1) *$H = H_1H_2$ with $H_1 = H \cap G_1$ and $H_2 = H \cap G_2$.*
(2) *$G_2 = A_2H_2N_2$, where N_2 is the nilradical of G, A_2H_2 is an abelian subgroup with A_2 acting on N_2 with $\operatorname{ad}\mathfrak{a}_2$ consisting of reductive operators with real eigenvalues.*
(3) *The group A_2N_2 with the submanifold geometry is an Einstein manifold itself.*
(4) *Writing $G_1 = G_cG_{nc}$ as a product of G_c the product of simple compact normal subgroups and G_{nc} the product of simple non-compact normal subgroups, we have $G_c \subset H$.*

Essentially, the above allows us to remove the compact normal subgroups from G_1 and reduce G_2 to the setting of completely solvable groups. At this point, what is left to resolve is how large the isotropy $H_1 \subset G_1$ is. If H_1 is a maximal compact subgroup, then one has that G/H is a solvmanifold.

Before moving on to the resolution of the Alekseevsky Conjecture, we comment briefly on a certain special case, namely, when G is unimodular.

Theorem 4.3 ([30]). *Consider a non-compact unimodular group G acting (effectively) on G/H. If G/H admits a G-invariant Einstein metric with negative scalar curvature, then G must be semi-simple and a closed subgroup of the isometry group.*

Remarkably, in this special case of G unimodular, there was little progress for 30 years until (4) of Theorem 4.2. Applying [21] in this special case, we have the following.

Theorem 4.4. *Consider G/H with G non-compact semi-simple. If G/H admits a G-invariant Einstein metric, then it is a symmetric space.*

Open Problem 4.5. *Find a simple, algebraic proof of the theorem above.*

In low dimensions, various algebraic tools have been used to verify the theorem above [5]. Finding an algebraic proof in this special case would likely lead to a whole new proof of the Alekseevsky Conjecture.

4.3. **Overview of the resolution of the conjecture.** The Alekseevsky Conjecture asserts that a (connected) non-compact homogeneous Einstein space with negative scalar curvature is diffeomorphic to Euclidean space. This conjecture was known to hold in low dimensions, namely dimensions 4, 5, and 7 from the works [55, 84, 5], respectively. Partial results in dimensions up to 10 were obtained in [5, 10].

We now layout a strategy for resolving the conjecture that is ahistorical—i.e., instead of presenting the results as they appear chronologically, we present a narrative which exploits the benefit that hindsight affords, with necessary structural results having appeared and been refined at various points in time.

First, we note that non-compact Einstein solvmanifolds of negative scalar curvature (i.e., $c < 0$) are simply-connected. By solvmanifold, we mean that it has a transitive, solvable group of isometries. A natural starting point would be to transform the original Alekseevsky Conjecture, which is a topological statement, into one whose outcome is Lie theoretic.

Theorem 4.6 ([20])**.** *Suppose $(G/H, g)$ is a simply-connected homogeneous Einstein space with negative scalar curvature. If G/H is diffeomorphic to $\mathbb{R}^n$, then G/H is a solvmanifold.*

Next, one can reduce to the simply-connected setting.

Theorem 4.7 ([50])**.** *Let $(G/H, g)$ be a homogeneous Einstein space whose simply-connected cover is a solvmanifold. Then G/H is a solvmanifold and the quotient $\widetilde{G/H} \to G/H$ is trivial.*

It then remains to prove the Alekseevsky Conjecture among simply-connected manifolds. This is the assertion in the recent work [21]. A new set of tools has been developed there which we will not comment on in this survey and we refer the interested reader to that work.

5. Classification questions

Given a homogeneous space G/H, we are interested in: (1) when there exists a G-invariant Einstein metric, and (2) when there exists a homogeneous Einstein metric (invariant under a possibly different transitive group on the manifold).

Open Problem 5.1. *Are there algebraic invariants on G and H which determine the existence of Einstein metrics as above?*

For compact homogeneous spaces, these are not necessarily the same problem. However, in the non-compact setting, all homogeneous Einstein spaces are solvmanifolds [21] and so the maximal symmetry theorem of Gordon and this author shows these two problems are one and the same. Even further, Einstein metrics are unique (up to isometry and scaling) on a given solvmanifold when they exist (see Theorem 3.6), and so it is reasonable to ask if we can classify the solvable Lie groups/homogeneous spaces that admit an Einstein metric.

In dimensions 7 and greater, the number of solvmanifolds admitting Einstein metrics is infinite [73] and we have to ask what it even means to classify these spaces. Any reasonable classification should

(1) be a list of the objects without redundancy, and
(2) provide a means for determining if an object you bring to the list is either on it or not.

Going further, we might even ask

(3) to identify exactly where in the list an object is.

The second point is the challenge as it is a difficult problem to know when any two given Lie algebras are isomorphic, and the third point might seem impossible. However, both can be achieved in reasonable sense for our problem.

5.1. **The classification list.** We follow [98] to produce a list for the classification of Einstein solvmanifolds.

The first step is to classify the Einstein nilradicals or equivalently nilsolitons, see Theorem 3.8. From geometric invariant theory (see Section 4.1), we know that the set of nilpotent Lie groups with Ricci soliton metrics is precisely the set of critical points of the function $F(\mu) = |m(\mu)|^2$, where $m(\mu)$ is the (normalized) moment map. Denoting the critical points of F by $\mathcal{C}$, we see that, for $\mu \in \mathcal{C}$,

$$\mathrm{GL}(n, \mathbb{R}) \cdot \mu \cap \mathcal{C} = \mathbb{R}(O(n) \cdot \mu).$$

In this way, we see that nilsoliton metrics on a given Lie algebra are unique up to scaling and isometry, see [61]. Thus, the set of Einstein nilradicals is in one-to-one correspondence with

$$\mathcal{C} \cap \mathcal{N}/(\mathbb{R} \times O(n)),$$

where $\mathcal{N}$ is the set of nilpotent Lie brackets in $V = \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

At each metric nilpotent Lie algebra μ above, we can consider a maximal abelian, symmetric subalgebra $\mathfrak{a}$ of $\mathrm{Der}(\mu)$. This subalgebra is unique up to conjugation in $\mathrm{Aut}(\mu)$ and so is essentially unique. Suppose $\dim \mu = n$ and $\dim \mathfrak{a} = a$. For each $1 \leq r \leq a$, we may consider the space of solvsolitons of dimension $r + n$ with a given nilradical μ. This space is parameterized by a quotient of the Grassmannian

$$\mathrm{Gr}_r(\mathfrak{a})/W,$$

where W is an explicit finite group [98]. In this way, one has a 'list' of the solvsoliton metrics or equivalently the algebras admitting such metrics. To obtain a list of the Einstein solvmanifolds, one simply restricts to the subspaces in $\mathrm{Gr}_r(\mathfrak{a})$ which contain the pre-Einstein derivation (see Equation (3.1)). In a sense, this resolves the first point of having a classification list.

5.2. **Determining if you are on the list.** Our next job is to determine whether or not a Lie algebra is on our list when one is handed to us. This can be achieved, in principle, but not in the most ideal of ways. Ideally, one would have a list of algebraic invariants which resolves the question of when a given Lie algebra admits a soliton or Einstein metric. This has been a long-standing open question for Einstein solvmanifolds [73].

Open Problem 5.2. *Determine a finite list of algebraic invariants which completely determines the existence of an Einstein metric on a given solvable Lie group.*

At this point in time, the best we have is a combination of

(1) algebraic invariants/measurements and
(2) 'local measurements' in the space of invariant metrics using any initial metric.

This has been achieved in [46]. See the discussion after Open Problem 3.11 for details.

5.3. **Determining where on the list you are.** Since our classification list is actually a list of metric Lie algebras (with the metric of preference), satisfying the third point is a matter of being able to find the soliton/Einstein metric on a given Lie algebra. Presently, there are few tools for finding these special metrics. In the Einstein setting, the Ricci flow is known to take any initial metric and produce the Einstein metric at time infinity, upon running the flow [18].

Open Problem 5.3. *Are there algebraic techniques for recovering the Einstein metric of a given solvable Lie group when one is known to exist?*

6. Special properties of Einstein spaces

Given that Einstein metrics on solvable Lie groups are unique (up to isometry and scaling), one might expect these metrics to have special properties. This is certainly the case.

6.1. **Ricci flow.** Since, by definition, Einstein metrics satisfy $\operatorname{ric}(g) = cg$, these metrics essentially do not change under the Ricci flow—under the flow they simply dilate. Recall that the Ricci flow is the differential equation on the space of Riemannian metrics given by

$$\frac{\partial}{\partial t} g = -2 \operatorname{ric}(g).$$

One can normalize by rescaling the Ricci flow to fix scalar curvature; for this normalized flow, Einstein metrics are then fixed points. We note that for compact manifolds, the isometry group is preserved and so homogeneity is preserved under the Ricci flow. For non-compact manifolds, the situation is more delicate, so one has to choose to restrict the Ricci flow to the set homogeneous metrics. In doing so, one has uniformly bounded curvature and so a unique solution to the Ricci flow at each homogeneous metric [25]. Among homogeneous metrics, the Ricci flow becomes an ODE.

Question 6.1. *Are the fixed points of the normalized Ricci flow (i.e., Einstein metrics) stable?*

Recall that for unimodular Lie groups G (including compact Lie groups) and on the space of G-invariant metrics on G/H, the Ricci tensor is the gradient of the scalar curvature function. Considering there are Einstein metrics of various energy levels for G compact, we cannot have all Einstein metrics be stable.

Theorem 6.2 ([58])**.** *$\mathbb{C}P^n$ with its Fubini–Study metric (which is Einstein) is not stable.*

In the work [58], criteria are given for determining when a compact Einstein manifold is stable. This work has been extended to other symmetric spaces [41]. On a fixed homogeneous space G/K, and looking at only G-invariant metrics, significantly more is known; we direct the interested reader to the recent work [71].

For non-compact spaces, we can be more optimistic because Einstein metrics are unique in this context.

Theorem 6.3 ([18])**.** *Let S be a solvable Lie group with left-invariant Einstein metric g. Among homogeneous metrics, g is stable under the scalar curvature-normalized Ricci flow. In fact, the scalar curvature-normalized Ricci flow starting at any left-invariant metric converges to the Einstein metric in the C^∞ topology.*

It should be no surprise that the proof of this result relies on the robust structure theory and GIT stratification on the space of Lie brackets; see Section 4.1. Since the Ricci flow preserves the isometry group of a metric, this gives a dynamical proof of maximal symmetry as the Einstein metric appears in the limit of the flow (see Theorem 6.10).

On homogeneous spaces, in general, the Ricci flow can be converted to a flow on the space of brackets [70]. This approach allows one to prove results on the long-term existence of the flow and more. For example, we have the following.

Theorem 6.4 ([68, 70])**.** *On nilpotent Lie groups with left-invariant metrics, the Ricci flow always evolves towards a Ricci soliton (defined below)—on a possibly different nilpotent group.*

The result above exploits GIT upon observing that the Ricci flow is essentially the negative gradient flow of the norm-squared of the moment map on the space of brackets. For low dimensions, we also refer the interested reader to [34]. More stability results are presented for nilpotent Lie groups with Ricci solitons below.

6.2. **Homogeneous Ricci solitons.** From the perspective of the Ricci flow, if one is interested in fixed points, one should also consider Ricci solitons as they are generalized fixed points in the sense that 'up to diffeomorphism' they are fixed.

A metric (M, g) is called a *Ricci soliton* if

$$\mathrm{ric}_g = cg + L_X g, \tag{6.1}$$

where L_X is the Lie derivative of some smooth vector field $X \in \mathfrak{X}(M)$. This is equivalent to g being the initial point of a solution to the Ricci flow of the form $g(t) = c(t)\phi(t)^* g$ for a 1-parameter family of diffeomorphisms $\phi(t)$ and a 1-parameter family of constants $c(t) > 0$.

Question 6.5. *What are the homogeneous Ricci solitons?*

Surprisingly, the answer to this question is quite simple and they are all born from the homogeneous Einstein metrics.

Theorem 6.6 ([78, 91])**.** *If G/H admits a Ricci soliton with non-negative cosmological constant $c \geq 0$, then G/H is (locally) the product of a compact homogeneous Einstein space with a Euclidean factor.*

Compact solitons are necessarily Einstein with $c \geq 0$ [49] and so we focus on the setting of non-compact homogeneous solitons with $c < 0$.

Theorem 6.7 ([48])**.** *Consider a (non-compact) homogeneous Ricci soliton M with negative cosmological constant c. Let G denote the isometry group of M. Then our Ricci soliton is algebraic with respect to G, i.e., there is a symmetric derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that the vector field X in Equation* (6.1) *is generated by the 1-parameter family of automorphisms* $\exp(tD)$*. Even further, the* $(1,1)$*-Ricci tensor is of the form*

$$\mathrm{Ric} = cId + D_{\mathfrak{p}},$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $T_eG/K \simeq \mathfrak{p}$, and $D_{\mathfrak{p}}$ is the projection of D onto $\mathfrak{p}$.

Remark 6.8. It is worth noting that the map $D_{\mathfrak{p}}$ above is symmetric. This subtle detail turns out to be essential for various structure results in the classification of Einstein and soliton metrics on non-compact homogeneous spaces.

As in the Einstein case, one can ask about the stability of homogeneous Ricci solitons under the Ricci flow. In [99], it is shown that linear stability of solitons implies dynamical stability. In [54, 53], it is shown that low-dimensional solitons are stable and all solitons on 2-step nilpotent Lie groups are stable (among all metrics, not just homogeneous metrics).

Open Problem 6.9. *Are non-compact homogeneous Ricci solitons all (linearly) stable under the Ricci flow?*

If one restricts to left-invariant metrics on a particular solvable Lie group which admits a soliton, then we know that the soliton is dynamically stable among these homogeneous metrics from [18]. In general, i.e., among all metrics, stability is not well understood. See also [76].

6.3. **Maximal symmetry.** Given the special curvature properties of Einstein and Ricci soliton metrics on solvmanifolds, along with their uniqueness on a given Lie group, one might wonder what other special properties these spaces enjoy.

Consider a Lie group S and a left-invariant metric g. Recall that for $\phi \in \mathrm{Aut}(S)$, $\phi^* g$ is another isometric left-invariant metric. We say that g has maximal symmetry if given an arbitrary left-invariant metric g' on S there is an automorphism $\phi \in \mathrm{Aut}(G)$ such that

$$\mathrm{Isom}(S, g') \subset \mathrm{Isom}(S, \phi^* g).$$

Theorem 6.10 ([36])**.** *Einstein metrics on solvable Lie groups have maximal symmetry.*

The proof of this result relies on the pre-Einstein derivation (see Equation (6.1)). A dynamical proof of the theorem above follows from the stability of Einstein solvmanifolds (among left-invariant metrics) under the Ricci flow. While the above result does hold for unimodular solitons on solvmanifolds [51], it cannot hold for all non-unimodular solitons. An infinitesimal version of it has been achieved in [37] where it is shown that one can get a containment of isometry algebras.

Open Problem 6.11. *Is there a dynamical proof of infinitesimal symmetry for solvsolitons?*

The technique of using Ricci flow that worked in the Einstein setting cannot work for solitons. New tools are needed.

Open Problem 6.12. *Are there algebraic criteria that determine when a given solvmanifold admits a maximal symmetry metric?*

There is an interesting algebraic question that arises from studying maximal symmetry. If $\mathfrak{g}$ is the isometry algebra of a completely solvable Lie algebra $\mathfrak{s}$, then decomposing $\mathfrak{g}$ into a Levi decomposition $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ we are able to see that

$$\mathfrak{s}\mathfrak{s}_1 \ltimes \mathfrak{s}_2,$$

where $\mathfrak{s}_1 \subset \mathfrak{g}_1$ is the Iwasawa subgroup of $\mathfrak{g}_1$, and the action of $\mathfrak{s}_1$ is the restriction of the adjoint action of $\mathfrak{g}_1$ on ideal $\mathfrak{s}_2$. We call the decomposition above a *pre-Levi decomposition* (pre-Levi as it extends to a Levi decomposition of some isometry group).

The existence of infinitesimal maximal symmetry metrics then becomes an algebraic problem for completely solvable $\mathfrak{s}$ of finding ‘maximal pre-Levi decompositions’.

Open Problem 6.13. *Does every solvable Lie algebra admit a maximal pre-Levi decomposition?*

For solvable Lie algebras with 2-step nilradicals, this has been considered in the forthcoming work [32]. There it is shown that these low step solvable Lie groups do admit metrics of infinitesimal maximal symmetry.

6.4. **Solitons as critical points of a geometric functional.** For compact homogeneous spaces, the Ricci flow is the negative gradient flow of the scalar curvature function and critical points of the scalar curvature function (restricted to the set of G-invariant, volume 1 metrics) are precisely the Einstein metrics. This is not true for non-compact homogeneous spaces, and another function has been proposed for study.

Motivated by GIT and the nilpotent setting (see Section 4.1), one may ask if homogeneous Ricci solitons on solvable groups are the maxima of the function

$$F(g) = \frac{sc(g)^2}{\operatorname{tr}\operatorname{Ric}_g^2}.$$

For unimodular solvmanifolds or those with codimension one nilradical, the answer is yes! (see [74, 75]).

Open Problem 6.14. *Are all Ricci solitons on solvable Lie groups maxima of the function* $F(g) = \frac{sc(g)^2}{\operatorname{tr}\operatorname{Ric}_g^2}$ *?*

For solitons in general, there is a function whose critical points are precisely the solitons—the so-called β-volume-normalized scalar curvature function. Even further, this function serves as a Lyapunov function of the Ricci flow. See [17] for more in this direction.

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