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HERMITE BESOV AND TRIEBEL–LIZORKIN SPACES AND APPLICATIONS

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In memory of Eleonor Harboure

Abstract. We present an overview of Besov and Triebel–Lizorkin spaces in the Hermite setting and applications on boundedness properties of Hermite pseudo-multipliers and fractional Leibniz rules in such spaces. We also give a new weighted estimate for Hermite multipliers for weights related to Hermite operators.

1. Introduction

In this survey, we give an overview of some recent work on function spaces and pseudo-differential type operators in the context of Hermite expansions. These expansions belong to the family of classical orthogonal expansions that include Laguerre, Jacobi and Chebyshev to name a few, and have been well studied as far back as the 18th century.

In one dimension, the Hermite function of degree $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is given by

$$h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2} \qquad \forall t \in \mathbb{R},$$

where $H_k(t) = (-1)^k e^{t^2} \partial_t^k (e^{-t^2})$ is the kth Hermite polynomial. In higher dimensions the Hermite functions h_ξ are defined over the multi-indices $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{N}_0^n$ as

$$h_\xi(x) = \prod_{j=1}^{n} h_{\xi_j}(x_j) \qquad \forall x \in \mathbb{R}^n.$$

These functions form an orthonormal basis for $L^2(\mathbb{R}^n)$ and also arise naturally as eigenfunctions of the harmonic oscillator $\mathcal{L} = -\Delta + |x|^2$ in the sense that

$$\mathcal{L}(h_\xi) = (2|\xi| + n) h_\xi,$$

where, for a multi-index $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{N}_0^n$, $|\xi| = \xi_1 + \cdots + \xi_n$. As such the Hermite functions have an important connection with mathematical physics.

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Partly for this reason, and partly due to their remarkable properties, they continue to possess an enduring role in various fields of mathematics.

The interest in Hermite expansions from the harmonic analysis viewpoint may perhaps be traced back to Stein and Muckenhoupt in the 1960s [34, 35, 36], who began exploring analogues of Fourier analytic results for orthogonal expansions including topics spanning conjugate functions, Hardy spaces, Littlewood–Paley theory, and multiplier theory, amongst other things. In the ensuing decades, the development of harmonic analysis for Hermite functions experienced several phases of activity and innovation, and Thangavelu's 1993 volume [43] marks a kind of capstone and state of the art for that period.

Since then there has been further progress and we shall now mention two particular themes of development that underpin the work in this survey, both of which have their roots in the work of Epperson from the mid 90s [17, 18, 19].

The first concerns the extension of the classical theory of function spaces in the spirit of Frazier–Jawerth [20, 21] to the Hermite context. As is well known, the φ-transform of Frazier–Jawerth provides a powerful way to represent functions or distributions (a so-called 'frame decomposition') via translates and dilations of a fixed Schwartz function φ, leading to a host of useful consequences and applications. In [17, 19] Epperson introduced the notion of a Triebel–Lizorkin space for Hermite expansions in dimension one and provided a frame decomposition. This was extended and generalised to higher dimensions and to the Besov scale a decade later by Petrushev and Xu [37].

The second theme concerns mapping properties of spectrally defined operators in the Hermite setting. For a bounded and measurable function $\sigma : \mathbb{N}_0^n \to \mathbb{C}$ one can define the Hermite multiplier on $L^2(\mathbb{R}^n)$ by

$$T_\sigma f = \sum_{\xi \in \mathbb{N}_0^n} \sigma(\xi) \langle f, h_\xi \rangle h_\xi .$$

These operators have been well studied (see [43]) and, in honor of Eleonor Harboure, we wish to especially highlight her work [22, 25] on the weighted L^p boundedness of T_σ. In [18], Epperson extended multiplier results from L^p to the Hermite–Triebel–Lizorkin spaces, and in the same work, introduced the notion of a new Hermite 'pseudo-multiplier', which is defined like a multiplier, but whose 'symbol' σ is allowed to also depend on the spatial variable. In this way these operators can be considered Hermite analogues of the classical pseudodifferential operators. Two decades later these objects were reinvestigated in [2], and has since sparked several new results, some of which will be described below.

These twin themes provide fertile ground for further exploration and, in the remainder of this survey, we describe some results and contributions to this growing area of investigation. We shall give a tour of some of the main features and results, omitting the proofs but providing references where appropriate. We will also, in the final section, give a new result that draws together several lines of research that Harboure was engaged with.

This article is organized as follows. In Section 2, we define Hermite Besov and Hermite Triebel–Lizorkin spaces and outline some respective decompositions including frame and new molecular decompositions. In Section 3, we describe some applications, including mapping properties of operators such as the multipliers, pseudo-multipliers and their bilinear counterparts, and other related consequences. Finally, in Section 4, we conclude our survey by giving a new result on Hermite multipliers and weights.

2. Hermite function spaces and their decompositions

In this section, we define Hermite Besov and Hermite Triebel–Lizorkin spaces and describe two decompositions, one through frames and one through smooth molecules. The full details and background can be found in [14, 30, 37] (see also [9, 12, 13]).

The definition of the Hermite Besov and Hermite Triebel–Lizorkin spaces that we employ utilises a Littlewood–Paley type construction. We say that φ is an *admissible function* if $\varphi \in C^\infty(\mathbb{R}_+)$ and

$$\operatorname{supp} \varphi \subset [\tfrac{1}{4}, 1], \quad |\varphi| > c > 0 \quad \text{on} \quad [2^{-7/4}, 2^{-1/4}] \tag{2.1}$$

for some $c > 0$. Given an admissible function φ, we set $\varphi_j(\lambda) = \varphi(2^{-j}\lambda)$ if $j \in \mathbb{N}_0$ and call the resulting collection $\{\varphi_j\}_{j\in\mathbb{N}_0}$ an *admissible system.* Since the Hermite functions h_ξ with $\xi \in \mathbb{N}_0^n$ are members of $\mathscr{S}(\mathbb{R}^n)$, then given any admissible system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ we may define the operators $\varphi_j(\sqrt{\mathcal{L}})$ on $\mathscr{S}'(\mathbb{R}^n)$ by

$$\varphi_j(\sqrt{\mathcal{L}})f(x) = \sum_{\xi\in\mathbb{N}_0^n} \varphi_j(\sqrt{2|\xi| + n})\, \langle f, h_\xi\rangle h_\xi(x) \qquad \forall f \in \mathscr{S}'(\mathbb{R}^n), x \in \mathbb{R}^n, \tag{2.2}$$

where $\langle f, \phi\rangle = f(\phi)$ for $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Let $\alpha \in \mathbb{R}$ and $0 < q \le \infty$. We say that a tempered distribution f belongs to the *Hermite Besov space* $B^{p,q}_\alpha = B^{p,q}_\alpha(\mathcal{L})$ for $0 < p \le \infty$ if

$$\|f\|_{B^{p,q}_\alpha} = \Big(\sum_{j\in\mathbb{N}_0} \big(2^{j\alpha}\|\varphi_j(\sqrt{\mathcal{L}})f\|_{L^p}\big)^q\Big)^{1/q} < \infty;$$

and to the *Hermite Triebel–Lizorkin space* $F^{p,q}_\alpha = F^{p,q}_\alpha(\mathcal{L})$ for $0 < p < \infty$ if

$$\|f\|_{F^{p,q}_\alpha} = \Big\|\Big(\sum_{j\in\mathbb{N}_0} \big(2^{j\alpha}|\varphi_j(\sqrt{\mathcal{L}})f|\big)^q\Big)^{1/q}\Big\|_{L^p} < \infty,$$

with the appropriate sup-norm replacement when p or q take the value of infinity. It turns out that these spaces are independent of the choice of φ; they are also in general different from the classical Triebel–Lizorkin and Besov spaces associated to the Laplacian operator in $\mathbb{R}^n$. For the details (as well as other related facts) see [8, 14, 30, 37] and earlier works cited there.

Throughout the rest of this article we will adopt the following notational conventions. We use $A^{p,q}_\alpha(\mathcal{L})$ (or $A^{p,q}_\alpha$) to refer to $B^{p,q}_\alpha(\mathcal{L})$ or $F^{p,q}_\alpha(\mathcal{L})$, with the understanding that $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $0 < p \le \infty$ if $A = B$ and $0 < p < \infty$ if $A = F$.

We also denote

$$n_{p,q} = \begin{cases} \dfrac{n}{\min\{1,p,q\}} & \text{if } A^{p,q}_\alpha(\mathcal{L}) = F^{p,q}_\alpha(\mathcal{L}), \\ \dfrac{n}{\min\{1,p\}} & \text{if } A^{p,q}_\alpha(\mathcal{L}) = B^{p,q}_\alpha(\mathcal{L}). \end{cases}$$

It is perhaps interesting to note that there is no notion of 'homogeneous' space $\dot{A}^{p,q}_\alpha(\mathcal{L})$ in the Hermite context (through taking summation over $j \in \mathbb{Z}$ in place of $j \in \mathbb{N}_0$ in the above definitions). This is due the fact that $\mathcal{L} = -\Delta + |x|^2$ has a spectral gap; that is, the eigenvalues of $\mathcal{L}$, being $2|\xi| + n$, lie away from zero. This implies that $\varphi_j(\sqrt{\mathcal{L}}) = 0$ for all $j < 0$ and thus it holds that

$$\dot{A}^{p,q}_\alpha(\mathcal{L}) = A^{p,q}_\alpha(\mathcal{L}).$$

It is well known that the classical Triebel–Lizorkin and Besov scales yield characterizations of many standard function spaces from analysis, such as the Lebesgue, Hardy, and Sobolev spaces. In an analogous way, we have the following identifications:

$$L^p(\mathbb{R}^n) \sim F^{p,2}_0(\mathcal{L}), \qquad 1 < p < \infty, \tag{2.3}$$

$$h^p(\mathcal{L}) \sim F^{p,2}_0(\mathcal{L}), \qquad 0 < p \le 1, \tag{2.4}$$

$$W^{s,p}(\mathcal{L}) \sim F^{p,2}_s(\mathcal{L}), \qquad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{2.5}$$

$$h^{s,p}(\mathcal{L}) \sim F^{p,2}_s(\mathcal{L}), \qquad 0 < p \le 1, \quad s \in \mathbb{R},$$

all with equivalent norms. The identification with $L^p(\mathbb{R}^n)$ was obtained in [8, 17, 37]. The spaces $h^p(\mathcal{L})$ are the atomic Hardy spaces associated to the Hermite operator introduced in [16] (see also [13]); the identification with the Triebel scale can be seen in [13, 24]. The spaces $W^{s,p}(\mathcal{L})$ and $h^{s,p}(\mathcal{L})$ are the Hermite Sobolev and Hermite Hardy–Sobolev spaces respectively; they are defined, for $s \in \mathbb{R}$, through

$$\|f\|_{W^{s,p}(\mathcal{L})} := \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{L^p} \qquad \text{and} \qquad \|f\|_{h^{s,p}(\mathcal{L})} := \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{h^p(\mathcal{L})},$$

for $f \in \mathscr{S}'(\mathbb{R}^n)$, where $\mathcal{D}^s_{\mathcal{L}}$ is the Hermite fractional differential operator

$$\mathcal{D}^s_{\mathcal{L}}(f) := \sum_{\xi \in \mathbb{N}^n_0} (2|\xi| + n)^{s/2} \langle f, h_\xi \rangle h_\xi,$$

(see [6, 26, 42]). Note that $\mathcal{D}^s_{\mathcal{L}}$ is well defined on $\mathscr{S}'(\mathbb{R}^n)$ since it preserves $\mathscr{S}(\mathbb{R}^n)$. The Hermite Sobolev spaces are strictly contained in the classical Sobolev spaces for $s > 0$ and, when s is a positive integer, $W^{s,p}(\mathcal{L})$ turns out to be the space of functions with 'Hermite derivatives up to order s' in $L^p(\mathbb{R}^n)$ (see [6, 42]). The identification above can be seen via the 'lifting property' (see [31, Proposition 2.1])

$$\|f\|_{F^{p,q}_\alpha} \sim \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{F^{p,q}_{\alpha-s}}, \qquad 0 < p, q < \infty, \alpha, s \in \mathbb{R}.$$

2.1. **Hermite sequence spaces and a frame decomposition.** In this section we describe how the frame decomposition of Frazier–Jawerth [20, 21] for classical function spaces has been adapted to the Hermite setting. The Frazier–Jawerth's setup utilises several key ingredients and we outline their analogues in the Hermite setting before providing the main result of this section (Theorem 2.1), a frame characterisation for Hermite Besov and Hermite Triebel–Lizorkin spaces.

2.1.1. *Hermite tiles.* A crucial ingredient in the classical setup is the family of dyadic cubes $\mathcal{Q}$ of $\mathbb{R}^n$. In the Hermite context the standard dyadic cubes are replaced by a notion of 'tiles' or 'rectangles'. They are constructed from the zeros of Hermite polynomials and we provide a brief description of their geometry here. Further details can be found in [14, 30, 37].

For each 'level' $j \in \mathbb{N}_0$ there exists a number $N_j \sim 4^j$ and a collection $\mathcal{X}_j$ of *nodes*, defined as the set of n-tuples of zeros of the Hermite polynomial H_{2N_j}. To each node in $\mathcal{X}_j$, we associate a *tile* R with sides parallel to the axes, so that each such tile contains precisely one node and any two different tiles with nodes in $\mathcal{X}_j$ have disjoint interiors. We set $\mathcal{E}_j$ to be the collection of all jth level tiles and define $\mathcal{E} := \bigcup_{j\geq 0} \mathcal{E}_j$ to be the collection of all tiles. It turns out that through this construction, $\mathcal{E}_j$ contains a finite number of tiles (approximately 4^{jn}), although every point in $\mathbb{R}^n$ is eventually contained in some tile in $\mathcal{E}$.

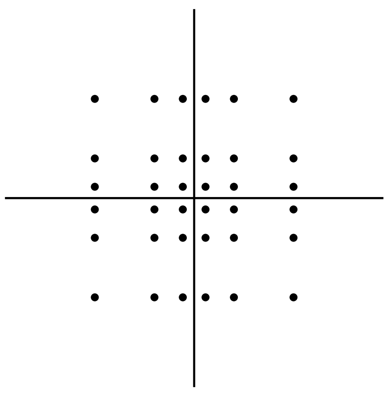
These tiles obey important properties, some of which we describe here. Roughly speaking the tiles are approximately cubes along the diagonals of $\mathbb{R}^n$, and are rectangular boxes off the diagonal. In fact there exists $2 < c_\star < 4$ such that

$$|R| \sim 2^{-jn} \qquad \text{if} \quad |x_R| \leq c_\star 2^j,$$

and

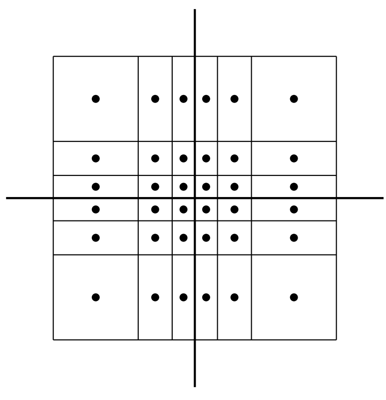
$$2^{-jn} \lesssim |R| \lesssim 2^{-jn/3} \qquad \text{if} \quad |x_R| > c_\star 2^j.$$

Here x_R denotes the 'node' of R. The diagrams below give a visual depiction of our tile construction.



Nodes from $\mathcal{X}_j$

$\#\mathcal{X}_j \sim 4^{jn}$



Tiles from $\mathcal{E}_j$

$\#\mathcal{E}_j \sim 4^{jn}$

2.1.2. *Needlets.* Another key ingredient of Frazier–Jawerth's theory is the system of 'canonical' frames $\{\varphi_Q\}_{Q\in\mathcal{Q}}$, which are translates and dilates of a fixed Schwartz function φ. Frames in the Hermite context are formed using the admissible function from (2.1), and in the literature they have been coined 'needlets' (see [37]).

If $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is an admissible system, then one can see from (2.2) that the kernels of the operators $\varphi_j(\sqrt{\mathcal{L}})$ are

$$\varphi_j(\sqrt{\mathcal{L}})(x,y) = \sum_{\xi\in\mathbb{N}_0^n} \varphi_j(\sqrt{2|\xi|+n})\, h_\xi(x)h_\xi(y) \qquad x,y\in\mathbb{R}^n.$$

Then for each tile $R\in\mathcal{E}_j$ we define the *needlet* φ_R by

$$\varphi_R(x) = \tau_R^{1/2}\varphi_j(\sqrt{\mathcal{L}})(x,x_R).$$

Here x_R is the node of R and $\tau_R = \tau_{x_R}$ is a structural constant that satisfies $\tau_R \sim |R|$. For more details concerning the numbers $\{\tau_R\}_{R\in\mathcal{E}}$ see [30, 37].

Crucial for the development of the subsequent theory outlined in this exposition are the following estimates on the needlets:

$$|\varphi_R(x)| \lesssim \frac{2^{jn}|R|^{-1/2}}{(1+2^j|x-x_R|)^\eta}\mathrm{e}_{\epsilon 4^j}(x)\mathrm{e}_{\epsilon 4^j}(x_R), \tag{2.6}$$

where $e_N(x) := e^{-c|x|^2}$ for some $c>0$ if $|x|^2\le N$, and $e_N(x) := 1$ otherwise. For further details and related estimates see [30, Lemma A.1] and [14, Propostion 2.2].

2.1.3. *Hermite sequence spaces.* The final key ingredient in Frazier–Jawerth theory is the notion of 'sequence spaces' $a_\alpha^{p,q}$, which are sequences defined over the set of all dyadic cubes. The Hermite analogues are as follows. Let $\alpha\in\mathbb{R}$ and $0<q\le\infty$. For $0<p\le\infty$, the *Hermite Besov sequence space* $b_\alpha^{p,q} = b_\alpha^{p,q}(\mathcal{L})$ is defined as the set of all complex sequences $s=\{s_R\}_{R\in\mathcal{E}}$ such that

$$\|s\|_{b_\alpha^{p,q}} = \bigg\{\sum_{j\in\mathbb{N}_0} 2^{j\alpha q}\Big(\sum_{R\in\mathcal{E}_j}\big(|R|^{1/p-1/2}|s_R|\big)^p\Big)^{q/p}\bigg\}^{1/q} < \infty;$$

for $0<p<\infty$, the *Hermite Triebel–Lizorkin sequence space* $f_\alpha^{p,q} = f_\alpha^{p,q}(\mathcal{L})$ is the set of all complex sequences $s=\{s_R\}_{R\in\mathcal{E}}$ such that

$$\|s\|_{f_\alpha^{p,q}} = \bigg\|\Big(\sum_{j\in\mathbb{N}_0} 2^{j\alpha q}\sum_{R\in\mathcal{E}_j}\big(\mathbf{1}_R(\cdot)|R|^{-1/2}|s_R|\big)^q\Big)^{1/q}\bigg\|_{L^p} < \infty.$$

Analogously to the function spaces $A_\alpha^{p,q}(\mathcal{L})$, we use $a_\alpha^{p,q}(\mathcal{L})$ (or just $a_\alpha^{p,q}$) to refer to $b_\alpha^{p,q}(\mathcal{L})$ or $b_\alpha^{p,q}(\mathcal{L})$, as appropriate to the context.

2.1.4. *Frame decomposition for Hermite function paces.* We are now ready to give the frame decomposition of Hermite function spaces. This essentially says that given a suitable pair of admissible functions φ and ψ, one can form maps S_φ and T_ψ between functions and sequences such that the following diagram commutes:

$$\begin{tikzcd} & a_\alpha^{p,q} \arrow[dr, "T_\psi"] & \\ A_\alpha^{p,q} \arrow[ur, "S_\varphi"] \arrow[rr, "\mathrm{Id}"'] & & A_\alpha^{p,q} \end{tikzcd}$$

In particular, the maps are continuous and one has that

$$\|f\|_{A^{p,q}_{\alpha}(\mathcal{L})} \sim \|S_{\varphi} f\|_{a^{p,q}_{\alpha}(\mathcal{L})}.$$

More precisely we have the following result.

Theorem 2.1 (Frame decomposition; [14, 37]). *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, and $0 < p < \infty$ if $A^{p,q}_{\alpha}(\mathcal{L}) = F^{p,q}_{\alpha}(\mathcal{L})$ or $0 < p \leq \infty$ if $A^{p,q}_{\alpha}(\mathcal{L}) = B^{p,q}_{\alpha}(\mathcal{L})$. Given any two admissible systems $\{\varphi_j\}_{j\in\mathbb{N}_0}$ and $\{\psi_j\}_{j\in\mathbb{N}_0}$ the 'analysis' operator $S_{\varphi} : f \longmapsto \{\langle f, \varphi_R\rangle\}_{R\in\mathcal{E}}$ and 'synthesis' operator $T_{\psi} : \{s_R\}_{R\in\mathcal{E}} \longmapsto \sum_{R\in\mathcal{E}} s_R \psi_R$ act as bounded maps between*

$$T_{\psi} : a^{p,q}_{\alpha}(\mathcal{L}) \to A^{p,q}_{\alpha}(\mathcal{L}) \qquad \text{and} \qquad S_{\varphi} : A^{p,q}_{\alpha}(\mathcal{L}) \to a^{p,q}_{\alpha}(\mathcal{L}).$$

Moreover, if

$$\sum_{j\geq 0} \psi_j(\lambda)\varphi_j(\lambda) = 1 \qquad \forall\, \lambda \geq \tfrac{1}{2},$$

then $T_{\psi} \circ S_{\varphi} = I$ on $A^{p,q}_{\alpha}(\mathcal{L})$ (with convergence in $\mathscr{S}'(\mathbb{R}^n)$).

Theorem 2.1 was proved in [37] (see also [14, Theorem 3.1]). An essential ingredient in the proof are the kernel and needlet estimates such as those in (2.6).

2.2. **Smooth molecular characterization.** Another cornerstone of the Frazier–Jawerth theory is the notion of smooth molecules, which provides an important tool in the study of operators between function spaces (see [21, 45]). In this section we present smooth molecules for function spaces in the Hermite context; they are used in obtaining some of our results in Section 3.

Molecules encode the intrinsic smoothness and cancellation properties of the function space scales. One important difference between the Hermite context with the classical situation relates to the required moment conditions on the molecules. Recall that the standard requirement is of the form,

$$\int_{\mathbb{R}^n} x^{\gamma} m(x)\, dx = 0, \quad \forall \gamma \in \mathbb{N}_0^n \text{ such that } |\gamma| \leq M, \tag{2.7}$$

where M is some positive number. This cancellation is satisfied, for example, by the canonical functions φ_Q used in the frame decomposition (which are prototypical examples of molecules), and follows from the fact that $\widehat{\varphi}$ is supported away from the origin.

Since $\int_{\mathbb{R}^n} h_{\xi} \neq 0$ in general, one should not expect (2.7) to hold for our needlets, φ_R. However the following property (see [40, Lemma 1.2])

$$\Big| \int_{\mathbb{R}^n} \chi(x) h_{\xi}(x)\, dx \Big| = O\big(|\xi|^{-N}\big), \qquad \chi \in C_0^{\infty}(\mathbb{R}^n),$$

hints at some form of inherent cancellation for the Hermite functions; this translates to a kind of 'approximate cancellation' for needlets, and forms the basis for the moment conditions of our Hermite molecules below (see [30, Lemma 3.3 and Lemma A.1]) .

We now present the definition of smooth molecules for the Hermite setting, which was introduced in [30, Section 3]. Let $(M, \theta) \in \{\mathbb{N}_0 \times (0, 1)\} \cup \{(-1, 1)\}$, $N \in \mathbb{N}_0$,

$0 \le \delta \le 1$ and $\mu \ge 1$. A function $m \in C^N(\mathbb{R}^n)$ is said to be a *smooth Hermite $(M, \theta, N, \delta, \mu)$-molecule* associated with a tile $R \in \mathcal{E}_j$ for some $j \in \mathbb{N}_0$ if

(i) for each multi-index γ with $0 \le |\gamma| \le N$ we have

$$|\partial^\gamma m(x)| \le |R|^{-1/2} 2^{j|\gamma|} (1 + 2^j |x - x_R|)^{-\mu} \Big(1 + \frac{|x|}{2^j}\Big)^{-N-\delta} \quad \forall x \in \mathbb{R}^n,$$

(ii) for each multi-index γ with $|\gamma| = N$ we have

$$|\partial^\gamma m(x) - \partial^\gamma m(y)| \le |R|^{-1/2} 2^{j|\gamma|} \Big(\frac{|x-y|}{2^{-j}}\Big)^\delta \big(1 + 2^j |x - x_R|\big)^{-\mu}$$

for every $x, y \in \mathbb{R}^n$ with $|x - y| \le 2^{-j}$.

(iii) for each multi-index γ with $0 \le |\gamma| \le M$ we have

$$\Big| \int_{\mathbb{R}^n} (y - x_R)^\gamma m(y)\, dy \Big| \le |R|^{-1/2} 2^{-j(n+|\gamma|)} \Big(\frac{1 + |x_R|}{2^j}\Big)^{M+\theta-|\gamma|}.$$

If $(M, \theta) = (-1, 1)$, part (iii) is taken to be void. Note also that property (i) for any N, implies (ii) for $N - 1$ (modulo a constant); see [30, Remark 3.2]. As already mentioned above, needlets φ_R are basic examples of smooth molecules (see [30, Lemma 3.3]).

The important fact here is that the Hermite function spaces can be characterised by smooth molecules. The following was obtained in [30] (see Theorems 3.5, 3.6 and Remark 3.7 (i) therein).

Theorem 2.2 (Molecular characterization; [30]). *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $0 < p < \infty$ if $A^{p,q}_\alpha(\mathcal{L}) = F^{p,q}_\alpha(\mathcal{L})$ or $0 < p \le \infty$ if $A^{p,q}_\alpha(\mathcal{L}) = B^{p,q}_\alpha(\mathcal{L})$.*

If $\mu \ge 1$, $(M, \theta) \in \{\mathbb{N}_0 \times (0, 1)\} \cup \{(-1, 1)\}$, $N \in \mathbb{N}_0$ and $0 \le \delta \le 1$ then there exists a family of $(M, \theta, N, \delta, \mu)$-molecules $\{m_R\}_{R\in\mathcal{E}}$ such that, for any $f \in A^{p,q}_\alpha(\mathcal{L})$, there is a sequence of scalars $\{s_R\}_{R\in\mathcal{E}}$ satisfying $f = \sum_R s_R m_R$ in $\mathscr{S}'(\mathbb{R}^n)$ and $\|s\|_{a^{p,q}_\alpha} \lesssim \|f\|_{A^{p,q}_\alpha}$.

Conversely if $\{m_R\}_{R\in\mathcal{E}}$ is a collection of $(M, \theta, N, \delta, \mu)$-molecules satisfying

$$N + \delta > \alpha, \quad n + M + \theta + \alpha > n_{p,q}, \quad \mu > \max\{n_{p,q}, n + M + \theta\},$$

then for any complex sequence $s = \{s_R\}_{R\in\mathcal{E}} \in a^{p,q}_\alpha(\mathcal{L})$, $\big\|\sum_{R\in\mathcal{E}} s_R m_R\big\|_{A^{p,q}_\alpha} \lesssim \|s\|_{a^{p,q}_\alpha}$.

It may be of interest to note that the proof requires an almost orthogonality type estimate:

$$|\varphi_j(\sqrt{\mathcal{L}}) m_R(x)| \lesssim \frac{|R|^{-1/2}}{(1 + 2^{j\wedge k}|x - x_R|)^\eta} 2^{-(n+M+\theta)[(k-j)\vee 0] - (N+\delta)[(j-k)\vee 0]}$$

for some $\eta < \mu$, and all $x \in \mathbb{R}^n$, $j, k \in \mathbb{N}_0$ and $R \in \mathcal{E}_k$ ([30, Lemma 3.4]).

3. Hermite multipliers and pseudo-multipliers

In this section we consider operators related to the Hermite operator derived through functions of its spectrum. These are analogues of the Fourier multipliers and pseudo-differential operators which, the reader may recall, can be defined for suitable functions f as operators of the form

$$\mathcal{F}^{-1}(\sigma\mathcal{F}(f)),$$

where $\mathcal{F}$ is the Fourier transform, and σ is a function depending on frequency or spatial variables.

In the Hermite context, one can study analogues using the Hermite–Fourier transform $\mathcal{F}_{\mathcal{L}} : f \mapsto \{\langle f, h_\xi\rangle\}_{\xi\in\mathbb{N}_0^n}$, and the inverse Hermite–Fourier transform $\mathcal{F}_{\mathcal{L}}^{-1} : \{s_\xi\}_{\xi\in\mathbb{N}_0^n} \mapsto \sum_{\xi\in\mathbb{N}_0^n} s_\xi h_\xi$. When $\sigma : \mathbb{N}_0 \to \mathbb{C}$ is a bounded function we obtain the Hermite multiplier

$$\mathcal{F}_{\mathcal{L}}^{-1}(\sigma\mathcal{F}_{\mathcal{L}}(f)) = \sum_{\xi\in\mathbb{N}_0^n} \sigma(2|\xi|+n)\langle f, h_\xi\rangle h_\xi. \tag{3.1}$$

The $L^2(\mathbb{R}^n)$ boundedness of such operators is immediate by invoking Parseval's identity. For the $L^p(\mathbb{R}^n)$ boundedness with $p \neq 2$ (and on other function spaces) satisfying answers have been given by [18, 22, 25, 32, 41, 43] to name a few. In Section 4 we will discuss an extension of one of Harboure et al.'s multiplier results; for now we turn to a new kind of operator that has recently been garnering increasing interest.

Consider again operators of the form (3.1) but where the symbol σ can also depend on the spatial variable. In this sense these are analogues of the usual pseudo-differential operators, and will be the main objects of study in the rest of this section. More precisely we will be considering the following linear and bilinear operators.

Pseudo-multipliers: Given $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$, we define the Hermite pseudo-multiplier by

$$T_\sigma f(x) = \sum_{\xi\in\mathbb{N}_0^n} \sigma(x, 2|\xi|+n)\langle f, h_\xi\rangle h_\xi(x). \tag{3.2}$$

Bilinear pseudo-multipliers: Given $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$, we define the bilinear Hermite pseudo-multiplier by

$$T_\sigma(f,g)(x) = \sum_{\xi,\eta\in\mathbb{N}_0^n} \sigma(x, 2|\xi|+n, 2|\eta|+n)\langle f, h_\xi\rangle\langle g, h_\eta\rangle h_\xi(x)h_\eta(x). \tag{3.3}$$

The operators (3.2) were introduced and first studied by Epperson [18], and their research has continued in [1, 2, 9, 15, 30, 29]; the operators (3.3) were introduced and investigated in [31]. Questions that have been tackled in these works include sufficient conditions for boundedness on $L^p(\mathbb{R}^n)$ and also for more general function spaces (which are more delicate than for the multiplier scenario (3.1)), investigating suitable analogues of the Hörmander symbols, and (in the bilinear case) establishing algebra properties and fractional Leibniz rules in the setting of Hermite function spaces.

In the remainder of this section we survey some of the results in this developing area, drawing mostly from [30, 31]. We will first discuss results for linear pseudo-multipliers (3.2) in Section 3.1, before presenting results for bilinear operators (3.3) in Section 3.2. Almost all proofs will be omitted but sketches are provided for the main results in Section 3.3; for everything else, relevant references will be given.

3.1. **Results for pseudo-multipliers.** Here we consider operators of the form (3.2). The main symbols we will consider are the following Hörmander-type symbols.

Definition 3.1 (Smooth symbols). *Let $m \in \mathbb{R}$, $\rho, \delta \geq 0$, and $\mathcal{N}, \mathcal{K} \in \mathbb{N}_0 \cup \{\infty\}$. The symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ belongs to $S^{m,\mathcal{K},\mathcal{N}}_{\rho,\delta}$ if $\sigma(\cdot, j) \in C^{\mathcal{N}}(\mathbb{R}^n)$ for each $j \in \mathbb{N}_0$ and there exists $C_{\nu,\kappa} > 0$ such that for each $(x, j) \in \mathbb{R}^n \times \mathbb{N}_0$ we have*

$$|\partial_x^\nu \triangle_j^\kappa \sigma(x, j)| \leq C_{\nu,\kappa} \, (1 + j)^{\frac{m}{2} - \rho|\kappa| + \frac{\delta}{2}|\nu|}$$

for $\nu \in \mathbb{N}_0^n$ and $\kappa \in \mathbb{N}_0$ satisfying $0 \leq |\nu| \leq \mathcal{N}$ and $0 \leq \kappa \leq \mathcal{K}$.

When $\mathcal{N} = \mathcal{K} = \infty$ we just write $S^{m,\infty,\infty}_{\rho,\delta} = S^m_{\rho,\delta}$. Note that here and in what follows the symbol $\triangle$ denotes the forward difference operator, that is, for a function f defined over the integers, $\triangle_j f(j) = f(j+1) - f(j)$ and $\triangle_j^\kappa f(j) = \triangle(\triangle^{\kappa-1} f)(j)$ for $\kappa \geq 2$, $\kappa \in \mathbb{N}$.

It is worth mentioning that many of the results below are also true when the symbols are allowed to have additional growth conditions and we direct the interested reader to [30] for further details.

We have the following result for spaces with smoothness index $\alpha > 0$.

Theorem 3.2 ([30]). *Let $m \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\mathcal{N}, \mathcal{K} \in \mathbb{N}$ and $\sigma \in S^{m,\mathcal{K},\mathcal{N}}_{1,\delta}$. Assume $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p < \infty$ for Triebel–Lizorkin spaces or $0 < p \leq \infty$ for Besov spaces satisfy*

$$n_{p,q} - n < \alpha < \mathcal{N} \qquad \text{and} \qquad n_{p,q} < \mathcal{K}.$$

Then the operator T_σ extends to a bounded operator from $A^{p,q}_{\alpha+m}(\mathcal{L})$ to $A^{p,q}_\alpha(\mathcal{L})$.

Theorem 3.2 furnishes an analogue of the classical results for pseudo-differential operators on Lipschitz and Sobolev spaces (see for example [39, Chapter 7, Sections 1.3 and 5.6]) and more generally on classical Besov and Triebel–Lizorkin spaces (see [38, 44]). For related results in the Hermite context see [9, 23].

Our next main result allows for function spaces with smoothness index $\alpha \leq 0$. It provides an avenue for considering questions of boundedness on, for instance, the $L^p(\mathbb{R}^n)$ scale. Before stating the result, let us define a new class of symbols.

Definition 3.3 (Cancellation class). *Let $m \in \mathbb{R}$, $M \in \mathbb{N}_0 \cup \{\infty\}$ and $\varrho(x) = (1 + |x|)^{-1}$. The symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{C}^{m,M}$ if*

$$\Big(\fint_{B(x,\varrho(x))} \big|\partial_y^\gamma \sigma(y, j)\big|^2 \, dy \Big)^{1/2} \lesssim (1 + j)^{\frac{m}{2}} \varrho(x)^{-|\gamma|} \quad \forall (x, j) \in \mathbb{R}^n \times \mathbb{N}_0$$

for $\gamma \in \mathbb{N}_0^n$ satisfying $0 \leq |\gamma| \leq 2\lfloor (n + M)/2 \rfloor + 2$ and where the implicit constant may depend on γ.

As is well known, there exist pseudo-differential operators with symbols from the corresponding Hörmander class for $S^0_{1,1}$ that are unbounded on $L^2(\mathbb{R}^n)$ (the so-called 'forbidden class'; see [39, Ch. 7, Proposition 2]). The class of symbols in Definition 3.3 enable us to consider the endpoint $(\rho,\delta)=(1,1)$.

We are now ready to state our second main result.

Theorem 3.4 ([30])**.** *Let $m\in\mathbb{R}$, $M\in\mathbb{N}_0$ and $\mathcal{N},\mathcal{K}\in\mathbb{N}$. Assume that $\sigma:\mathbb{R}^n\times\mathbb{N}_0\to\mathbb{C}$ satisfies one of the following conditions:*
(a) $\sigma\in S^{m,\mathcal{K},\mathcal{N}}_{1,1}\cap\mathcal{C}^{m,M}$,
(b) $\sigma\in S^{m,\mathcal{K},\mathcal{N}}_{1,\delta}$ *for some* $0\le\delta<1$ *and* $\mathcal{N}\ge 2\lceil\frac{n+M+1}{2(1-\delta)}\rceil$.
Suppose $\alpha\in\mathbb{R}$, $0<q\le\infty$, $0<p<\infty$ for Triebel–Lizorkin spaces or $0<p\le\infty$ for Besov spaces satisfy

$$n_{p,q}-n-M-1<\alpha<\mathcal{N}\qquad\text{and}\qquad \max\{n_{p,q},n+M\}<\mathcal{K}.$$

Then the operator T_σ extends to a bounded operator from $A^{p,q}_{\alpha+m}(\mathcal{L})$ to $A^{p,q}_{\alpha}(\mathcal{L})$.

It is worth pointing out that the 'approximate cancellation' property of our smooth molecules from Section 2.2 plays an important role in the proof of Theorem 3.4.

When $\mathcal{N}=\mathcal{K}=\infty$, one can summarise Theorems 3.2 and 3.4 more simply as follows.

Corollary 3.5 ([30])**.** *Let $m\in\mathbb{R}$, $\alpha\in\mathbb{R}$, $0<q\le\infty$, $0<p<\infty$ for Triebel–Lizorkin spaces or $0<p\le\infty$ for Besov spaces. Assume that $\sigma:\mathbb{R}^n\times\mathbb{N}_0\to\mathbb{C}$ satisfies one of the following conditions:*
(a) $\sigma\in S^m_{1,1}\cap\mathcal{C}^{m,\infty}$,
(b) $\sigma\in S^m_{1,\delta}$ *for some* $0\le\delta<1$,
(c) $\sigma\in S^m_{1,\delta}$ *for some* $0\le\delta\le1$ *and* $\alpha>n_{p,q}-n$.
Then the operator T_σ extends to a bounded operator from $A^{p,q}_{\alpha+m}(\mathcal{L})$ to $A^{p,q}_{\alpha}(\mathcal{L})$.

3.1.1. *Applications to boundedness on L^p and Sobolev scales.* In this section we give some applications of Theorems 3.2 and 3.4 to the boundedness of pseudo-multipliers on particular function spaces. We only focus on certain cases, aiming to be indicative rather than exhaustive.

We first discuss the question of L^p boundedness. The earliest result in this direction is due to Epperson [18], who studied the case $p<2$ in dimension 1 under the condition

$$\|\Delta^\kappa_j\sigma(\cdot,j)\|_{L^\infty}\lesssim(1+j)^{-\kappa},\qquad 0\le\kappa\le5,$$

and assuming a-priori boundedness in $L^2(\mathbb{R})$. Epperson showed that under these conditions, T_σ is of weak type $(1,1)$ and hence preserves $L^p(\mathbb{R})$ for all $1<p<2$. Bagchi-Thangavelu, [2], extended this to dimension $n\ge2$ assuming $|\kappa|\le n+1$. The case $p>2$ can be obtained by assuming additional regularity in the spatial variable: it was shown in [2, Theorem 1.4]) that if T_σ is bounded on $L^2(\mathbb{R}^n)$ and $\sigma\in S^{0,n+1,1}_{1,0}$ then T_σ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

The reader will observe that the a-priori $L^2(\mathbb{R}^n)$ boundedness was assumed in the results above. While the $L^2(\mathbb{R}^n)$ boundedness for multipliers (3.1) is immediate, the situation is far from clear for pseudo-multipliers. The question of sufficient conditions ensuring $L^2(\mathbb{R}^n)$ boundedness of pseudo-multipliers was raised in [2] along with the possibility of a Calderón–Vaillencourt type theorem for the Hermite context. The result below, which is a consequence of Theorem 3.4 and (2.3), provides one answer to this question.

Corollary 3.6 (L^p boundedness; [30])**.** *Assume that* $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ *satisfies one of the following conditions:*
(a) $\sigma \in S^{0,n+1,1}_{1,1} \cap \mathcal{C}^{0,0}$,
(b) $\sigma \in S^{0,n+1,\mathcal{N}}_{1,\delta}$ *for some* $0 \le \delta < 1$ *and* $\mathcal{N} \ge 2\lceil \frac{n+1}{2(1-\delta)} \rceil$.
Then T_σ *extends to a bounded operator on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

Additional answers are given in [1, 15, 29] under various interesting conditions on the symbols. We will not give an account here but refer the reader to the relevant works for the details. It is however worth pointing out here that under the hypotheses of Corollary 3.6, the operators T_σ are in fact Calderón–Zygmund operators ([29, Theorem 1.5]). This parallels the situation for pseudo-differential operators with the Hörmander class $S^0_{1,\delta}$ (see [39, p. 322]), and as a by-product, also extends the results for multipliers on weighted L^p ([22, 43]) to the case of pseudo-multipliers.

In passing, let us conclude this section with some additional consequences of Theorems 3.2 and 3.4 for Hardy and Sobolev spaces. From Theorem 3.4 and (2.4) we have the following result.

Corollary 3.7 (h^p boundedness)**.** *Let* $0 < p \le 1$, $0 \le \delta < 1$ *and* $\mathcal{N} \ge 2\lceil \frac{\lfloor n/p \rfloor + 1}{2(1-\delta)} \rceil$. *If* $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ *satisfies* $\sigma \in S^{0,\lfloor \frac{n}{p} \rfloor + 1, \mathcal{N}}_{1,\delta}$ *then* T_σ *extends to a bounded operator on* $h^p(\mathcal{L})$.

From Theorem 3.2 and (2.5) we have the following result.

Corollary 3.8 (Boundedness on Sobolev spaces)**.** *Let* $s > 0$ *and* $1 < p < \infty$. *If* $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \to \mathbb{C}$ *satisfies* $\sigma \in S^{0,n+1,\lfloor s \rfloor + 1}_{1,1}$ *then* T_σ *extends to a bounded operator on* $W^{s,p}(\mathcal{L})$.

3.1.2. *Applications to properties of function spaces.* In this section we present some properties of Hermite function spaces that follow as consequences of Theorems 3.2 and 3.4. The first fact is that Hermite Besov spaces and Hermite Triebel–Lizorkin spaces are closed under non-linearities.

Theorem 3.9 (Closure under non-linearities; [30])**.** *Assume* $0 < p < \infty$, $0 < q < \infty$, $\alpha > n_{p,q} - n$ *and* $H \in \mathcal{C}^\infty(\mathbb{R})$ *is such that* $H(0) = 0$. *If* $f \in A^{p,q}_\alpha(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ *is real-valued, then* $H(f) \in A^{p,q}_\alpha(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$.

This result uses ideas from Bony [7] and Meyer [33], and can be obtained from Theorem 3.2 together with the following linearization formula from [30]: Let $H \in$

$\mathcal{C}^\infty(\mathbb{R})$ be such that $H(0) = 0$. If $f \in \mathscr{S}(\mathbb{R}^n)$ is real-valued, there exists $\sigma_f \in S^{0,\infty,\infty}_{1,1}$ such that $H(f) = T_{\sigma_f}(f)$.

Note that Theorem 3.9 implies that Hermite spaces form an algebra under pointwise products. More precisely, if $0 < p < \infty$, $0 < q < \infty$, $\alpha > n_{p,q} - n$ and $f, g \in A^{p,q}_\alpha(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$, then $fg \in A^{p,q}_\alpha(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ (see [30, Remark 5.14]). For a different approach to this fact that also yields 'Leibniz rules', see Corollary 3.13 below.

The final property we wish to highlight is that the Hermite spaces admit the so-called 'lifting property'; this follows from Corollary 3.5.

Theorem 3.10 (Lifting property for Hermite function spaces; [31])**.** *Assume $0 < p < \infty$ for Triebel–Lizorkin spaces or $0 < p \leq \infty$ for Besov spaces, $0 < q \leq \infty$ and $\alpha, s \in \mathbb{R}$. Then the operator $\mathcal{D}^s_{\mathcal{L}}$ maps $A^{p,q}_\alpha(\mathcal{L})$ isomorphically onto $A^{p,q}_{\alpha-s}(\mathcal{L})$ and $\|f\|_{A^{p,q}_\alpha} \sim \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{A^{p,q}_{\alpha-s}}$.*

This result will be important in Section 3.2 below.

3.2. **Results for bilinear pseudo-multipliers.** In this section we consider operators of the form (3.3) and the symbols we will consider are the following bilinear Hörmander-type symbols.

Definition 3.11 (Smooth bilinear symbols)**.** *Let $m \in \mathbb{R}$, $\rho, \delta \geq 0$, and $\mathcal{N}, \mathcal{K} \in \mathbb{N}_0 \cup \{\infty\}$. The symbol $\sigma : \mathbb{R}^n \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ belongs to $BS^{m,\mathcal{N},\mathcal{K}}_{\rho,\delta}$ if $\sigma(\cdot, j, \ell) \in C^{\mathcal{N}}(\mathbb{R}^n)$ for all $j, \ell \in \mathbb{N}_0$, and there exists $C_{\nu,\kappa,\vartheta} > 0$ such that for each $(x, j, \ell) \in \mathbb{R}^n \times \mathbb{N}_0 \times \mathbb{N}_0$ we have*

$$|\partial^\nu_x \triangle^\kappa_j \triangle^\vartheta_\ell \sigma(x, j, \ell)| \leq C_{\nu,\kappa,\vartheta} \, (1 + j + \ell)^{\frac{m}{2} - \rho(\kappa+\vartheta) + \frac{\delta}{2}|\nu|}$$

for $\nu \in \mathbb{N}^n_0$ and $\kappa, \vartheta \in \mathbb{N}_0$ satisfying $0 \leq |\nu| \leq \mathcal{N}$ and $0 \leq \kappa, \vartheta \leq \mathcal{K}$.

As before when $\mathcal{N} = \mathcal{K} = \infty$ we just write $BS^{m,\infty,\infty}_{\rho,\delta} = BS^m_{\rho,\delta}$.

The main result of this section is the following generalized Leibniz-type rules.

Theorem 3.12 (Fractional Leibniz-type rules for bilinear Hermite pseudo-multipliers; [31])**.** *Let $m \in \mathbb{R}$, $0 \leq \delta \leq 1$ and $\sigma \in BS^m_{1,\delta}$. Assume $0 < p < \infty$ for Triebel–Lizorkin spaces or $0 < p \leq \infty$ for Besov spaces, $0 < q \leq \infty$ and $\alpha > n_{p,q} - n$. Then it holds that*

$$\|T_\sigma(f, g)\|_{A^{p,q}_\alpha} \lesssim \|f\|_{A^{p,q}_{\alpha+m}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{A^{p,q}_{\alpha+m}}.$$

3.2.1. *Applications to properties of function spaces.* Let us highlight two main consequences of Theorem 3.12. The first concerns pointwise products and algebra properties of our function spaces. Taking $\sigma \equiv 1$, Theorem 3.12 yields the following.

Corollary 3.13 (Algebra property for $A^{p,q}_\alpha$; [31])**.** *Assume $0 < p < \infty$ for Triebel–Lizorkin spaces or $0 < p \leq \infty$ for Besov spaces, $0 < q \leq \infty$ and $\alpha > n_{p,q} - n$. Then $A^{p,q}_\alpha(\mathcal{L}) \cap L^\infty(\mathbb{R}^n)$ is a quasi-Banach algebra and*

$$\|fg\|_{A^{p,q}_\alpha} \lesssim \|f\|_{A^{p,q}_\alpha} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{A^{p,q}_\alpha}. \tag{3.4}$$

Note that as alluded earlier, closure under pointwise products can also be obtained through Theorem 3.9. However that approach does not seem to yield (3.4).

The second consequence concerns extensions to the Hermite setting of the well known fractional Leibniz rules on $\mathbb{R}^n$, namely,

$$\|(-\Delta)^s(fg)\|_{L^p} \lesssim \|(-\Delta)^s(f)\|_{L^p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|(-\Delta)^s(g)\|_{L^p}, \tag{3.5}$$

for $1 < p < \infty$ and $s > 0$. Here $(-\Delta)^s$ is the homogeneous fractional differential operator given by $\widehat{(-\Delta)^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$. The 'lifting property' of Theorem 3.10 and the estimates in Corollary 3.13 along with (2.3) and (2.4) give the following Hermite analogues of (3.5).

Corollary 3.14 (Hermite fractional Leibniz rules; [31]). *Let $0 < p < \infty$ and $s > 0$.*

(a) *If $1 < p < \infty$ and $s > 0$, it holds that*

$$\|\mathcal{D}^s_{\mathcal{L}}(fg)\|_{L^p} \lesssim \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{L^p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\mathcal{D}^s_{\mathcal{L}}(g)\|_{L^p}.$$

(b) *If $0 < p \leq 1$ and $s > n(1/p - 1)$, it holds that*

$$\|\mathcal{D}^s_{\mathcal{L}}(fg)\|_{h^p(\mathcal{L})} \lesssim \|\mathcal{D}^s_{\mathcal{L}}(f)\|_{h^p(\mathcal{L})}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\mathcal{D}^s_{\mathcal{L}}(g)\|_{h^p(\mathcal{L})}.$$

3.3. **Sketch of proofs of the results for pseudo-multipliers.** In this section we outline some of the main steps in the proofs of Theorems 3.2, 3.4 and 3.12.

3.3.1. *Steps in the proofs of Theorems 3.2 and 3.4.* Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ and $\{\psi_j\}_{j\in\mathbb{N}_0}$ be admissible systems satisfying $\sum_{j\geq 0}\psi_j(\lambda)\varphi_j(\lambda) = 1$ for $\lambda \geq 0$. Under the hypothesis of Theorems 3.2 and 3.4 it follows that if $R \in \mathcal{E}_j$, then $2^{-jm}T_\sigma\psi_R$ satisfies (i) and (ii) in the definition of molecule (smoothness conditions), with appropriate parameters. This requires the following size, smoothness and cancellation estimates on the action of T_σ on needlets: for $|\gamma| \leq \mathcal{N}$, $1 \leq N \leq \mathcal{K}$ and $\beta \geq 0$

$$|\partial^\gamma_x T_\sigma\varphi_R(x)| \lesssim \frac{|R|^{-1/2}2^{j(m+|\gamma|)}}{(1+2^j|x-x_R|)^N}\left(1+\frac{|x|}{2^j}\right)^{-\beta},$$

and

$$\Big|\int_{\mathbb{R}^n}(y-x_R)^\gamma T_\sigma\varphi_R(y)\,dy\Big| \lesssim |R|^{-1/2}2^{j(m-n-|\gamma|)}\Big(\frac{1+|x_R|}{2}\Big)^{M+\theta-|\gamma|}$$

for $j\in\mathbb{N}_0, R\in\mathcal{E}_j, x\in\mathbb{R}^n$. Using the frame characterization (Theorem 2.1) we can write

$$T_\sigma f = \sum_{R\in\mathcal{E}} 2^{jm}\langle f,\varphi_R\rangle 2^{-jm}T_\sigma(\psi_R);$$

then use molecular and frame characterization again (Section 2) to obtain

$$\|T_\sigma f\|_{A^{p,q}_\alpha} \lesssim \big\|\{2^{jm}\langle f,\varphi_R\rangle\}_{R\in\mathcal{E}}\big\|_{a^{p,q}_\alpha} = \big\|\{\langle f,\varphi_R\rangle\}_{R\in\mathcal{E}}\big\|_{a^{p,q}_{\alpha+m}} \lesssim \|f\|_{A^{p,q}_{\alpha+m}}.$$

3.3.2. *Steps in the proof of Theorem 3.12.* Besides the frame and molecular characterizations from Section 2, the proof of Theorem 3.12 requires a certain decomposition of the bilinear pseudo-multiplier $T_\sigma = T_{\sigma^1} + T_{\sigma^2}$ (with $\sigma^1, \sigma^2 \in BS^m_{1,\delta}$) along with suitable molecular estimates of the following form: for any family of needlets $\{\psi_R\}_R$, $\mu, \beta > 0$ and $\gamma \in \mathbb{N}_0^n$,

$$|\partial_x^\gamma T_{\sigma^1}(\psi_R, g)(x)| + |\partial_x^\gamma T_{\sigma^2}(g, \psi_R)(x)| \lesssim \frac{|R|^{-1/2} 2^{j(m+|\gamma|)}}{(1+2^j|x-x_R|)^\mu} \left(1 + \frac{1+|x|}{2^j}\right)^{-\beta} \|g\|_{L^\infty},$$

where the implicit constant is independent of $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, $R \in \mathcal{E}_j$ and $g \in \mathcal{S}(\mathbb{R}^n)$. In particular this shows that the functions $2^{-jm} T_{\sigma^1}(\psi_R, g)/\|g\|_{L^\infty}$ and $2^{-jm} T_{\sigma^2}(g, \psi_R)/\|g\|_{L^\infty}$ are multiples of smooth $(-1, 1, N, \epsilon, \mu)$-molecules for any N, ϵ.

Then using linearity and the frame decomposition we may write

$$T_\sigma(f,g) = T_{\sigma^1}(f,g) + T_{\sigma^2}(f,g) = \sum_{R\in\mathcal{E}} \langle f, \varphi_R\rangle T_{\sigma^1}(\psi_R, g) + \sum_{R\in\mathcal{E}} \langle f, \varphi_R\rangle T_{\sigma^2}(g, \psi_R).$$

Finally, molecular characterization and frame decomposition give us

$$\begin{aligned} \|T_{\sigma^1}(f,g)\|_{A^{p,q}_\alpha} \lesssim \|\{2^{jm}\langle f, \varphi_R\rangle\}\|_{a^{p,q}_\alpha} \|g\|_{L^\infty} &\sim \|\{\langle f, \varphi_R\rangle\}\|_{a^{p,q}_{\alpha+m}} \|g\|_{L^\infty} \\ &\sim \|f\|_{A^{p,q}_{\alpha+m}} \|g\|_{L^\infty}, \end{aligned}$$

with a similar calculation for $T_{\sigma^2}(f,g)$.

4. A new weighted estimate for Hermite multipliers

We conclude this exposition with a new result that brings together several lines of inquiry that Harboure was involved with, namely her work on multipliers and weighted estimates in [3, 4, 5, 22, 25].

Let $\sigma \in \ell^\infty(\mathbb{N}_0^n)$ with

$$|\Delta^\kappa \sigma(\xi)| \lesssim (1+|\xi|)^{-|\kappa|}, \qquad |\kappa| \leq \mathcal{K}, \tag{4.1}$$

where here $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}_0^n$ and the symbol $\Delta^\kappa = \triangle_1^{\kappa_1} \triangle_2^{\kappa_2} \ldots \triangle_n^{\kappa_n}$ denotes the multivariate forward difference operator of order $\kappa \in \mathbb{N}_0^n$; that is, for a function f defined over $\mathbb{N}_0^n$, $\triangle_i f(\xi) = f(\xi + e_i) - f(\xi)$ and $\triangle_i^{\kappa_i} f = \triangle_i(\triangle_i^{\kappa_i - 1} f)$ for $\kappa_i \geq 2$.

Consider the Hermite multiplier given by $T_\sigma = \mathcal{F}_\mathcal{L}^{-1}(\sigma \mathcal{F}_\mathcal{L})$ from (3.1). The L^p boundedness was studied by various authors including Mauceri [32] and Thangavelu [41, 43]; in particular, Thangavelu [41] showed that if $\mathcal{K} = \lfloor n/2 \rfloor + 1$ then T_σ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Subsequently, weighted estimates were considered by Thangavelu [43] and by Harboure and her collaborators [25, 22]. The latter showed that by increasing the number of derivatives from $\lfloor n/2 \rfloor + 1$ to $n+1$, one can obtain weighted estimates. More precisely the following result holds.

Theorem 4.1 (Weighted estimates for Hermite multipliers; [22, Theorem 1.6])**.** *Let $\sigma \in \ell^\infty(\mathbb{N}_0^n)$ satisfy* (4.1) *with $\mathcal{K} = n+1$. If $1 < p < \infty$, the Hermite multiplier T_σ extends to a bounded operator on $L^p(w)$ for each $w \in A_p$.*

Here A_p is the standard class of Muckenhoupt weights. Note that this result can be recovered from the theory of pseudo-multipliers in Section 3 since it can be readily seen that σ belongs to the counterpart of $S^{0,n+1,\infty}_{1,0}$ defined in terms of the multivariate forward difference operator Δ, which turns out to generate Calderón–Zygmund operators (see Corollary 3.6 and the comments following).

In another direction, Harboure and co-authors introduced, in the seminal work [5], a new class of weights which has generated an extensive line of inquiry (see [3, 4, 11, 14, 28, 46] for a recent selection). These weights extend the Muckenhoupt weights and their appeal appears to be their suitability in the study of certain Schrödinger-type operators. They are defined via a so-called 'critical radius' function $\varrho : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies, at a minimum, the following growth property: there exist constants $C, k_0 > 0$ such that for every $x, y \in \mathbb{R}^n$

$$\varrho(y) \leq C\varrho(x)\Big(1 + \frac{|x-y|}{\varrho(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{4.2}$$

Such functions are built from and are typically used in the analysis of the Schrödinger operators under consideration; for the Hermite operator $\mathcal{L} = -\Delta + |x|^2$, it is standard to take $\varrho(x) := (1+|x|)^{-1}$.

One can now define the class of weights $\mathbb{A}^\theta_p = \mathbb{A}^\theta_p(\mathcal{L})$ related to $\mathcal{L}$ as follows. For $1 < p < \infty$ and $\theta \geq 0$ we say that $w \in \mathbb{A}^\theta_p$ if

$$[w]_{\mathbb{A}^\theta_p} := \sup_{B:\text{balls}} \Big(\fint_B w\Big)\Big(\fint_B w^{-\frac{1}{p-1}}\Big)^{p-1}\Big(1 + \frac{r_B}{\varrho(x_B)}\Big)^{-\theta} < \infty,$$

where r_B and x_B are the radius and center of B, respectively. See [5] for further details and properties of these weight classes.

The main aim of this section is to show that the weighted estimates of Theorem 4.1 can be extended to the class of weights $\mathbb{A}^\theta_p$, in some sense unifying the two lines of Harboure's work described above. More precisely we shall prove the following result.

Theorem 4.2 (Weighted estimates for Hermite multipliers for $\mathbb{A}^\theta_p$)**.** *Let $1 < p < \infty$, $\theta \geq 0$, and $\sigma : \mathbb{N}^n_0 \to \mathbb{C}$ satisfy* (4.1) *for $\mathcal{K} = n + 1 + \lfloor \theta_p \rfloor$, where $\theta_p = \frac{\theta}{p-1}\max\{1 + k_0, k_0 - n/2\}$ and k_0 is a positive constant from* (4.2)*. Then the Hermite multiplier T_σ extends to a bounded operator on $L^p(w)$ for every $w \in \mathbb{A}^\theta_p$ with*

$$\|T_\sigma f\|_{L^p(w)} \lesssim [w]_{\mathbb{A}^\theta_p}^{\max\{\frac{3}{2(p-1)}, \frac{p}{p-1}, \frac{3}{2}\}} \|f\|_{L^p(w)}.$$

We will adopt the approach from [22, 43] which relies on Littlewood–Paley 'g-functions'. Let ψ be an even function in $\mathscr{S}(\mathbb{R}^n)$ with $\psi(0) = 0$. Then consider the following generalized g and g^*-functions associated to the Hermite operator:

$$g_{\mathcal{L},\psi}(f) = \Big(\int_0^\infty \big|\psi(t\sqrt{\mathcal{L}})f\big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}},$$

and

$$g^*_{\mathcal{L},\psi,\lambda}(f) = \Big(\int_0^\infty \int_{\mathbb{R}^n} \Big(1 + \frac{|x-y|}{t}\Big)^{-\lambda n} \big|\psi(t\sqrt{\mathcal{L}})f\big|^2 \frac{dy\,dt}{t^{1+n}}\Big)^{\frac{1}{2}}, \qquad \lambda > 0.$$

Observe that if we take $\psi_\ell(z) = z^{2\ell}e^{-z^2}$ then, up to a harmless constant, $g_{\mathcal{L},\psi_\ell}$ and $g^*_{\mathcal{L},\psi_1,\lambda}$ coincide respectively with the g_ℓ and g^*_λ functions in [22].

Let us now gather some of the necessary facts that will be needed in the proof of Theorem 4.2.

Proposition 4.3 (Weighted estimates for $g^*_{\mathcal{L},\psi,\lambda}$; [11, Theorem 1.2])**.** *Let $\psi \in \mathscr{S}(\mathbb{R}^n)$ be an even function with $\psi(0) = 0$. Let $1 < p < \infty$, $\theta \geq 0$ and $\lambda > 2+2\theta_p/n$. Then for any $w \in \mathbb{A}_p^\theta$ we have*

$$\|g^*_{\mathcal{L},\psi,\lambda}(f)\|_{L^p(w)} \lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

Proposition 4.4 ([22, Proposition 2.4])**.** *Let $\lambda > 1$ and $\sigma : \mathbb{N}_0^n \to \mathbb{C}$ satisfy* (4.1) *with $\mathcal{K} = \lfloor \lambda n/2\rfloor + 1$. Then for any $\ell \geq \lambda n/2 + 1$ we have*

$$g_{\mathcal{L},\psi_\ell}(T_\sigma f) \lesssim g^*_{\mathcal{L},\psi_1,\lambda}(f)(x), \qquad a.e.\ x \in \mathbb{R}^n.$$

Proposition 4.5 (Weighted estimates for $g_{\mathcal{L},\psi}$)**.** *Let $\psi \in \mathscr{S}(\mathbb{R}^n)$ be an even function with $\psi(0) = 0$. Then for any $1 < p < \infty$, $\theta \geq 0$ and $w \in \mathbb{A}_p^\theta$ we have $\|g_{\mathcal{L},\psi}(f)\|_{L^p(w)} \sim \|f\|_{L^p(w)}$. In particular it holds that*

$$[w]_{\mathbb{A}_p^\theta}^{-\max\{\frac{1}{2(p-1)},1\}}\|f\|_{L^p(w)} \lesssim \|g_{\mathcal{L},\psi}(f)\|_{L^p(w)} \lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}\|f\|_{L^p(w)}. \tag{4.3}$$

The right-hand side inequality in (4.3) was obtained in [11, Theorem 1.2]. We shall provide a proof of the left-hand side inequality at the end of this section.

We will also need to use the following 'duality' fact concerning the $\mathbb{A}_p^\theta$ weights: it holds that $w \in \mathbb{A}_p^\theta$ if and only if $w^{1-p'} \in \mathbb{A}_{p'}^{\theta(p'-1)}$. Indeed, it can be checked easily from the definition that

$$\left[w^{1-p'}\right]_{\mathbb{A}_{p'}^{\theta(p'-1)}} = [w]_{\mathbb{A}_p^\theta}^{\frac{1}{p-1}}. \tag{4.4}$$

We are now ready to give the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $\lambda = 2 + 2\theta_p/n + \varepsilon$ where ε is small enough such that $\frac{\varepsilon n}{2} \in (0, 1)$ and $\lfloor\theta_p + \frac{\varepsilon n}{2}\rfloor = \lfloor\theta_p\rfloor$. Then we see that

$$\mathcal{K} = n + 1 + \lfloor\theta_p\rfloor = \lfloor n + \theta_p + \tfrac{\varepsilon n}{2}\rfloor + 1 = \lfloor \tfrac{\lambda n}{2}\rfloor + 1,$$

which means that the conditions of Proposition 4.4 are satisfied.

Now let $\psi_\ell(z) = z^{2\ell}e^{-z^2}$ with $\ell = n + 2 + \lfloor\theta_p\rfloor$. Then by Propositions 4.5, 4.4 and 4.3 we have

$$\begin{aligned}\|T_\sigma f\|_{L^p(w)} &\lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2(p-1)},1\}}\|g_{\mathcal{L},\psi_\ell}(T_\sigma f)\|_{L^p(w)}\\ &\lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2(p-1)},1\}}\|g^*_{\mathcal{L},\psi_1,\lambda}(f)\|_{L^p(w)}\\ &\lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2(p-1)},1\}+\max\{\frac{1}{2},\frac{1}{p-1}\}}\|f\|_{L^p(w)}.\end{aligned}$$

By a simple calculation one finds that

$$\max\{\tfrac{1}{2(p-1)}, 1\} + \max\{\tfrac{1}{2}, \tfrac{1}{p-1}\} = \max\{\tfrac{3}{2(p-1)}, \tfrac{p}{p-1}, \tfrac{3}{2}\},$$

which completes our proof. □

Proof of Proposition 4.5. We note that the second inequality in (4.3) is directly from equation (15) of [11, Theorem 1.2] and we only need to obtain the first inequality. That is, we will show

$$\|f\|_{L^p(w)} \lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2(p-1)},1\}} \|g_{\mathcal{L},\psi}(f)\|_{L^p(w)}. \tag{4.5}$$

To prove (4.5) we will follow the approach from [43, Theorem 4.1.2].

Firstly we claim that

$$|\langle f_1, f_2\rangle| \le c_\psi^{-1} \int_{\mathbb{R}^n} g_{\mathcal{L},\psi}(f_1)\, g_{\mathcal{L},\psi}(f_2)\, dx, \qquad f_1 \in L^p(\mathbb{R}^n), f_2 \in L^{p'}(\mathbb{R}^n), \tag{4.6}$$

where $c_\psi = \int_0^\infty \psi(s)^2 ds/s$.

Assuming (4.6) for the moment we proceed with the proof of (4.5). Let $w \in \mathbb{A}_p^\theta$. Then by (4.6) and Hölder's inequality we obtain

$$|\langle f w^{\frac{1}{p}}, \widetilde{f}\rangle| \lesssim \int_{\mathbb{R}^n} g_{\mathcal{L},\psi}(f)\, g_{\mathcal{L},\psi}(w^{\frac{1}{p}}\widetilde{f})\, dx \le \big\|g_{\mathcal{L},\psi}(f)\big\|_{L^p(w)} \big\|g_{\mathcal{L},\psi}(w^{\frac{1}{p}}\widetilde{f}) w^{-\frac{1}{p}}\big\|_{L^{p'}}.$$

Now since $w^{1-p'} \in \mathbb{A}_{p'}^{\theta(p'-1)}$ (see (4.4)) then we may apply the right hand inequality of (4.3) on $L^{p'}(w^{1-p'})$ to see that

$$\begin{aligned}\big\|g_{\mathcal{L},\psi}(w^{\frac{1}{p}}\widetilde{f}) w^{-\frac{1}{p}}\big\|_{L^{p'}} &= \big\|g_{\mathcal{L},\psi}(w^{\frac{1}{p}}\widetilde{f})\big\|_{L^{p'}(w^{1-p'})} \\ &\lesssim \big[w^{1-p'}\big]_{\mathbb{A}_{p'}^{\theta(p'-1)}}^{\max\{\frac{1}{2},\frac{1}{p'-1}\}} \big\|w^{\frac{1}{p}}\widetilde{f}\big\|_{L^{p'}(w^{1-p'})}.\end{aligned}$$

Inserting this calculation into the previous inequality, and then invoking (4.4) along with the fact that $\big\|w^{\frac{1}{p}}\widetilde{f}\big\|_{L^{p'}(w^{1-p'})} = \|\widetilde{f}\|_{L^{p'}}$ we arrive at

$$|\langle f w^{\frac{1}{p}}, \widetilde{f}\rangle| \lesssim [w]_{\mathbb{A}_p^\theta}^{\max\{\frac{1}{2(p-1)},1\}} \big\|g_{\mathcal{L},\psi}(f)\big\|_{L^p(w)} \|\widetilde{f}\|_{L^{p'}}.$$

Taking supremum over all $\|\widetilde{f}\|_{L^{p'}} \le 1$ in this latter expression we arrive at (4.5).

To complete the proof of Proposition 4.5 we need to show (4.6) which we now proceed with. Firstly observe that $g_{\mathcal{L},\psi}$ is an 'isometry' on $L^2(\mathbb{R}^n)$. That is,

$$\|g_{\mathcal{L},\psi}(f)\|_{L^2} = c_\psi \|f\|_{L^2}, \qquad \forall f \in L^2(\mathbb{R}^n) \tag{4.7}$$

Indeed, from the orthogonality of the Hermite functions we have

$$\|\psi(t\sqrt{\mathcal{L}})f\|_{L^2}^2 = \sum_{\xi\in\mathbb{N}_0^n} \psi(t\sqrt{2|\xi|+n})^2 \langle f, h_\xi\rangle^2.$$

This fact along with a change of variable gives

$$\begin{aligned}\|g_{\mathcal{L},\psi}(f)\|_{L^2}^2 = \int_0^\infty \|\psi(t\sqrt{\mathcal{L}})f\|_{L^2}^2 \frac{dt}{t} &= \sum_{\xi\in\mathbb{N}_0^n} \Big(\int_0^\infty \psi(s)^2 \frac{ds}{s}\Big) |\langle f, h_\xi\rangle|^2 \\ &= c_\psi \|\{\langle f, h_\xi\rangle\}\|_{\ell^2}^2.\end{aligned}$$

Invoking Parseval's identity then yields (4.7).

Continuing, we apply (4.7) with the parallelogram law to obtain the following 'polarization identity':

$$\langle f_1, f_2 \rangle = c_\psi^{-1} \int_0^\infty \left\langle \psi(t\sqrt{\mathcal{L}})f_1, \psi(t\sqrt{\mathcal{L}})f_2 \right\rangle \frac{dt}{t}. \tag{4.8}$$

Finally (4.6) follows from applying the Cauchy–Schwarz inequality to (4.8). This completes the proof of (4.6) and hence that of Proposition 4.5. □

Remark 4.6. The weighted quantitative estimates for Hermite multipliers for the classes $\mathbb{A}_p^\theta$ as given in Theorem 4.2 seem to be the first ones of their kind. The works [11, 10, 28, 46] contain weighted quantitative estimates related to the classes A_p^θ for other operators that include maximal and square functions, Riesz transforms, and spectral multipliers of Laplace transform type.

It is unclear whether the dependence of the $[w]_{\mathbb{A}_p^\theta}$ constant in Theorem 4.2 is sharp. For a comparison with the classical setting, we refer the reader to [27, Theorem 1.2], where weighted quantitative estimates are given for Marcinkiewicz multiplier operators in the context of Muckenhoupt weights.

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