REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 67, No. 1, 2024, Pages 15–25
Published online: February 21, 2024
https://doi.org/10.33044/revuma.2842

ON AN EXTENSION OF THE NEWTON POLYGON TEST FOR POLYNOMIAL REDUCIBILITY

BRAHIM BOUDINE

Abstract. Let R be a commutative local principal ideal ring which is not integral, f a polynomial in $R[x]$ such that $f(0) \neq 0$ and $N(f)$ its Newton polygon. If $N(f)$ contains r sides of different slopes, we show that f has at least r different pure factors in $R[x]$. This generalizes the Newton polygon method over a ring which is not integral.

1. Introduction

Let $(R, \pi R, k)$ be a commutative local principal ideal ring which is not integral, where πR is its maximal ideal for an element $\pi \in R$, and k its residual field. It is easy to show that R is a chain ring (all its ideals form a chain under inclusion) and its ideals are powers of πR. Then, πR is the unique prime ideal in R, and it follows that R is a special principal ideal ring which is not a field (see [2, Definition 14.3]). Therefore, $\pi R = \mathrm{Nil}(R)$, and π is nilpotent. Let e be the index of nilpotency of π. By abuse of notation, we shall often write k in place of $U(R) \cup \{0\}$, where $U(R)$ is the set of units of R. Then, we get the same result as that obtained by Dinh and Lopez-Permouth in [6], that the ideals of R are

$$(0) \subset \pi^{e-1}R \subset \ldots \subset \pi R \subset R.$$

Further, it is easy to show that the result obtained by McDonald in [12, pp. 339–341] holds in our case:

$$\forall x \in R,\ \exists!(u_0, \ldots, u_{e-1}) \in k^e, \quad x = u_0 + u_1\pi + \ldots + u_{e-1}\pi^{e-1}.$$

$K[X]/(f(X)^n)$ is an example of an special principal ideal ring which is not integral, where K is a field, f is an irreducible polynomial in $K[X]$, and n is a positive integer.

Since R contains a finite number of ideals, it is a complete ring (see [7, p. 182]). As well, Theorem 2.3 in [14] shows that R is a Henselian ring, that is, a ring in which Hensel's lemma holds [1, p. 134].

2020 *Mathematics Subject Classification.* 13P05, 11C08, 11S05.
Key words and phrases. Newton polygon, polynomials, special principal ideal ring.

Lemma 1.1 (Hensel's lemma)**.** *Let R be a complete local Noetherian ring and let f be in $R[x]$ such that $\overline{f} = g_1 \times \ldots \times g_k$ in $k[x]$, where $g_1, \ldots, g_k$ are pairwise coprime polynomials in $k[x]$. Then, there are $G_1, \ldots, G_k \in R[x]$ such that*

$$\begin{cases} f = G_1 \times \ldots \times G_k & \text{in } R[x], \\ \overline{G_i} = g_i & \forall i \in \{1, \ldots, k\}. \end{cases}$$

We already proved in [4] that every polynomial in $R[x]$ can be written as $\pi^v f$, where v is an integer and f is a primitive polynomial. Moreover, in the same paper, we proved that every primitive polynomial is associated with a monic polynomial. Then, the study of the polynomial factorization will be reduced to the case of monic polynomials.

In this paper, we investigate the factorization of monic polynomials which satisfy $f(0) \in \pi R$. So we can easily generalize the Eisenstein criterion.

Lemma 1.2. *Let $f(x) = a_0 + \ldots + a_n x^n$ be a monic polynomial in $R[x]$, where $a_0 \notin \pi^2 R$ and $a_i \in \pi R$ for every $i \in \{0, \ldots, n-1\}$. Then, f is irreducible.*

Proof. Assume that $f = gh$. We can assume that g and h are monic polynomials. Moreover, $\overline{f} = \overline{x^n} = \overline{gh}$, then $\overline{g} = \overline{x^{n-s}}$ and $\overline{h} = \overline{x^s}$ for some positive integer s. Thus, $g = \pi g' + x^{n-s}$ and $h = \pi h' + x^s$ for some polynomials g' and h' in $R[x]$. Therefore, $a_0 = g(0)h(0) \in \pi^2 R$, which contradicts our assumption. □

That may be sufficient when $a_0 \notin \pi^2 R$, but we need something stronger for the general case of monic polynomials satisfying $f(0) \in \pi R$. Therefore we extend the Newton polygon method.

The Newton polygon was introduced by Ore [13] over a field of p-adic numbers, generalized later to any valued field by Cohen et al. [5] and fantastically developed by Guardia et al. [10], Khanduja and Kumar [11], and El Fadil [8]. In this work, we generalize the techniques of Newton polygons over a ring not even integral.

The second section will be devoted to presenting all the necessary tools and interesting lemmas that we will need to prove our main result, which will be presented in the last section.

2. Preliminaries and lemmas

Throughout this paper, R means the special principal ideal ring $(R, \pi R, k, e)$ which is not a field, πR its maximal ideal, k its residual field and e the index of nilpotency of π.

We define V as follows:

$$V(x) = \begin{cases} \max\{k \in \mathbb{N} \mid x \in \pi^k R\} & \text{if } x \neq 0, \\ +\infty & \text{if } x = 0. \end{cases}$$

We remark the following statements:

- $V(xy) \geq V(x) + V(y)$ for every $x, y \in R$, and the equality holds when $xy \neq 0$.
- $V(x+y) \geq \min(V(x), V(y))$ for every $x, y \in R$, and the equality holds when $V(x) \neq V(y)$.

Let $f(X) = a_0 + a_1X + \ldots + a_{n-1}X^{n-1} + X^n$ be a monic polynomial in $R[X]$ such that $a_0 \neq 0$. The Newton polygon $N(f)$ of f is the lower boundary of the convex hull of the set $\{(i, V(a_i)) \mid i \in \{0, \ldots, n\}$ and $a_i \neq 0\}$ (see [8]).

If $N(f)$ contains the sides $S_1, \ldots, S_r$ of several slopes $0 \geq -\lambda_1 > \ldots > -\lambda_r$ respectively, where for each $i \in \{1, \ldots, r\}$ the initial point of S_i is (x_{i-1}, y_{i-1}) and its final point is (x_i, y_i), then f is called of type $(l_1, -\lambda_1; l_2, -\lambda_2; \ldots; l_r, -\lambda_r)$, where $l_i = x_i - x_{i-1}$ is the length of S_i for every $i \in \{1, \ldots, r\}$ (see [3]). Furthermore, if $N(f)$ has only one side, then f is called a pure polynomial [3]. Notice that the following statements are equivalent:

(1) f is pure and the slope of $N(f)$ is equal to $-\lambda$.
(2) $V(a_0) = n\lambda$, and $V(a_i) \geq (n-i)\lambda$ for each $i \in \{0, \ldots, n\}$.

Lemma 2.1. *Let f and g be two pure monic polynomials in $R[X]$ for which $N(f)$ and $N(g)$ have the same slope $-\lambda$ and $f(0).g(0) \neq 0$. Then, fg is also a pure monic polynomial and $N(fg)$ has the same slope $-\lambda$.*

Proof. Let

$$\begin{cases} f(x) = a_0 + a_1X + \ldots + a_{n-1}X^{n-1} + X^n, \\ g(x) = b_0 + b_1X + \ldots + b_{m-1}X^{m-1} + X^m. \end{cases}$$

Since f and g are pure,

$$\begin{cases} V(a_0) = n\lambda, \\ \forall i \in \{0, \ldots, n\},\ V(a_i) \geq (n-i)\lambda, \end{cases} \quad , \quad \begin{cases} V(b_0) = m\lambda, \\ \forall i \in \{0, \ldots, m\},\ V(b_i) \geq (m-i)\lambda. \end{cases}$$

Set $f(x)g(x) = c_0 + c_1X + \ldots + c_{n+m-1}X^{n+m-1} + X^{n+m}$, where $c_i = \sum_{k=0}^{i} a_k b_{i-k}$ with $a_k = 0$ if $k > n$ or $k < 0$ and $b_k = 0$ if $k > m$ or $k < 0$.

(1) $V(c_0) = V(a_0b_0) = V(a_0) + V(b_0) = n\lambda + m\lambda = (n+m)\lambda$ since $a_0b_0 \neq 0$.
(2) $V(c_i) = V\left(\sum_{k=0}^{i} a_k b_{i-k}\right) \geq \min(V(a_k b_{i-k}) \mid k \in \{0, \ldots, i\})$. For $k \in \{0, \ldots, i\}$, $V(a_k b_{i-k}) \geq V(a_k) + V(b_{i-k}) \geq (n-k)\lambda + (m-i+k)\lambda = (n+m-i)\lambda$. Then, $V(c_i) \geq \min(V(a_k b_{i-k}) \mid k \in \{0, \ldots, i\}) \geq (n+m-i)\lambda$.

Therefore, fg is pure and the slope of $N(fg)$ is $-\lambda$. □

Lemma 2.2. *Let $f \in R[x]$ be a monic polynomial of type $(l_1, -\lambda_1; l_2, -\lambda_2; \ldots; l_r, -\lambda_r)$, where $\deg(f) = \sum_{i=1}^{r} l_i$, and let $g \in R[x]$ be a pure monic polynomial of type $(l_{r+1}, -\lambda_{r+1})$ where $\lambda_r > \lambda_{r+1}$ and $\deg(g) = l_{r+1}$. If $f(0)g(0) \neq 0$, then fg is a monic polynomial of type $(l_1, -\lambda_1; l_2, -\lambda_2; \ldots; l_r, -\lambda_r; l_{r+1}, -\lambda_{r+1})$.*

Proof. Set $s_i = \sum_{k=1}^{i} l_i$ and

$$\begin{cases} f(x) = a_0 + a_1X + \ldots + a_{n-1}X^{n-1} + X^n, \\ g(x) = b_0 + b_1X + \ldots + b_{m-1}X^{m-1} + X^m. \end{cases}$$

Then:

- $\forall i \in \{1,\dots,r\}$, $V(a_{s_i}) = \sum\limits_{k=i+1}^{r} l_k\lambda_k$.
- $\forall i \in \{1,\dots,r\}$, if $j < s_i$, then $V(a_j) \geq V(a_{s_i}) + (s_i - j)\lambda_i$.
- $\forall i \in \{1,\dots,r-1\}$, if $j > s_i$, then $V(a_j) \geq V(a_{s_i}) - (j - s_i)\lambda_{i+1}$.
- $V(b_0) = l_{r+1}\lambda_{r+1}$.
- $\forall i \in \{0,\dots,m\}$, $V(b_i) \geq (l_{r+1} - i)\lambda_{r+1}$.

Set $t_i = \sum\limits_{k=i+1}^{r+1} l_i\lambda_i$ and $f(x)g(x) = c_0 + c_1X + \ldots + c_{n+m-1}X^{n+m-1} + X^{n+m}$, where $c_i = \sum\limits_{k=0}^{i} a_k b_{i-k}$ with $a_k = 0$ if $k > n$ or $k < 0$ and $b_k = 0$ if $k > m$ or $k < 0$.

We should prove the following statements:

(1) $\forall i \in \{1,\dots,r\}$, $V(c_{s_i}) = \sum\limits_{k=i+1}^{r+1} l_k\lambda_k$.
(2) $\forall i \in \{1,\dots,r+1\}$, if $j < s_i$, then $V(c_j) \geq V(c_{s_i}) + (t_i - j)\lambda_i$.
(3) $\forall i \in \{1,\dots,r\}$, if $j > t_i$, then $V(c_j) \geq V(c_{s_i}) - (j - t_i)\lambda_{i+1}$.

Let us proceed.

(1) Let $c_{s_i} = \sum\limits_{k=0}^{s_i} a_k b_{s_i-k}$. Notice that $c_{s_{r+1}} = a_n b_m = 1$. Therefore, $V(c_{s_{r+1}}) = 0$. Suppose now that $i < r+1$ and set $k \in \{0,\dots,s_i\}$. If $k = s_i$, we get $V(a_k b_{s_i-k}) = V(a_{s_i} b_0) = \sum\limits_{j=i+1}^{r+1} l_j\lambda_j$ since $a_{s_i} b_0 \neq 0$ ($V(a_{s_i}) \leq V(a_0)$ and $a_0 b_0 \neq 0$). Else, let $k \in \{0,\dots,s_i - 1\}$. We have

$$\begin{cases} V(a_k) \geq V(a_{s_i}) + (s_i - k)\lambda_i, \\ V(b_{s_i-k}) \geq (l_{r+1} - (s_i - k))\lambda_{r+1}. \end{cases}$$

Therefore, $V(a_k b_{s_i-k}) \geq V(a_k) + V(b_{s_i-k}) \geq \sum\limits_{k=i+1}^{r+1} l_k\lambda_k + (s_i - k)(\lambda_i - \lambda_{r+1})$. Since $\lambda_i > \lambda_{r+1}$, $V(a_k b_{s_i-k}) > \sum\limits_{k=i+1}^{r+1} l_k\lambda_k$. Thus, $V(c_{s_i}) = \sum\limits_{k=i+1}^{r+1} l_k\lambda_k$.

(2) Let $i \in \{1,\dots,r+1\}$ and $j < s_i$; then $c_j = \sum\limits_{k=0}^{j} a_k b_{j-k}$. For any $k \in \{0,\dots,j\}$, we have the following inequalities:

$$\begin{cases} V(a_k) \geq V(a_{s_i}) + (s_i - k)\lambda_i, \\ V(b_{j-k}) \geq (l_{r+1} - (j - k))\lambda_{r+1}. \end{cases}$$

Therefore, $V(a_k b_{j-k}) \geq V(a_k) + V(b_{j-k}) \geq \sum\limits_{k=i+1}^{r+1} l_k\lambda_k + (s_i - k)\lambda_i - (j - k)\lambda_{r+1}$. However, $(s_i - k)\lambda_i - (j-k)\lambda_{r+1} = (s_i - j)\lambda_i + (j-k)(\lambda_i - \lambda_{r+1}) \geq$

$(s_i - j)\lambda_i$. Thus, $V(a_k b_{j-k}) \geq \sum_{k=i+1}^{r+1} l_k\lambda_k + (s_i - j)\lambda_i$ for every $k \in \{0, \ldots, j\}$. Hence, $V(c_j) \geq \sum_{k=i+1}^{r+1} l_k\lambda_k + (s_i - j)\lambda_i = V(c_{s_i}) + (s_i - j)\lambda_i$.

(3) Let $i \in \{1, \ldots, r\}$ and $j > s_i$; then $c_j = \sum_{k=0}^{j} a_k b_{j-k}$. If $k \leq s_i$, we have the following inequalities:

$$\begin{cases} V(a_k) \geq V(a_{s_i}) + (s_i - k)\lambda_i, \\ V(b_{j-k}) \geq (l_{r+1} - (j - k))\lambda_{r+1}. \end{cases}$$

Then we get like in the previous part, $V(a_k b_{j-k}) \geq V(c_{s_i})$. If $k > s_i$, we have the following inequalities:

$$\begin{cases} V(a_k) \geq V(a_{s_i}) - (k - s_i)\lambda_{i+1}, \\ V(b_{j-k}) \geq (l_{r+1} - (j - k))\lambda_{r+1}. \end{cases}$$

Then, $V(a_k b_{j-k}) \geq V(a_k) + V(b_{j-k}) \geq V(c_{s_i}) - (k - s_i)\lambda_{i+1} - (j - k)\lambda_{r+1}$. However, $-(k - s_i)\lambda_{i+1} - (j - k)\lambda_{r+1} = -(j - s_i)\lambda_{i+1} + (j - k)(\lambda_{i+1} - \lambda_{r+1}) \geq -(j - s_i)\lambda_{i+1}$. Therefore, $V(a_k b_{j-k}) \geq V(c_{s_i}) - (j - s_i)\lambda_{i+1}$ for any $k \in \{0, \ldots, j\}$. Thus, $V(c_j) \geq V(c_{s_i}) - (j - s_i)\lambda_{i+1}$.

This proves that fg is a monic polynomial of type $(l_1, -\lambda_1; l_2, -\lambda_2; \ldots; l_{r+1}, -\lambda_{r+1})$. ∎

Definition 2.3. Let $\lambda \in \mathbb{Q}^+$ and let $\frac{p}{q}$ be its irreducible form. We define the function V_λ by

$$V_\lambda : \quad R[x] \to \mathbb{N} \cup \{+\infty\}$$
$$f(x) = \sum_{k=0}^{n} a_k x^k \mapsto V_\lambda(f(x)) = \min\{qV(a_k) + pk \mid k \in \{0, \ldots, n\}\}.$$

Lemma 2.4. *The function V_λ satisfies the following properties:*

- $V_\lambda(f) = +\infty$ *if and only if* $f = 0$.
- $\forall f, g \in R[x]$, $V_\lambda(f + g) \geq \min(V_\lambda(f), V_\lambda(g))$.
- $\forall f, g \in R[x]$, *if* $f(0)g(0) \neq 0$, *then* $V_\lambda(fg) = V_\lambda(f) + V_\lambda(g)$.

Proof. Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ and $g(x) = b_0 + b_1x + \ldots + b_mx^m$, where $m \leq n$, $b_k = 0$ if $k > m$ and $a_k = 0$ if $k > n$.

- $V_\lambda(P) = +\infty$ if and only if for every $k \in \{0, \ldots, n\}$, $qV(a_k) + pk = +\infty$. This means that $V(a_k) = +\infty$. Thus, $a_k = 0$ for every $k \in \{0, \ldots, n\}$.
- For any $k \in \{0, \ldots, n\}$, we have $V(a_k + b_k) \geq \min(V(a_k), V(b_k))$.
  Then, $V_\lambda(f + g) \geq \min\{q\min(V(a_k), V(b_k)) + pk \mid k \in \{0, \ldots, n\}\} \geq \min(V_\lambda(f), V_\lambda(g))$.

- Set $f(x)g(x) = \sum_{i=0}^{n+m} c_i x^i$, where $c_i = \sum_{k=0}^{i} a_k b_{i-k}$ for every $i \in \{0, \ldots, n+m\}$. Assume that
$$\begin{cases} r = \min\{k \in \{0, \ldots, n\} \mid V_\lambda(f) = qV(a_k) + k.p\}, \\ s = \min\{k \in \{0, \ldots, m\} \mid V_\lambda(g) = qV(b_k) + k.p\}. \end{cases}$$
Then,
$$\begin{cases} V(a_k) \geq V(a_r) + (r-k)\lambda \forall k \in \{0, \ldots, n\}, \\ V(b_k) \geq V(b_s) + (s-k)\lambda \forall k \in \{0, \ldots, m\}. \end{cases}$$
Thus, $V(a_k b_{i-k}) \geq V(a_k) + V(b_{i-k}) \geq V(a_r) + V(b_s) + (r+s-i)\lambda$ for every $k \in \{0, \ldots, i\}$. Therefore, $qV(a_k b_{i-k}) + p.i \geq q(V(a_r) + V(b_s) + (r+s-i)\lambda) + p.i = V_\lambda(f) + V_\lambda(g)$. Thus, $qV(c_i) + p.i \geq V_\lambda(f) + V_\lambda(g)$ for each $i \in \{0, \ldots, m+n\}$. Hence, $V_\lambda(fg) \geq V_\lambda(f) + V_\lambda(g)$. Then, for $i = r+s$, if $k \in \{0, \ldots, r+s\}$, we distinguish some cases:

  Case 1: $k < r$. In this case, by the definition of r, we get $V(a_k) > V(a_r) + (r-k)\lambda$. Thus, $qV(a_k b_{r+s-k}) + p(r+s) \geq qV(a_k) + qV(b_{r+s-k}) + p(r+s) > V_\lambda(f) + V_\lambda(g)$.

  Case 2: $k > r$. Likewise, by the definition of s, we get $V(b_{r+s-k}) > V(b_s) + (-r+k)\lambda$. Thus, $qV(a_k b_{r+s-k}) + p(r+s) > V_\lambda(f) + V_\lambda(g)$.

  Case 3: $k = r$. Notice that
$$\begin{cases} qV(a_r) + rp \leq qV(a_0) \Rightarrow V(a_r) \leq V(a_0) - r\lambda, \\ qV(b_s) + sp \leq qV(b_0) \Rightarrow V(b_s) \leq V(b_0) - s\lambda. \end{cases}$$
Then, $V(a_r b_s) \leq V(a_r) + V(b_s) < V(a_0) + V(b_0) < e$ since $a_0 b_0 \neq 0$. Thus, $qV(a_r b_s) + p(r+s) = qV(a_r) + qV(b_s) + p(r+s) = V_\lambda(f) + V_\lambda(g)$.

  Therefore, $qV(c_{r+s}) + p(r+s) = V_\lambda(f) + V_\lambda(g)$. Hence, $V_\lambda(fg) = V_\lambda(f) + V_\lambda(g)$. ☐

Corollary 2.5. *Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$ be a monic polynomial. Suppose that $\deg(f) = n = \min\{k \in \mathbb{N} \mid V_\lambda(f) = qV(a_k) + pk\}$. Then for every polynomial $g = b_0 + b_1 x + \ldots + b_N x^N \in R[x]$, $V_\lambda(fg) = V_\lambda(f) + V_\lambda(g)$. Furthermore, if $j = \min\{k \in \mathbb{N} \mid qV(b_k) + pk\}$, then $V_\lambda(fg) = qV\big(\sum_{k=0}^{n+j} a_k b_{n+j-k}\big) + p(n+j)$.*

Proof. If we use the same notation as in the proof of the third statement of the previous lemma, we get $r = n$ and by the same method we obtain $V_\lambda(fg) = V_\lambda(f) + V_\lambda(g)$. The fact that $a_n = 1$, means that $V(a_r b_s) = V(a_n b_s) = V(a_n) + V(b_s) = V(b_s)$ and this allows us to avoid the assumption that $f(0)g(0) \neq 0$. ☐

Lemma 2.6. *Let $f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n$ be a monic polynomial in $R[x]$ of type $(l_1, -\lambda_1; \ldots; l_r, -\lambda_r)$ satisfying $a_0 \neq 0$ and $\lambda = \frac{p}{q} \in \mathbb{Q}^+$ such that p and q are coprime and $\lambda_r < \lambda < \lambda_{r-1}$. Then:*

(1) *If $i < \sum_{i=1}^{r-1} l_i = N$, then $V_\lambda(a_i x^i) > V_\lambda(a_N x^N)$.*

(2) *If* $i > \sum_{i=1}^{r-1} l_i = N$, *then* $V_\lambda(a_i x^i) > V_\lambda(a_N x^N)$.

Furthermore, $V_\lambda(f) = V_\lambda(a_N x^N)$.

Proof. Set $N = \sum_{i=1}^{r-1} l_i$.

(1) Assume that $i < N$:
We have $V_\lambda(a_i x^i) = qV(a_i) + p.i \geq q(V(a_N) + (N-i)\lambda_{r-1}) + p.i$.
Since $\lambda < \lambda_{r-1}$, we get

$$q(V(a_N) + (N-i)\lambda_{r-1}) + p.i = qV(a_N) + Np + (N-i)(\lambda_{r-1}q - p) > V_\lambda(a_N x^N).$$

Therefore, $V_\lambda(a_i x^i) > V_\lambda(a_N x^N)$.

(2) Assume that $i > N$:
We have $V_\lambda(a_i x^i) = qV(a_i) + p.i \geq q(V(a_N) - (i-N)\lambda_r) + p.i$.
Since $\lambda > \lambda_r$, we get

$$q(V(a_N) - (i-N)\lambda_r) + p.i = qV(a_N) + Np - (i-N)(\lambda_r q - p) > V_\lambda(a_N x^N).$$

Therefore, $V_\lambda(a_i x^i) > V_\lambda(a_N x^N)$. ☐

Lemma 2.7 ([9, Lemma 15.9 (i)])**.** *Let* f *be a monic polynomial of degree* n *and* g *be a monic polynomial of degree* N *in* $R[x]$. *Then, there exist polynomials* q *and* r *such that* $\deg(r) < N$ *and* $f = qg + r$.

Lemma 2.8. *Let* f *be a monic polynomial of degree* n, $g = g_0 + g_1 x + \ldots + g_N x^N$ *be a monic polynomial of degree* $N < n$ *with* $g_N = 1$, q *and* r *be polynomials in* $R[x]$ *such that* $f = qg + r$, $\deg(r) < \deg(g)$ *and* $\deg(g) = N = \min\{k \in \mathbb{N} \mid V_\lambda(g) = V_\lambda(g_k x^k)\}$. *Then,* $V_\lambda(q) \geq V_\lambda(f) - V_\lambda(g)$ *and* $V_\lambda(r) \geq V_\lambda(f)$.

Proof. Step 1: We prove first that $V_\lambda(q) \geq V_\lambda(f) - V_\lambda(g)$ is equivalent to $V_\lambda(r) \geq V_\lambda(f)$.

By Corollary 2.5, $V_\lambda(qg) = V_\lambda(q) + V_\lambda(g)$. Then, $V_\lambda(q) \geq V_\lambda(f) - V_\lambda(g)$ implies that $V_\lambda(qg) \geq V_\lambda(f)$. However, $r = f - qg$ shows that $V_\lambda(r) = V_\lambda(f - gq) \geq \min(V_\lambda(f), V_\lambda(gq)) = V_\lambda(f)$. Conversely, suppose that $V_\lambda(r) \geq V_\lambda(f)$. Since $qg = f - r$, $V_\lambda(qg) \geq \min(V_\lambda(f), V_\lambda(r)) = V_\lambda(f)$, so we get the result by Corollary 2.5.

Step 2: Suppose that $V_\lambda(q) < V_\lambda(f) - V_\lambda(g)$ and $V_\lambda(r) < V_\lambda(f)$. Then, $V_\lambda(qg) < V_\lambda(f)$. It follows that $V_\lambda(gq + r) > \max\{V_\lambda(qg), V_\lambda(r)\}$.

Set $q(x) = q_0 + \ldots + q_{n-N-1}x^{n-N-1} + x^{n-N}$ and $q(x)g(x) = c_0 + \ldots + c_{n-1}x^{n-1} + x^n$.

By Corollary 2.5, $V_\lambda(qg) = V_\lambda(c_{N+j}x^{N+j})$, where $j = \min\{k \in \mathbb{N} \mid V_\lambda(q) = V_\lambda(q_k x^k)\}$. However, $\deg(r) < N \leq N + j$, then we have $V_\lambda(f) = V_\lambda(qg + r) \leq V_\lambda(a_{N+j}x^{N+j}) = V_\lambda(c_{N+j}x^{N+j}) = V_\lambda(qg)$, which is a contradiction. ☐

3. Main results

Theorem 3.1. *Let* f *be a monic polynomial in* $R[x]$ *such that* $f(0) \neq 0$. *If* f *is irreducible, then* $N(f)$ *has only one side.*

Proof. Assume that $f(x) = a_0 + \ldots + a_{n-1}x^{n-1} + x^n$ of type $(l_1, -\lambda_1; \ldots; l_r, -\lambda_r)$, where $0 \leq \lambda_r < \ldots < \lambda_1$. Let $\frac{p}{q}$ be the irreducible form of λ which satisfies $\lambda_r < \lambda < \lambda_{r-1}$. Set $\delta = p - q\lambda_r > 0$ and $N = \sum_{k=1}^{r-1} l_k$. Then, by Lemma 2.6 we get $V_\lambda(f) = V_\lambda(a_N x^N) = qV(a_N) + pN$.

For $g_1(x) = \sum_{k=0}^{N} a_k x^k$ and $h_1(x) = 1$, we get

$$\begin{cases} \deg(g_1) = N \quad \text{and} \quad \deg(h_1) \leq N - n, \\ V_\lambda(f - g_1) \geq \delta + V_\lambda(f) \quad \text{and} \quad V_\lambda(h_1 - 1) \geq \delta, \\ V_\lambda(f - g_1 h_1) \geq 1 \times \delta + V_\lambda(f), \\ V(g_1) = V(lc(g_1)) = a_N, \\ N = \min\{k \in \{0, \ldots, N\} \mid V_\lambda(g_1) = qV(a_k) + pk\}. \end{cases}$$

Indeed, $a_N \neq 0$ because the bottom part of $N(f)$ has a slope change at $(N, V(a_N))$.

Moreover, $V_\lambda(f - g_1) = qV(a_i) + p.i$ for a certain $i > N$. Then, $V_\lambda(f - g_1) \geq q(V(a_N) - (i-N)\lambda_r) + p_i = V_\lambda(f) + (i-N)\delta \geq V_\lambda(f) + \delta$. Finally, the last property is given by the fact that for every $k < N$, we have $qV(a_k) + kp > q(V(a_N) + (N-k)\lambda) + kp \geq qV(a_N) + pN$.

By induction, suppose that there are $g_k = b_0 + b_1 x + \ldots + b_N x^N$ and h_k such that

$$\begin{cases} \deg(g_k) = N \quad \text{and} \quad \deg(h_k) \leq n - N, \\ V_\lambda(f - g_k) \geq \delta + V_\lambda(f) \quad \text{and} \quad V_\lambda(h_k - 1) \geq \delta, \\ V_\lambda(f - g_k h_k) \geq k \times \delta + V_\lambda(f), \\ V(g_k) = V(lc(g_k)) = a_N, \\ N = \min\{j \in \{0, \ldots, N\} \mid V_\lambda(g_k) = qV(b_j) + pj\}. \end{cases}$$

We need to prove the existence of g_{k+1} and h_{k+1} satisfying the same properties above.

Let $v = V(a_N) = l_r\lambda_r$. Then, $lc(g_k) = \pi^v u$, where $u \notin \pi R$. Let $\tilde{g_k} = \frac{g_k}{\pi^v u}$ (by abuse of notation). Then, $\tilde{g_k}$ is a monic polynomial of degree N and $V_\lambda(\tilde{g_k}) = V_\lambda(g_k) - qv$.

We also have that $V(lc(g_k)) = V(a_N) > 0$. This means that $u'(f - g_k h_k)$ is a monic polynomial for a certain $u' \notin \pi R$. So we can apply the Euclidean division of Lemma 2.7: there are a polynomial q of degree $n - N$ and a polynomial r such that $\deg(r) < N$ and $f - g_k h_k = q\tilde{g_k} + r$. Set $g_{k+1} = g_k + r$ and $h_{k+1} = h_k + q$:

(1) Since $\deg(r) < N$ and $\deg(g_k) = N$, $\deg(g_{k+1}) = N$. Since $\deg(q) \leq n - N$ and $\deg(h_k) \leq n - N$, $\deg(h_{k+1}) \leq n - N$.
(2) We have

$$\begin{cases} V_\lambda(f - g_{k+1}) = V_\lambda(f - g_k - r) \geq \min(V_\lambda(f - g_k), V_\lambda(r)), \\ V_\lambda(h_{k+1} - 1) = V_\lambda(h_k - 1 + q) \geq \min(V_\lambda(h_k - 1), V_\lambda(q)). \end{cases}$$

By Lemma 2.8, we get

$$\begin{cases} V_\lambda(q) \geq V_\lambda(f - g_k h_k) - V_\lambda(\tilde{g_k}) \geq k\delta + V_\lambda(f) - V_\lambda(g_k) + qv = k\delta + qv \geq \delta, \\ V_\lambda(r) \geq V_\lambda(f - g_k h_k) \geq k\delta + V_\lambda(f) \geq \delta + V_\lambda(f). \end{cases}$$

Thus,

$$\begin{cases} V_\lambda(f - g_{k+1}) \geq \delta + V_\lambda(f), \\ V_\lambda(h_{k+1} - 1) \geq \delta. \end{cases}$$

(3) Remark that $f - g_{k+1}h_{k+1} = r(1 - h_k) - rq$. Then, $V_\lambda(f - g_{k+1}h_{k+1}) = V_\lambda(r(1-h_k)-rq) \geq \min(V_\lambda(r(1-h_k)), V_\lambda(rq))$. We have $V_\lambda(rq) \geq V_\lambda(r) + V_\lambda(q) \geq k\delta + V_\lambda(f) + \delta = (k+1)\delta + V_\lambda(f)$. Moreover, $V_\lambda(r(1 - h_k)) \geq V_\lambda(r) + V_\lambda(1 - h_k) \geq k\delta + V_\lambda(f) + \delta = (k+1)\delta + V_\lambda(f)$. Therefore, $V_\lambda(f - g_{k+1}h_{k+1}) \geq (k+1)\delta + V_\lambda(f)$.

(4) Let $r(x) = r_0 + \ldots + r_j x^j$ for some $j < N$, and let $g_{k+1}(x) = c_0 + \ldots + c_N x^N$. We have $V_\lambda(r) > V_\lambda(f)$; then, for every $0 \leq k \leq N-1$, $qV(r_k) + pk > qV(a_N) + pN$ and $qV(b_k) + pk > qV(a_N) + pN$, which implies that $V(r_k) > V(a_N) + (N-k)\lambda$ and $V(b_k) > V(a_N) + (N-k)\lambda$, thus $V(c_k) = V(r_k + b_k) > V(a_N) + (N-k)\lambda$. Therefore, $V(g_{k+1}) = V(lc(g_{k+1})) = V(a_N)$ and $N = \min\{j \in \{0, \ldots, N\} \mid V_\lambda(g_{k+1}) = qV(c_k) + pk\}$.

When k tends to $+\infty$, $V_\lambda(f - g_k h_k)$ tends to $+\infty$. Thus, $f - g_k h_k = 0$ for some integer k large enough. Therefore, there are g and h such that $f = gh$. □

The converse is not true in general.

Example 3.2. The ring $R = \mathbb{Z}/3^5\mathbb{Z}$ is a special principal ideal ring where $3R$ is its maximal ideal, $k = R/3R \sim \mathbb{Z}/3\mathbb{Z}$ is its residual field and $e = 5$ the index of nilpotency of $\pi = 3$.

Set the polynomial $f(x) = 135 + 99x + 21x^2 + x^3$.

$N(f)$ has only one side. However f is not irreducible since $f(x) = g(x)h(x)$, where

$$\begin{cases} g(x) = 9 + 6x + x^2, \\ h(x) = 15 + x. \end{cases}$$

Corollary 3.3. *Let f be a monic polynomial in $R[x]$ of type $(l_1, -\lambda_1; l_2, -\lambda_2; \ldots; l_r, -\lambda_r)$, with $f(0) \neq 0$. Then, there are some pure monic polynomials $g_1, \ldots, g_r$ in $R[x]$ such that $f = g_1 \times \ldots \times g_r$ and the slope of $N(g_i)$ is $-\lambda_i$ for every $i \in \{1, \ldots, r\}$.*

Proof. By Theorem 3.1, f is not irreducible. Then, $f = gh$ for some polynomials $g, h \in R[x]$. If either g or h is not pure, we can factorize it too. We continue until we get a product of pure polynomials $h_1, \ldots, h_s$ for some $s \in \mathbb{N}$. Lemma 2.2 shows that the slopes of the Newton polygons of these factors h_i belong to $\{-\lambda_1, \ldots, -\lambda_r\}$. Then we take g_i to be the product of all h_k for which the slope of the Newton polygon is $-\lambda_i$. As well Lemma 2.1 shows that the slope of $N(g_i)$ is $-\lambda_i$. □

Example 3.4. The ring $R = \mathbb{Z}/3^5\mathbb{Z}$ is a special principal ideal ring where $3R$ is its maximal ideal, $k = R/3R \sim \mathbb{Z}/3\mathbb{Z}$ is its residual field and $e = 5$ the index of nilpotency of $\pi = 3$.

Set the polynomial $f(x) = 81 + 27x + 189x^2 + 12x^3 + 36x^4 + x^5$.

Neither the Eisenstein criterion 1.2 nor Hensel's lemma 1.1 can assure if f is irreducible or not in $R[x]$. However, the Newton polygon method can do it.

The Newton polygon shows that f is of type $(3, -1; 2, -\frac{1}{2})$. Then, Corollary 3.3 assures that there are two pure monic polynomials $g, h \in R[x]$ such that $f = gh$. Indeed,

$$\begin{cases} g(x) = 27 + 9x + 27x^2 + x^3 \Rightarrow N(g) = (3, -1), \\ h(x) = 3 + 9x + x^2 \Rightarrow N(h) = (2, -\frac{1}{2}). \end{cases}$$

Example 3.5. The ring $R = \mathbb{R}[t]/t^8\mathbb{R}[t]$ is a special principal ideal ring where tR is its maximal ideal, $k = R/tR$ is its residual field and $e = 8$ the index of nilpotency of $\pi = t$.

Set the polynomial $f(x) = t^7 + 2t^6x + (t^5 + t^7)x^3 + 2t^4x^4 + t^2x^5 + (t^5 + 3t^6)x^6 + x^8$.

Neither the Eisenstein criterion 1.2 nor Hensel's lemma 1.1 can assure if f is irreducible or not in $R[x]$. However, the Newton polygon method can do it.

The Newton polygon shows that f is of type $(5, -1; 3, -\frac{2}{3})$. Then, Corollary 3.3 assures that there are two pure monic polynomials $g, h \in R[x]$ such that $f = gh$. Indeed,

$$\begin{cases} g(x) = t^5 + 2t^4x + (t^5 + 3t^6)x^3 + x^5 \Rightarrow N(g) = (5, -1), \\ h(x) = t^2 + x^3 \Rightarrow N(h) = (3, -\frac{2}{3}). \end{cases}$$

Acknowledgements

I would like to express many thanks to the anonymous referees and the editors for their corrections and editorial suggestions, which improved the comprehensibility and quality of the manuscript.

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Brahim Boudine
Faculty of sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah university, Fez, Morocco
brahimboudine.bb@gmail.com

Received: September 1, 2021
Accepted: June 11, 2022