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UPPER ENDPOINT ESTIMATES AND EXTRAPOLATION FOR COMMUTATORS

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ABSTRACT. In this note we revisit the upper endpoint estimates for commutators following the line by Harboure, Segovia, and Torrea [*Illinois J. Math.* **41** no. 4 (1997), 676–700]. Relying upon the suitable BMO subspace suited for the commutator that was introduced in Accomazzo's PhD thesis (2020), we obtain a counterpart for commutators of the upper endpoint extrapolation result by Harboure, Macías and Segovia [*Amer. J. Math.* **110** no. 3 (1988), 383–397]. Multilinear counterparts are provided as well.

1. Introduction and main results

Extrapolation has been a fruitful area of research since the 80s. The first results in that direction were due to Rubio de Francia [14, 13]. We briefly discuss the general principle behind that kind of results in the following lines.

We say that w is a weight if it is a non-negative locally integrable function on $\mathbb{R}^n$. Recall that $w \in A_p$ for $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \fint_Q w \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty$$

and that $w \in A_1$ if

$$[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty} < \infty,$$

where M stands for the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{x\in Q} \fint_Q |f(y)|dy,$$

where each Q is a cube of $\mathbb{R}^n$ with its sides parallel to the axis.

A fundamental property of the A_p classes is that they characterize the weighted L^p boundedness of the Hardy–Littlewood maximal operator and they are good

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weights for a number of operators in the theory such as singular integrals, commutators and some further ones.

The Rubio de Francia extrapolation results say that if T is a sublinear operator such that for some $1 < p_0 < \infty$

$$\|Tf\|_{L^{p_0}(w)} \leq c_{w,T,p_0}\|f\|_{L^{p_0}(w)} \tag{1.1}$$

for every $w \in A_{p_0}$, then

$$\|Tf\|_{L^p(w)} \leq c_{w,T,p}\|f\|_{L^p(w)} \tag{1.2}$$

for every $w \in A_p$ and every $1 < p < \infty$.

This approach has been extensively studied by a number of authors in a wide variety of settings. For instance, in the linear setting there are fundamental works due to Cruz-Uribe, Martell, and Pérez [9, 4, 5, 8, 7, 6], Duoandikoetxea [11, 12], Dragicevic, Grafakos, Petermichl, Pereyra [10], Harboure, Macías, Segovia [18, 17]. After a number of intermediate results in the multilinear setting (see for instance [16, 2, 7]) the question was succesfully solved in the last years, in works such as [26, 25, 28].

A useful development in the area since Rubio de Francia's pioneering works consisted in learning that the operator involved in (1.1) and (1.2) actually plays no role. To be more precise, it can be replaced by a condition on pairs of functions. Assume that $\mathcal{F}$ is a family of pairs of functions such that for some $1 < p_0 < \infty$

$$\|f\|_{L^{p_0}(w)} \leq c_{w,T,p_0}\|g\|_{L^{p_0}(w)}$$

for every $(f,g) \in \mathcal{F}$ and every $w \in A_{p_0}$, then

$$\|f\|_{L^p(w)} \leq c_{w,T,p}\|g\|_{L^p(w)}$$

for every $(f,g) \in \mathcal{F}$, for every $w \in A_p$ and every $1 < p < \infty$.

Another line of research would consist in considering the endpoints, namely $p_0 = \infty$ or $p_0 = 1$ as a "departing" point for extrapolation. For instance, the following result was obtained in [15, 9]

Theorem 1. *Let (f,g) be a pair of functions and suppose that*

$$\|gw\|_{L^\infty} \leq c_w\|fw\|_{L^\infty}$$

holds for all w with $w^{-1} \in A_1$, where c_w depends only on $[w^{-1}]_{A_1}$. Then for all $1 < p < \infty$ and all $w \in A_p$, we have

$$\|g\|_{L^p(w)} \leq \tilde{c}_w\|f\|_{L^p(w)},$$

where $\tilde{c}_w$ depends only on $[w]_{A_p}$.

There are a number of operators that do not map L^∞ into L^∞ such as the Hilbert transform. However for the Hilbert transform H itself and even for a larger class of operators, the Calderón–Zygmund operators, it is possible to show that they map L^∞ into BMO. Weighted versions of that result were studied first in [27]. There it was shown that if $w \in A_1$, then

$$\fint_Q |Hf - (Hf)_Q| \leq C\left\|\frac{f}{w}\right\|_{L^\infty} \operatorname*{ess\,inf}_Q w(x). \tag{1.3}$$

In view of this estimate, it seems natural to think about extending this result to Calderón–Zygmund operators, and also, within the framework of extrapolation whether it would be possible to extrapolate from that weighted $L^\infty \to BMO$ bound in order to obtain weighted L^p estimates. Those questions were answered in the positive in the inspiring paper [18] by Harboure, Macías, and Segovia. In that work the following extrapolation result was settled.

Theorem 2. *Let T be a sublinear operator defined on $\mathcal{C}_0^\infty(\mathbb{R}^n)$ satisfying that for any cube $Q \subset \mathbb{R}^n$ and any $w \in A_1$*

$$\fint_Q |Tf - (Tf)_Q| \leq c_{w,T} \left\| \frac{f}{w} \right\|_{L^\infty} \operatorname*{ess\,inf}_Q w.$$

Then for every $1 < p < \infty$ and every $w \in A_p$ we have that

$$\|Tf\|_{L^p(w)} \leq c_w \|f\|_{L^p(w)}.$$

Quite recently in [3], a quantitative version of this result was obtained. In that paper it was shown that if $\delta \in (0,1)$ and

$$\inf_{c \in \mathbb{R}} \left(\fint_Q |Tf - c|^\delta \right)^{\frac{1}{\delta}} \leq c_{n,\delta,T} \varphi([w]_{A_1}) \left\| \frac{f}{w} \right\|_{L^\infty} \operatorname*{ess\,inf}_Q w,$$

then

$$\|Tf\|_{L^p(w)} \leq c_n \varphi(\|M\|_{L^p(w)}) \|M\|_{L^{p'}(\sigma)} \|f\|_{L^p(w)},$$

where $\sigma = w^{-\frac{1}{p-1}}$. Note that since $\|M\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}$ such an estimate yields

$$\|Tf\|_{L^p(w)} \leq c_n \varphi([w]_{A_p}^{\frac{1}{p-1}}) [w]_{A_{p'}} \|f\|_{L^p(w)}.$$

In the same paper it is shown that for Calderón–Zygmund operators

$$\inf_{c \in \mathbb{R}} \left(\fint_Q |Tf - c|^\delta \right)^{\frac{1}{\delta}} \leq c_{n,\delta,T} [w]_{A_1} \left\| \frac{f}{w} \right\|_{L^\infty} \operatorname*{ess\,inf}_Q w, \tag{1.4}$$

namely $\varphi(t) = t$ and hence the sharp exponent for the A_p constant $\max\left\{1, \frac{1}{p-1}\right\}$ for that class is not recovered. Such a fact is not surprising since the current best known extrapolation argument from the lower endpoint neither recovers the sharp estimate. At this point we would like to note that a way more general version of the aforementioned extrapolation result, replacing $L^p(w)$ spaces by function Banach spaces and the A_p constant by suitable boundedness constants of the maximal function over those spaces, was obtained very recently in [29]. Also a quantitative multilinear result in that direction was provided in [28, Corollary 4.14]

Now we turn our attention to our contribution in this work. We recall that given $b \in BMO$, the Coifman–Rochberg–Weiss commutator is defined as

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

It is well known that $[b,T]$ is bounded on $L^p(w)$ and that, as Pérez showed in [30], $[b,T]$ is not of weak type $(1,1)$ but it satisfies the following estimate instead:

$$w\left(\{|[b,T]f(x)|>t\}\right)\lesssim [w]_{A_1}^2\log(e+[w]_{A_1})\int_{\mathbb{R}^n}\Phi\left(\frac{|f|}{t}\right)w,$$

where $\Phi(t)=t\log(e+t)$. The quantitative dependence was obtained in [23].

In view of (1.3) and (1.4) one may wonder what can be said about commutators. In [19, Theorem A], Harboure, Segovia, and Torrea provided the following result.

Theorem 3. *Let T be a Calderón–Zygmund operator and let $b\in BMO$. Then the following statements are equivalent:*

(1) *For every ball B and every $f\in L_c^\infty(\mathbb{R}^n)$,*

$$\fint_B |[b,T]f(x)-([b,T]f)_B|dx\lesssim \|f\|_{L^\infty}. \tag{1.5}$$

(2) *The function b satisfies the following condition. For any cube Q and $u\in Q$*

$$\Big(\fint_Q |b-b_Q|\Big)T(f\chi_{(2Q)^c})(u)\leq C\|f\|_{L^\infty(\mathbb{R}^n)}$$

for every $f\in L_c^\infty(\mathbb{R}^n)$.

Also in [19] the authors point out that if T is the Hilbert transform and any of the conditions in the preceding Theorem are satisfied, then necessarily b is constant and hence $[b,H]=0$. This fact leads us to think about the possibility of considering a "smaller" oscillation in the left hand side of (1.5).

Aiming for a dual of the Hardy spaces for commutators studied by Pérez [30] and Ky [21], Accomazzo introduced in [1] the spaces BMO_b^q which are defined as follows. Given a function b, and $q\in[1,\infty)$ we have that $f\in BMO_b^q$ if

$$\|f\|_{BMO_b^q}:=\sup_B\left(\inf_{c_0,c_1\in\mathbb{R}}\fint_B|f(x)-c_0+c_1b(x)|^q\right)^{\frac{1}{q}}.$$

Note that $\|f\|_{BMO_b^q}=0$ if and only if $f=\alpha+\beta b$ and hence in order to consider $\|f\|_{BMO_b^q}$ as a norm one needs to take quotient by the subspace $\langle 1,b\rangle$ (the space of linear combinations of 1 and b). It readily follows from the definition that $BMO\subset BMO_b^q$ for every q. It is also easy to show that if $b\in BMO$ then $b^2\in BMO_b^q$. And for instance choosing $b(x)=\log(x)$, we have that $\log(x)^2\in BMO_b^q\setminus BMO$.

Inspired by the definition of BMO_b^q we provide the following result for commutators.

Theorem 4. *Let $b\in BMO$ and T a Calderón–Zygmund operator satisfying a* log-*Dini regularity condition. Then for every ball B, if $\delta\in(0,1)$ and $r>1$ we have that*

$$\inf_{c_1\in\mathbb{R}}\left(\fint_B|[b,T]f(x)-c_1-T(f\chi_{(2B)^c})(c_B)b(x)|^\delta dx\right)^{\frac{1}{\delta}} \lesssim r'\left\|\frac{f}{w}\right\|_{L^\infty}\|b\|_{BMO}\inf_{z\in B}M_rw(z).$$

Consequently,

$$\inf_{c_1,c_2\in\mathbb{R}}\left(\fint_B |[b,T]f(x)-c_1-c_2b(x)|^\delta dx\right)^{\frac{1}{\delta}} \lesssim r'\left\|\frac{f}{w}\right\|_{L^\infty}\|b\|_{BMO}\inf_{z\in B}M_r w(z).$$

The next natural question would be whether it is possible to extrapolate from the condition above. We show that that is the case under some additional conditions.

Theorem 5. *Let T be a linear operator such that for every $b\in BMO$ and every $w\in A_1$*

$$\inf_{c_1\in\mathbb{R}}\left(\fint_B |[b,T]f(x)-c_1-T(f\chi_{(2B)^c})(c_B)b(x)|^\delta dx\right)^{\frac{1}{\delta}} \le c_{w,T}\left\|\frac{f}{w}\right\|_{L^\infty}\|b\|_{BMO}\inf_{z\in B}w(z)$$

and such that Lerner's grand maximal operator

$$\mathcal{M}_T f(x)=\sup_{x\in B}\operatorname*{ess\,sup}_{z\in B} T(f\chi_{(2B)^c})(z)$$

is bounded on $L^p(v)$ for some $p\in(1,\infty)$ and some $v\in A_p$. Then

$$\|[b,T]f\|_{L^p(v)}\le c_{v,T}\|b\|_{BMO}\|f\|_{L^p(v)}.$$

Observe that the operator $\mathcal{M}_T$ was introduced in [22] in order to study sparse domination. There it was shown that in the case of T being a Calderón–Zygmund operator

$$\mathcal{M}_T f(x)\lesssim Mf(x)+T^*f(x),$$

where T^* stands for the maximal Calderón–Zygmund operator. Since both M and T^* are bounded on $L^p(w)$ for $w\in A_p$, the result above combined with the estimate in Theorem 1.4 allows to provide an alternative proof of the weighted L^p boundedness of the commutator $[b,T]$.

Here we just presented the results in the linear setting. However results in the multilinear setting are feasible as well and will be obtained in Section 4.

The remainder of the paper is organized as follows. In Section 2 we gather some preliminaries.

In Section 3 we settle Theorems 4 and 5. Finally in Section 4 we present and settle the multilinear counterparts of the main results.

2. Preliminaries

We recall that T is a Calderón–Zygmund operator if T is a linear operator that is bounded on L^2 and it admits a representation in terms of a kernel K

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy \qquad x\notin \operatorname{supp} f,$$

where K satisfies the following properties:

- Size condition: $|K(x,y)| \leq C_K|x-y|^{-n}$.
- Smoothness condition: Provided that $|x-y| \geq 2|x-z|$,
$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)| \leq \omega\left(\frac{|x-z|}{|x-y|}\right)\frac{1}{|x-y|^n},$$
where ω is a continuous subadditive function such that
$$\int_0^1 \omega(t)\log\left(\frac{1}{t}\right)\frac{dt}{t} < \infty.$$

In the definition of commutators we used BMO functions. We recall that $b \in BMO$ if
$$\|b\|_{BMO} = \sup_B \fint_B |b-b_B| < \infty.$$
A fundamental property of this space of functions is the well-known John–Nirenberg that says that the integrability of the oscillations self-improves to exponential integrability, namely, there exist constants $\lambda, c > 0$ such that for every ball B and every BMO function
$$|\{x \in B : |b-b_B| > \lambda\}| \lesssim e^{-c\lambda/\|b\|_{BMO}}|B|.$$
Note that this in turn implies that
$$\fint_B |b-b_B|^\alpha \lesssim \max\{\alpha,1\}\|b\|_{BMO} \tag{2.1}$$
for every $\alpha > 0$. Another fact that we will use in what follows is that if B is a ball then
$$|b_{2^jB}-b_B| \lesssim j\|b\|_{BMO}. \tag{2.2}$$
We remit the interested reader to [20] for more details on BMO.

Quite related to the definition of BMO is that of the sharp maximal function. Given $\delta > 0$, we define
$$M_{\sharp,\delta}(f)(x) = \sup_{x\in B}\inf_{c\in\mathbb{R}}\left(\fint_B |f-c|^\delta\right)^{\frac{1}{\delta}}.$$

We would like to end this preliminaries section by gathering some basic facts about multilinear theory. We recall that a linear operator T is an m-linear Calderón–Zygmund operator if $T : L^{p_1}\times\cdots\times L^{p_m} \to L^p$ for some $1 < p_1,\ldots,p_m < \infty$ with $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and it admits the following representation:
$$T(\vec{f})(x) = \int_{\mathbb{R}^{nm}} K(x,y_1,\ldots,y_m)f(y_1)\ldots f(y_m)dy_1\ldots dy_m,$$
where $x \notin \operatorname{supp}(f_i)$ for any $i \in \{1,\ldots,m\}$, in terms of a kernel K that satisfies the following properties:

- Size condition: $|K(x,\vec{y})| \le C_K(\sum_{i=1}^m |x-y_i|)^{-mn}$.
- Smoothness condition: Given ω a continuous subadditive function such that $\int_0^1 \omega(t)\log\left(\frac{1}{t}\right)\frac{dt}{t} < \infty$, the following conditions hold

$$|K(x,\vec{y}) - K(z,\vec{y})| \le \omega\left(\frac{|x-z|}{\max_{i\in\{1,\dots,m\}}|x-y_i|}\right)\frac{1}{\left(\sum_{i=1}^m |x-y_i|\right)^{mn}}$$

provided that $\max_{i\in\{1,\dots,m\}}|x-y_i| \ge 2|x-z|$, and also, for any $j \in \{1,\dots,m\}$

$$\begin{aligned}|K(x,y_1,\dots,y_j,\dots,y_m) &- K(x,y_1,\dots,y_j',\dots,y_m)| \\ &\le \omega\left(\frac{|y_j-y_j'|}{\max_{i\in\{1,\dots,m\}}|x-y_i|}\right)\frac{1}{\left(\sum_{i=1}^m |x-y_i|\right)^{mn}},\end{aligned}$$

where $\max_{i\in\{1,\dots,m\}}|x-y_i| \ge 2|y_j-y_j'|$.

Note that in this context the commutator $[b,T]_j\vec{f}(x)$ is defined as

$$[b,T]_j\vec{f}(x) = b(x)T(\vec{f})(x) - T(f_1,\dots,f_jb,\dots,f_m).$$

Note that the definition is essentially equivalent whichever index we commute in. Hence throughout the remainder of this work we will consider just the case $[b,T]_1$.

Let us also recall that we say $\vec{w} = (w_1,\dots,w_m) \in A_{\vec{p}}$, if

$$\sup_Q\left(\fint_Q w^p\right)^{\frac{1}{p}}\prod_{i=1}^m\left(\fint_Q w_i^{-p_i'}\right)^{\frac{1}{p_i'}} < \infty, \qquad w := \prod_{i=1}^m w_i,$$

where $\vec{p} = (p_1,\dots,p_m)$ with $1 \le p_i \le \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. A consequence of the multilinear extrapolation result that appeared first in [28, Theorem 4.12] (see as well [25]) states the following.

Theorem 6. *Let $(f,f_1,\dots,f_m)$ be an $(m+1)$-tuple of functions. Suppose that*

$$\|fw\|_{L^\infty} \le c_{\vec{w}}\prod_{i=1}^m \|f_iw_i\|_{L^\infty}$$

holds for all $\vec{w}$ with $\vec{w} \in A_{(\infty,\dots,\infty)}$, where $c_{\vec{w}}$ depends only on $[\vec{w}]_{A_{(\infty,\dots,\infty)}}$. Then for all $\vec{p}$ with $p_i > 1$, $i = 1,\dots,m$, and all $\vec{w} \in A_{\vec{p}}$, we have

$$\|fw\|_{L^p} \le c_{\vec{w}}\prod_{i=1}^m \|f_iw_i\|_{L^{p_i}},$$

where $\tilde{c}_{\vec{w}}$ depends only on $[\vec{w}]_{A_{\vec{p}}}$.

3. Proofs of the main results

3.1. **Proof of Theorem 4.** Let B be a ball and c_2, λ constants to be chosen. Let

$$c_1 = -\lambda Tf_2(c_B) - T((b-\lambda)f_2)(c_B),$$

where $f_2 = f\chi_{\mathbb{R}^n\setminus 2B}$. Then we begin arguing as follows:

$$
\begin{aligned}
&\Big(\fint_B |[b,T]f(x)-c_1-c_2b(x)|^\delta dx\Big)^{\frac{1}{\delta}}\\
&=\left(\fint_B |[b-\lambda,T]f(x)-c_1-c_2b(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\lesssim \left(\fint_B |(b(x)-\lambda)Tf(x)+\lambda Tf_2(c_B)-c_2b(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\fint_B |T((b-\lambda)f)(x)-T((b-\lambda)f_2)(c_B)|^\delta dx\right)^{\frac{1}{\delta}}\\
&=: L_1+L_2.
\end{aligned}
$$

Note that for L_1, choosing $\lambda=b_{2B}$ we have that for $\delta<\varepsilon<1$, calling $f_1=f\chi_{2B}$,

$$
\begin{aligned}
L_1&=\left(\fint_B |(b(x)-b_{2B})Tf(x)+b_{2B}Tf_2(c_B)-c_2b(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\lesssim \left(\fint_B |(b(x)-b_{2B})Tf_1(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\fint_B |(b(x)-b_{2B})Tf_2(x)+b_{2B}Tf_2(c_B)-c_2b(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&=: L_{11}+L_{12}.
\end{aligned}
$$

First we focus on L_{11}. We argue as follows:

$$
\begin{aligned}
L_{11}&=\left(\fint_B |(b(x)-b_{2B})Tf_1(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\le \left(\fint_B |b(x)-b_{2B}|^{\delta(\frac{\varepsilon}{\delta})'} dx\right)^{\frac{1}{\delta(\frac{\varepsilon}{\delta})'}}\left(\fint_B |Tf_1|^\varepsilon dx\right)^{\frac{1}{\varepsilon}}\\
&\lesssim \|b\|_{BMO}\fint_{2B}|f|=\|b\|_{BMO}\fint_{2B}\frac{|f|}{w}w\\
&\le \|b\|_{BMO}\left\|\frac{f}{w}\right\|_{L^\infty}\inf_{z\in B} Mw.
\end{aligned}
$$

Now we turn to L_{12}. Choosing $c_2=Tf_2(c_B)$ we have that

$$
\begin{aligned}
L_{12}&=\left(\fint_B |(b(x)-b_{2B})Tf_2(x)-Tf_2(c_B)b(x)+b_{2B}Tf_2(c_B)|^\delta dx\right)^{\frac{1}{\delta}}\\
&=\left(\fint_B |(b(x)-b_{2B})Tf_2(x)-(b(x)-b_{2B})Tf_2(c_B)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\le \left(\fint_B |b(x)-b_{2B}||Tf_2(x)-Tf_2(c_B)|dx\right).
\end{aligned}
$$

From this point taking into account the smoothness condition of the kernel we may argue as follows:

$$
\begin{aligned}
&\left(\fint_B |b(x)-b_{2B}||Tf_2(x)-Tf_2(c_B)|dx\right)\\
&\quad\le \fint_B |b(x)-b_{2B}| \int_{\mathbb{R}^n\setminus 2B} |K(x,y)-K(c_B,y)||f(y)|dydx\\
&\quad\le \fint_B |b(x)-b_{2B}| \int_{\mathbb{R}^n\setminus 2B} \frac{1}{|x-y|^n}\omega\left(\frac{|x-c_B|}{|x-y|}\right)|f(y)|dydx\\
&\quad\le \fint_B |b(x)-b_{2B}| \sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB} \frac{1}{|x-y|^n}\omega\left(\frac{|x-c_B|}{|x-y|}\right)|f(y)|dydx\\
&\quad\le \fint_B |b(x)-b_{2B}|dx \sum_{j=1}^{\infty}\frac{1}{2^{jn}l(B)^n}\omega\left(\frac{l(B)}{2^jl(B)}\right)\int_{2^{j+1}B\setminus 2^jB}|f(y)|dy\\
&\quad\lesssim \|b\|_{BMO}\sum_{j=1}^{\infty}\omega(2^{-j})\fint_{2^{j+1}B}|f(y)|dy\\
&\quad\lesssim \|b\|_{BMO}\left\|\frac{f}{w}\right\|_{L^\infty}\inf_{z\in B} Mw.
\end{aligned}
$$

We continue bounding L_2. Note that

$$
\begin{aligned}
L_2 &\lesssim \left(\fint_B |T((b-b_{2B})f_1)(x)|^\delta dx\right)^{\frac{1}{\delta}}\\
&\qquad+\left(\fint_B |T((b-b_{2B})f_2)(x)-T((b-b_{2B})f_2)(c_B)|^\delta dx\right)^{\frac{1}{\delta}}\\
&= L_{21}+L_{22}.
\end{aligned}
$$

For L_{21} by Kolmogorov's inequality,

$$
\begin{aligned}
L_{21} &= \left(\fint_B |T((b-b_{2B})f_1)(x)|^\delta dx\right)^{\frac{1}{\delta}} \lesssim \fint_{2B}|b-b_{2B}||f|dx\\
&\le \left\|\frac{f}{w}\right\|_{L^\infty}\fint_{2B}|b-b_{2B}|wdx \lesssim r'\left\|\frac{f}{w}\right\|_{L^\infty}\|b\|_{BMO}\inf_{z\in B} M_rw(z),
\end{aligned}
$$

where in the last step we have used Hölder's inequality and (2.1). For L_{22}, we have that, using the smoothness condition of the kernel,

$$
\begin{aligned}
L_{22} &\le \left(\fint_B |T((b-b_{2B})f_2)(x)-T((b-b_{2B})f_2)(c_B)|dx\right)\\
&\le \fint_B\int_{\mathbb{R}^n\setminus 2B}|K(x,y)-K(c_B,y)||b(y)-b_{2B}||f(y)|dydx\\
&\le \fint_B\int_{\mathbb{R}^n\setminus 2B}\omega\left(\frac{|x-c_B|}{|x-y|}\right)\frac{1}{|x-y|^n}|b(y)-b_{2B}||f(y)|dydx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{\infty} \omega(2^{-j}) \frac{1}{(2^j l(B))^n} \int_{2^{j+1}B \setminus 2^j B} |b(y) - b_{2B}||f(y)| dy \\
&\lesssim \left\| \frac{f}{w} \right\|_{L^\infty} \sum_{j=1}^{\infty} \omega(2^{-j}) \fint_{2^{j+1}B} |b(y) - b_{2B}| w(y) dy \\
&\leq \left\| \frac{f}{w} \right\|_{L^\infty} \sum_{j=1}^{\infty} \omega(2^{-j}) \fint_{2^{j+1}B} |b(y) - b_{2^{j+1}B}| w(y) dy \\
&\qquad + \left\| \frac{f}{w} \right\|_{L^\infty} \sum_{j=1}^{\infty} \omega(2^{-j}) |b_{2^{j+1}B} - b_B| \fint_{2^{j+1}B} w(y) dy \\
&\lesssim r' \left\| \frac{f}{w} \right\|_{L^\infty} \|b\|_{BMO} \operatorname*{ess\,inf}_{z \in B} M_r w(z),
\end{aligned}
$$

where in the last step we have used Hölder's inequality, (2.1) and (2.2). This ends the proof.

3.2. **Proof of Theorem 5.** Let us fix a ball B and $x \in B$. Following the same notation as that in the proof of Theorem 4, if we choose $c = c_1 + Tf_2(c_B) b_{2B}$,

$$
\begin{aligned}
&\left(\fint_B |[b,T]f(y) - c|^\delta dy \right)^{\frac{1}{\delta}} \\
&= \left(\fint_B |[b,T]f(y) - c_1 - Tf_2(c_B) b(y) - Tf_2(c_B) b_{2B} + Tf_2(c_B) b(y)|^\delta dy \right)^{\frac{1}{\delta}} \\
&\lesssim \left(\fint_B |[b,T]f(y) - c_1 - Tf_2(c_B) b(y)|^\delta dy \right)^{\frac{1}{\delta}} + |Tf_2(c_B)| \left(\fint_B |b(y) - b_{2B}|^\delta dy \right)^{\frac{1}{\delta}} \\
&\lesssim \left(\fint_B |[b,T]f(y) - c_1 - Tf_2(c_B) b(y)|^\delta dy \right)^{\frac{1}{\delta}} + \mathcal{M}_T f(x) \|b\|_{BMO}.
\end{aligned}
$$

Note that this yields

$$
\begin{aligned}
M_{\sharp,\delta}([b,T]f)(x) \lesssim \sup_{x \in B} \inf_{c_1 \in \mathbb{R}} & \left(\fint_B |[b,T]f(y) - c_1 - Tf_2(c_B) b(y)|^\delta dy \right)^{\frac{1}{\delta}} \\
& + \mathcal{M}_T f(x) \|b\|_{BMO}.
\end{aligned}
$$

Observe that if we call

$$
g(x) := \sup_{x \in B} \inf_{c_1 \in \mathbb{R}} \left(\fint_B |[b,T]f(y) - c_1 - Tf_2(c_B) b(y)|^\delta dy \right)^{\frac{1}{\delta}},
$$

by hypothesis we have that

$$
\|gw\|_{L^\infty} \leq c_w \|b\|_{BMO} \|fw\|_{L^\infty}
$$

and hence by Theorem 1 we have that for all $1 < q < \infty$ and every $w \in A_q$

$$
\|g\|_{L^q(w)} \leq \tilde{c}_w \|b\|_{BMO} \|f\|_{L^q(w)}.
$$

Since by hypothesis as well we know that

$$
\|\mathcal{M}_T f\|_{L^p(v)} \leq \tilde{c}_v \|f\|_{L^p(v)}.
$$

We have that combining the estimates above,

$$\|M_{\sharp,\delta}([b,T]f)\|_{L^p(v)} \leq \tilde{c}_v\|b\|_{BMO}\|f\|_{L^p(v)}.$$

Then the desired estimate

$$\|[b,T]f\|_{L^p(v)} \leq \tilde{c}_v\|b\|_{BMO}\|f\|_{L^p(v)}$$

follows from the Fefferman–Stein's inequality.

4. Multilinear counterparts

In this section we present multilinear versions of the results presented above. We begin providing a counterpart of Theorem 4.

Theorem 7. *Let $b \in BMO$ and T be an m-linear CZO. Then for every ball B, if $\delta \in (0,1/m)$ then*

$$\begin{aligned}&\inf_{c_1\in\mathbb{R}}\left(\fint_B |[b,T]_1(\vec{f})(y) - c_1 - \left(T(\vec{f}) - T(\vec{f}\chi_{2B})\right)b(y)|^\delta dy\right)^{\frac{1}{\delta}}\\ &\lesssim c_w\|b\|_{BMO}\prod_{i=1}^{\infty}\|f_iw_i\|_{L^\infty}\operatorname*{ess\,inf}_{x\in B}\frac{1}{w(x)},\end{aligned}$$

where $(w_1,\dots,w_m) \in A_{(\infty,\dots,\infty)}$ and $w = \prod_{i=1}^m w_i$. Consequently, the following inequality holds as well

$$\inf_{c_1,c_2\in\mathbb{R}}\left(\fint_B |[b,T]_1(\vec{f})(y) - c_1 - c_2b(y)|^\delta dy\right)^{\frac{1}{\delta}} \lesssim c_w\|b\|_{BMO}\prod_{i=1}^{\infty}\|f_iw_i\|_{L^\infty}\operatorname*{ess\,inf}_{x\in B}\frac{1}{w(x)}.$$

Proof of Theorem 7. For notational convenience we may denote

$$T_B(\vec{f}) = T(\vec{f}) - T(\vec{f}\chi_{2B}).$$

Let

$$c_1(B) = -b_{2B}T_B(\vec{f})(c_B) - T_B\big((b-b_{2B})f_1, f_2,\dots,f_m\big)(c_B)$$

and

$$c_2(B) = T_B(\vec{f})(c_B).$$

Having that notation in mind we have that

$$\begin{aligned}&\left(\fint_B |[b,T]_1(\vec{f})(y) - c_1(B) - c_2(B)b(y)|^\delta dy\right)^{\frac{1}{\delta}}\\ &\quad= \left(\fint_B |[b-b_{2B},T]_1(\vec{f})(y) - c_1(B) - c_2(B)b(y)|^\delta dy\right)^{\frac{1}{\delta}}\\ &\quad\lesssim \left(\fint_B |(b-b_{2B})T(\vec{f})(y) - (b-b_{2B})T_B(\vec{f})(c_B)|^\delta dy\right)^{\frac{1}{\delta}}\\ &\qquad\quad+ \left(\fint_B |T((b-b_{2B})f_1,\dots,f_m)(y) - T_B((b-b_{2B})f_1,\dots,f_m)(c_B)|^\delta dy\right)^{\frac{1}{\delta}}\\ &\quad=: I_1 + I_2.\end{aligned}$$

For I_1, we have

$$\begin{aligned} I_1 &\lesssim \Big(\fint_B |b-b_{2B}|^\delta \cdot |T_B(\vec{f})(y) - T_B(\vec{f})(c_B)|^\delta dy\Big)^{\frac{1}{\delta}} \\ &\quad + \Big(\fint_B |b-b_{2B}|^\delta \cdot |T(\vec{f}\chi_{2B})(y)|^\delta dy\Big)^{\frac{1}{\delta}} \\ &\lesssim \|b\|_{BMO}\Big(\sup_{y\in B} |T_B(\vec{f})(y) - T_B(\vec{f})(c_B)| + \|T(\vec{f}\chi_{2B})\|_{L^{\frac{1}{m},\infty}(B,\frac{dx}{|B|})}\Big) \\ &\lesssim \|b\|_{BMO}\Big(\sum_{k=1}^\infty \omega(2^{-k}) \prod_{i=1}^m \fint_{2^kB} |f_i| + \prod_{i=1}^m \fint_{2B}|f_i|\Big) \\ &\le \|b\|_{BMO}[\vec{w}]_{A_{(\infty,\dots,\infty)}} \Big(\prod_{i=1}^\infty \|f_i w_i\|_{L^\infty}\Big) \operatorname*{ess\,inf}_{x\in B} \frac{1}{w(x)}, \end{aligned}$$

where we have used the weak endpoint estimate of T. Now we turn to estimate I_2. We argue as follows:

$$\begin{aligned} I_2 &\lesssim \Big(\fint_B |T_B((b-b_{2B})f_1,\dots,f_m)(y) - T_B((b-b_{2B})f_1,\dots,f_m)(c_B)|^\delta dy\Big)^{\frac{1}{\delta}} \\ &\quad + \Big(\fint_B |T((b-b_{2B})f_1\chi_{2B},\dots,f_m\chi_{2B})(y)|^\delta dy\Big)^{\frac{1}{\delta}} \\ &=: I_{21} + I_{22}. \end{aligned}$$

The estimate of I_{22} can be handled similarly as before, that is, we use Kolmogorov's inequality and then the weak type endpoint estimate

$$\begin{aligned} I_{22} &\lesssim \Big(\fint_{2B} |b-b_{2B}||f_1|\Big) \prod_{i=2}^m \fint_{2B} |f_i| \\ &\le [\vec{w}]_{A_{(\infty,\dots,\infty)}} \Big(\prod_{i=1}^m \|f_i w_i\|_{L^\infty}\Big) \operatorname*{ess\,inf}_{x\in B} \frac{1}{w(x)} \cdot \frac{1}{w_1^{-1}(2B)} \int_{2B} |b-b_{2B}| w_1^{-1} \\ &\lesssim \|b\|_{BMO}[\vec{w}]_{A_{(\infty,\dots,\infty)}} \Big(\prod_{i=1}^m \|f_i w_i\|_{L^\infty}\Big) \operatorname*{ess\,inf}_{x\in B} \frac{1}{w(x)}, \end{aligned}$$

where the last inequality holds since $w_1^{-1} \in A_\infty$.

It remains to consider I_{21}. Similarly as before, we have

$$\begin{aligned} I_{21} &\lesssim \sum_{k=1}^\infty \omega(2^{-k}) \fint_{2^kB} |b-b_{2B}||f_1| \prod_{i=2}^m \fint_{2^kB} |f_i| \\ &\le \sum_{k=1}^\infty \omega(2^{-k}) \fint_{2^kB} |b-b_{2^kB}||f_1| \prod_{i=2}^m \fint_{2^kB} |f_i| \\ &\quad + \sum_{k=1}^\infty \omega(2^{-k}) |b_{2B} - b_{2^kB}| \prod_{i=1}^m \fint_{2^kB} |f_i| \end{aligned}$$

$$\lesssim \|b\|_{BMO}[\vec{w}]_{A_{(\infty,\dots,\infty)}} \Big(\prod_{i=1}^{m} \|f_i w_i\|_{L^\infty} \Big) \operatorname*{ess\,inf}_{x\in B} \frac{1}{w(x)},$$

where we have used that

$$|b_{2B} - b_{2^k B}| \lesssim k\|b\|_{BMO}.$$

This requires that the kernel satisfies the log-Dini condition. This completes the proof. □

Having the Theorem above at our disposal we can obtain the following result.

Theorem 8. *Let T be an m-linear operator such that for every $b \in BMO$ and every $\vec{w} \in A_{(\infty,\dots,\infty)}$,*

$$\inf_{c_1\in\mathbb{R}} \Big(\fint_B |[b,T]_1(\vec{f})(y) - c_1 - \Big(T(\vec{f}) - T(\vec{f}\chi_{2B})\Big)(y) b(y)|^\delta dy \Big)^{\frac{1}{\delta}}$$
$$\lesssim c_w \|b\|_{BMO} \prod_{i=1}^{\infty} \|f_i w_i\|_{L^\infty} \operatorname*{ess\,inf}_{x\in B} \frac{1}{w(x)}$$

and such that Lerner's grand maximal operator

$$\mathcal{M}_T(\vec{f})(x) = \sup_{x\in B} \operatorname*{ess\,sup}_{z\in B} T(\vec{f}\chi_{(2B)^c})(z)$$

satisfies

$$\|\mathcal{M}_T(\vec{f})v\|_{L^p} \le c_{\vec{v}} \prod_{i=1}^{m} \|f_i v_i\|_{L^{p_i}}$$

for some $\vec{p}$ with $p_i > 1$, $i = 1,\dots,m$, and some $\vec{v} \in A_{\vec{p}}$. Then

$$\|[b,T]_1(\vec{f})v\|_{L^p} \le c_{\vec{v},T} \|b\|_{BMO} \prod_{i=1}^{m} \|f_i v_i\|_{L^{p_i}}.$$

Proof. The argument is analogous to the one given for the linear case. Let us define

$$g(x) = \sup_{B\ni x} \inf_{c_1\in\mathbb{R}} \Big(\fint_B |[b,T]_1(\vec{f})(y) - c_1 - \Big(T(\vec{f}) - T(\vec{f}\chi_{2B})\Big)(y) b(y)|^\delta dy \Big)^{\frac{1}{\delta}};$$

then, arguing as in the linear case, we have that

$$\begin{aligned} M_{\sharp,\delta}([b,T]_1(\vec{f}))(x) &\le \sup_{B\ni x} \Big(\fint_B |[b,T]_1(\vec{f})(y) - c(B)|^\delta dy \Big)^{\frac{1}{\delta}} \\ &\lesssim g(x) + \sup_{B\ni x} \Big(\fint_B |b(y) - b_{2B}|^\delta dy \Big)^{\frac{1}{\delta}} |T_B(\vec{f})(c_B)| \\ &\lesssim g(x) + \|b\|_{BMO} \mathcal{M}_T(\vec{f})(x). \end{aligned}$$

To deal with g we argue as in the linear setting. $\mathcal{M}_T$ is bounded by hypothesis. Hence we are done. □

Note that in the case of T being a Calderón–Zygmund operator, in the bilinear case, a careful calculus to bound $\mathcal{M}_T$ was presented in [24]. Such an estimate, that we recall in the following line, can be extended to the multilinear case directly. Therefore, we have

$$\mathcal{M}_T(\vec{f})(x) \lesssim M(\vec{f})(x) + M_s(T(\vec{f}))(x)$$

for every $0 < s < \frac{1}{m}$. Of course, since M_s is increasing with s, the inequality holds for all $s > 0$. In particular, we can choose $s = \frac{1}{m}$, which in turn allows us to show that

$$\|M_{1/m}(T(\vec{f}))w\|_{L^p} = \|M(T(\vec{f})^{1/m})\|_{L^{mp}(w^p)}^m \lesssim \|T(\vec{f})w\|_{L^p} \lesssim \prod_{i=1}^m \|f_i w_i\|_{L^p},$$

where we have used the fact that if $(w_1, \dots, w_m) \in A_{\vec{p}}$ then $w^p \in A_{mp}$ (when $p < \infty$). This ends the argument and completes an alternative proof of the boundedness of $[b, T]_i$ in the multilinear setting.

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