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ON THE SECOND $\mathfrak{osp}(1|2)$-RELATIVE COHOMOLOGY OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON $\mathcal{C}^{1|1}$

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ABSTRACT. Let $\mathcal{K}(1)$ be the Lie superalgebra of contact vector fields on the $(1,1)$-dimensional complex superspace; it contains the Möbius superalgebra $\mathfrak{osp}(1|2)$. We classify $\mathfrak{osp}(1|2)$-invariant superanti-symmetric binary differential operators from $\mathcal{K}(1) \wedge \mathcal{K}(1)$ to $\mathfrak{D}_{\lambda,\mu}$ vanishing on $\mathfrak{osp}(1|2)$, where $\mathfrak{D}_{\lambda,\mu}$ is the superspace of linear differential operators acting on the superspaces of weighted densities. This result allows us to compute the second differential $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}$.

1. Introduction

Let $\mathfrak{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{C}$. Consider the 1-parameter deformation of the $\mathfrak{vect}(1)$-action on $\mathbb{C}[x]$:

$$L^{\lambda}_{X\frac{d}{dx}}(f) = Xf' + \lambda X' f,$$

where $X, f \in \mathbb{C}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathfrak{vect}(1)$-module structure on $\mathbb{C}[x]$ defined by L^λ for a fixed λ. Geometrically, $\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in \mathbb{C}[x] \right\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{C}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1$, 0 and 1, respectively.

Denote by $\mathrm{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\mathfrak{vect}(1)$-module of linear differential operators with the natural $\mathfrak{vect}(1)$-action denoted $L^{\lambda,\mu}_X(A)$. If we restrict ourselves to the Lie subalgebra of $\mathfrak{vect}(1)$ generated by $\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite-dimensional $\mathfrak{sl}(2)$-modules, still denoted by $\mathcal{F}_\lambda$ and $\mathrm{D}_{\lambda,\mu}$. Bouarroudj [6] computed the space

$$\mathrm{H}^2_{\mathrm{diff}}\left(\mathfrak{vect}(1), \mathfrak{sl}(2); \mathrm{D}_{\lambda,\mu}\right),$$

where $\mathrm{H}^i_{\mathrm{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the $\mathfrak{sl}(2)$-trivial deformations of the $\mathfrak{vect}(1)$-module $\mathcal{S}_{\mu-\lambda} =$

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$\bigoplus_{k=0}^{\infty}\mathcal{F}_{\mu-\lambda-k}$, the space of symbols of differential operators (for example, see [3, 16]).

The purpose of this paper is to study the simplest super analogue of the problem solved in [6], namely, we consider the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the linear supermanifold, i.e., the ringed space $\mathcal{C}^{1|1} := (\mathbb{C}, \mathbb{C}[x,\theta])$ associated with the linear superspace $\mathbb{C}^{1|1}$, where x is a coordinate on $\mathbb{C}$ and θ is a generator of the Grassmann algebra, x is supposed to be even and θ odd. We also consider the $\mathcal{K}(1)$-module $\mathfrak{F}_\lambda$ of λ-densities on $\mathcal{C}^{1|1}$ and the $\mathcal{K}(1)$-module of linear differential operators $\mathfrak{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\lambda, \mathfrak{F}_\mu)$, which are super analogues of the spaces $\mathcal{F}_\lambda$ and $\mathrm{D}_{\lambda,\mu}$, respectively. The Lie superalgebra $\mathfrak{osp}(1|2)$, a super analogue of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(1)$, see equation (2.1) below.

It was discovered in [10] that $\mathfrak{D}_{\lambda,\mu}$ has an important refinement to a $\mathcal{K}(1)$-invariant $\mathbb{N}/2$-filtration:

$$\mathfrak{D}^0_{\lambda,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^1_{\lambda,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu} \subset \cdots \subset \mathfrak{D}^{i-\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^i_{\lambda,\mu} \cdots \tag{1.1}$$

The quotient module $\mathfrak{D}^i_{\lambda,\mu}/\mathfrak{D}^{i-\frac{1}{2}}_{\lambda,\mu}$ is isomorphic to the module of weighted densities $\Pi^{2i}(\mathfrak{F}_{\mu-\lambda-i})$ (see, e.g., [10]), where Π is the change of parity map. Thus, the graded $\mathcal{K}(1)$-module $\mathrm{gr}\,\mathfrak{D}_{\lambda,\mu}$ associated with the filtration (1.1) is a direct sum of density modules:

$$\mathrm{gr}\,\mathfrak{D}_{\lambda,\mu} = \bigoplus_{i=0}^{\infty} \Pi^i\Big(\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}\Big).$$

Note that this module depends only on the shift, $\mu-\lambda$, of the weights and not on μ and λ independently. We call this $\mathcal{K}(1)$-module the space of symbols of differential operators and denote it by $\mathfrak{S}_{\mu-\lambda}$, a super analogue of $\mathcal{S}_{\mu-\lambda}$. For generic values of λ and μ, $\mathfrak{S}_{\mu-\lambda}$ and $\mathfrak{D}_{\lambda,\mu}$ are isomorphic as $\mathfrak{osp}(1|2)$-modules (cf. [10]).

In this paper, we classify all $\mathfrak{osp}(1|2)$-invariant superanti-symmetric binary differential operators from $\mathcal{K}(1)\wedge\mathcal{K}(1)$ to $\mathfrak{D}_{\lambda,\mu}$. We use the result to compute $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$. We show that the nonzero cohomology $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ only appears for resonant values of weights that satisfy $\mu-\lambda \in \frac{1}{2}\mathbb{N}+3$. These spaces allow us to study the generic formal $\mathfrak{osp}(1|2)$-trivial deformations of the natural action of $\mathcal{K}(1)$ on the superspace of symbols $\mathfrak{S}_{\mu-\lambda}$ and classify the nontrivial projectively invariant extensions of the Lie superalgebra $\mathcal{K}(1)$ by the module $\mathfrak{D}_{\lambda,\mu}$. Recall that the same problem was considered in [18] for the case of $(1,2)$-dimensional real superspace and was solved for values of the weights that satisfy $\mu-\lambda \le 6$ or $\mu-\lambda$ being a semi-integer, and conjecturally for $\mu-\lambda \in \mathbb{N}+7$. In the case of $(1,n)$-dimensional complex superspace, $n \ge 3$, this problem is related to the classification of $\mathfrak{osp}(n|2)$-invariant bilinear differential operators acting on the superspaces of weighted densities, where $\mathfrak{osp}(n|2)$ is the orthosymplectic Lie superalgebra—superization of Cohen–Rankin operators related with n-extended superstrings (see [12, 13]), and still out of reach.

2. Definitions and notations

Let $\mathcal{C}^{1|1} := (\mathbb{C}, \mathbb{C}[x,\theta])$ be the ringed space associated with the linear superspace $\mathbb{C}^{1|1}$, where x is a coordinate on $\mathbb{C}$ and θ is a generator of the Grassmann algebra; x is assumed to be even, and θ odd. Here $\theta^2 = 0$, so $\mathbb{C}[x,\theta]$ has a basis $\{1,\theta\}$ over $\mathbb{C}[x]$. On the space $\mathbb{C}[x,\theta]$, we consider the contact bracket

$$\{F,G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\cdot\overline{\eta}(G),$$

where the superscript $'$ stands for $\frac{\partial}{\partial x}$, $|F|$ is the parity of F and $\overline{\eta} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}$. Note that the derivation $\overline{\eta}$ is the generator of 1-extended supersymmetry and generates the kernel of the contact 1-form

$$\alpha = dx + \theta\, d\theta$$

as a module over the ring of polynomial functions. Let $\mathfrak{vect}(1|1)$ be the superspace of polynomial vector fields on $\mathcal{C}^{1|1}$:

$$\mathfrak{vect}(1|1) = \{F_0\partial_x + F_1\partial_\theta \mid F_0, F_1 \in \mathbb{C}[x,\theta]\},$$

where $\partial_\theta = \frac{\partial}{\partial\theta}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(1)$ of contact polynomial vector fields on $\mathcal{C}^{1|1}$. That is, $\mathcal{K}(1)$ is the superspace of vector fields on $\mathcal{C}^{1|1}$ preserving the distribution singled out by the 1-form α:

$$\mathcal{K}(1) = \big\{X \in \mathfrak{vect}(1|1) \mid \text{there exists } F \in \mathbb{C}[x,\theta] \text{ such that } L_X(\alpha) = F\alpha\big\},$$

where L_X is the Lie derivative along the vector field X. The Lie superalgebra $\mathcal{K}(1)$ is spanned by the fields of the form

$$X_F = F\partial_x - \frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\overline{\eta}, \quad \text{where } F \in \mathbb{C}[x,\theta].$$

Of course, $\mathcal{K}(1)$ is a subalgebra of $\mathfrak{vect}(1|1)$, and $\mathcal{K}(1)$ acts on $\mathbb{C}[x,\theta]$ through

$$\mathfrak{L}_{X_F}(G) = FG' - \frac{1}{2}(-1)^{|F|}\overline{\eta}(F)\cdot\overline{\eta}(G).$$

The bracket in $\mathcal{K}(1)$ can be written as $[X_F, X_G] = X_{\{F,G\}}$.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ can be realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \operatorname{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta). \tag{2.1}$$

The space of even elements is isomorphic to $\mathfrak{sl}(2)$, while the space of odd elements is two-dimensional:

$$(\mathfrak{osp}(1|1))_{\overline{1}} = \operatorname{Span}(X_\theta, X_{x\theta}).$$

We define the space of λ-densities as

$$\mathfrak{F}_\lambda = \big\{F(x,\theta)\alpha^\lambda \mid F(x,\theta) \in \mathbb{C}[x,\theta]\big\}.$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $\mathbb{C}[x,\theta]$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is now

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}^\lambda_{X_F}(G)\alpha^\lambda, \quad \text{with } \mathfrak{L}^\lambda_{X_F}(G) = \mathfrak{L}_{X_F}(G) + \lambda F'G.$$

A differential operator on $\mathbb{C}^{1|1}$ is an operator on $\mathbb{C}[x,\theta]$ of the form

$$A=\sum_{k=0}^{M}\sum_{\varepsilon}a_{k,\varepsilon}(x,\theta)\partial_x^k\partial_\theta^\varepsilon;\quad \varepsilon=0,1;\quad M\in\mathbb{N}.$$

Of course, any differential operator defines a linear mapping $F\alpha^\lambda\mapsto(AF)\alpha^\mu$ from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any λ, $\mu\in\mathbb{C}$, thus the space of differential operators becomes a family of $\mathcal{K}(1)$-modules $\mathfrak{D}_{\lambda,\mu}$ for the natural action

$$\mathfrak{L}_{X_F}^{\lambda,\mu}(A)=\mathfrak{L}_{X_F}^{\mu}\circ A-(-1)^{|A||F|}A\circ\mathfrak{L}_{X_F}^{\lambda}.$$

2.1. **Lie superalgebra cohomology.** Let us first recall some fundamental concepts from cohomology theory (see, e.g., [8, 9]). Let $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be a Lie superalgebra acting on a superspace $V=V_{\bar 0}\oplus V_{\bar 1}$ and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it is assumed to be $\{0\}$.) The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g},\mathfrak{h};V):=\mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h});V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g},\mathfrak{h};V)\longrightarrow C^{n+1}(\mathfrak{g},\mathfrak{h};V)$ is an even map satisfying $\delta_n\circ\delta_{n-1}=0$ (see for instance, [14]): for $\phi\in C^n(\mathfrak{g},\mathfrak{h};V)$,

$$\begin{aligned}(\delta_n\phi)(g_0,\dots,g_n)=&\sum_{i=0}^{n}(-1)^i(-1)^{|g_i|(|\phi|+|g_0|+\cdots+|g_{i-1}|)}g_i\phi(g_0,\dots,\hat{\imath},\dots,g_n)\\&+\sum_{0\le i<j\le n}(-1)^{i+j}(-1)^{|g_i|(|g_0|+\cdots+|g_{i-1}|)}(-1)^{|g_j|(|g_0|+\cdots+\hat{\imath}+\cdots+|g_{j-1}|)}\\&\qquad\times\phi([g_i,g_j],g_0,\dots,\hat{\imath},\dots,\hat{\jmath},\dots,g_n).\end{aligned}$$

The kernel of δ_n, denoted by $Z^n(\mathfrak{g},\mathfrak{h};V)$, is the space of $\mathfrak{h}$-relative *n-cocycles*; among them, the elements in the range of δ_{n-1} are called $\mathfrak{h}$-relative *n-coboundaries*. We denote by $B^n(\mathfrak{g},\mathfrak{h};V)$ the space of n-coboundaries.

By definition, the n-th $\mathfrak{h}$-relative cohomology space is the quotient space

$$\mathrm{H}^n(\mathfrak{g},\mathfrak{h};V)=Z^n(\mathfrak{g},\mathfrak{h};V)/B^n(\mathfrak{g},\mathfrak{h};V).$$

We can also define a $\mathfrak{g}$-action π on $C^n(\mathfrak{g},V)$ by setting, for any $g\in\mathfrak{g}$,

$$\begin{aligned}&(\pi(g)\phi)(g_1,\dots,g_n)\\&\quad=g\phi(g_1,\dots,g_n)-\sum_{i=1}^{n}(-1)^{|g|(|\phi|+|g_1|+\cdots+|g_{i-1}|)}\phi(g_1,\dots,[g,g_i],\dots,g_n),\end{aligned}$$

and a contraction operator $\iota(g)$ from C^n to C^{n-1} by

$$(\iota(g)\phi)(g_1,\dots,g_{n-1})=(-1)^{|g||\phi|}\phi(g,g_1,\dots,g_{n-1}).$$

A direct computation gives the classical formula

$$\pi(g)\phi=(\delta_{n-1}\circ\iota(g)+\iota(g)\circ\delta_n)\phi,$$

and thus $\delta_n(\pi(g)\phi)=\pi(g)(\delta_n\phi)$; that is, δ_n is a $\mathfrak{g}$-map. Note that $C^n(\mathfrak{g},\mathfrak{h};V)$ may be viewed as the subspace of $C^n(\mathfrak{g},V)$ annihilated by both $\iota(\mathfrak{h})$ and $\pi(\mathfrak{h})$. We will only need the formula of δ_n (which will be simply denoted by δ) in degrees 0, 1 and 2: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{h}\}.$$

3. The $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$ acting on $\mathfrak{D}_{\lambda,\mu}$

3.1. **$\mathfrak{osp}(1|2)$-invariant binary differential operators.** The following steps to compute the relative cohomology have been used extensively in [18, 17, 4, 5, 7, 6, 15]. First, we classify $\mathfrak{osp}(1|2)$-invariant differential operators, then we isolate among them those that are 2-cocycles. To do that, we need the following lemma.

Lemma 3.1. *Any 2-cocycle vanishing on the subalgebra $\mathfrak{osp}(1|2)$ of $\mathcal{K}(1)$ is $\mathfrak{osp}(1|2)$-invariant.*

Proof. The 2-cocycle condition reads as follows:

$$\begin{aligned}(-1)^{|X||c|}\mathfrak{L}_X^{\lambda,\mu} c(Y,Z) - (-1)^{|Y|(|X|+|c|)}\mathfrak{L}_Y^{\lambda,\mu} c(X,Z)&\\ + (-1)^{|Z|(|X|+|Y|+|c|)}\mathfrak{L}_Z^{\lambda,\mu} c(X,Y) - c([X,Y],Z)&\\ + (-1)^{|Y||Z|} c([X,Z],Y) + c(X,[Y,Z]) = 0&\end{aligned}$$

for every $X, Y, Z \in \mathcal{K}(1)$. Now, if $X \in \mathfrak{osp}(1|2)$, then the equation above becomes

$$c([X,Y],Z) - (-1)^{|Y||Z|} c([X,Z],Y) = (-1)^{|X||c|}\mathfrak{L}_X^{\lambda,\mu} c(Y,Z).$$

This relation is nothing but the $\mathfrak{osp}(1|2)$-invariance property of the bilinear map c. □

As our 2-cocycles vanish on $\mathfrak{osp}(1|2)$, we will investigate $\mathfrak{osp}(1|2)$-invariant superanti-symmetric binary differential operators that vanish on $\mathfrak{osp}(1|2)$. The following proposition is known and it is a corollary of the results by Gieres and Theisen, see [12].

Proposition 3.2. *The space of superanti-symmetric bilinear differential operators $\mathcal{K}(1) \wedge \mathcal{K}(1) \to \mathfrak{D}_{\lambda,\mu}$, which are $\mathfrak{osp}(1|2)$-invariant and vanish on $\mathfrak{osp}(1|2)$, is purely even if $\mu - \lambda$ is integer and it is purely odd if $\mu - \lambda$ is semi-integer; moreover, it is as follows:*

- (i) *It is $(2p-5)$-dimensional if $(\mu - \lambda) = 2p - 2$ and $p \geq 3$.*
- (ii) *It is $(2p-3)$-dimensional if $(\mu - \lambda) = 2p - 1$ or $(\mu - \lambda) = 2p - \frac{1}{2}$ and $p \geq 2$.*
- (iii) *It is $(2p-4)$-dimensional if $(\mu - \lambda) = 2p - \frac{3}{2}$ and $p \geq 3$.*
- (iv) *It is 0-dimensional otherwise.*

Proof. The general form of any such differential operator is

$$c(X_F, X_G) = \sum_{\substack{\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\\ 0\leq k_1,k_2,k_3\leq M}} c_\varepsilon^{k_1,k_2,k_3}(x,\theta,|F|,|G|)\, \overline{\eta}^{\varepsilon_1}(F^{(k_1)})\overline{\eta}^{\varepsilon_2}(G^{(k_2)})\overline{\eta}^{\varepsilon_3}\partial_x^{k_3},$$

where $\varepsilon_i = 0$ or 1, $M \in \mathbb{N}, \varepsilon_1 + k_1 \geq 3$ and $\varepsilon_2 + k_2 \geq 3$. The invariance property of c with respect to X_1 and X_x implies that

$$\frac{d}{dx} c_\varepsilon^{k_1,k_2,k_3} = 0, \quad \overline{\eta}(c_\varepsilon^{k_1,k_2,k_3}) = 0 \quad \text{and} \quad \sum_{i=1}^{3} (\varepsilon_i + 2k_i) = 2(\mu - \lambda) + 4.$$

Therefore, the coefficients $c_\varepsilon^{k_1,k_2,k_3}$ are functions of $|F|$ and $|G|$ and the parameters λ and μ must satisfy $2(\mu - \lambda) + 4 = n$, where $n \in \mathbb{N}$. The corresponding operator can be expressed as

$$c(X_F, X_G) = \sum_{\varepsilon, k_1, k_2} c_\varepsilon^{k_1,k_2,n}(|F|, |G|)\, \overline{\eta}^{\varepsilon_1}(F^{(k_1)}) \overline{\eta}^{\varepsilon_2}(G^{(k_2)}) \overline{\eta}^{\varepsilon_3} \partial_x^{\frac{1}{2}\left(n - \sum_{i=1}^{3} \varepsilon_i\right) - k_1 - k_2}. \tag{3.1}$$

The superanti-symmetric condition $c(X_F, X_G) = -(-1)^{|F||G|} c(X_G, X_F)$ leads to the following relation:

$$c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|, |G|) = -(-1)^{\varepsilon_2 .|F| + \varepsilon_1 .|G| + \varepsilon_1 .\varepsilon_2} c_{\varepsilon_2,\varepsilon_1,\varepsilon_3}^{k_2,k_1,n}(|G|, |F|). \tag{3.2}$$

Now, for the sake of completeness, we consider the invariance property with respect to $X_{x\theta}$. According to the parity of n, we distinguish two cases.

The case where n is even.

In this case, the invariance property of c with respect to $X_{x\theta}$ leads to the following relations:

$$c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|, |G|) = (-1)^{\varepsilon_1} c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F| + 1, |G|) = (-1)^{\varepsilon_1 + \varepsilon_2} c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|, |G| + 1), \tag{3.3}$$

and

$$\begin{aligned}
&\Lambda_{k_1,k_2}^n c_{0,0,0}^{k_1,k_2,n} - (k_1 - 2) c_{1,0,1}^{k_1,k_2,n} - (-1)^{|F|}(k_2 - 2) c_{0,1,1}^{k_1,k_2,n} = 0,\\
&\qquad\qquad\qquad\qquad k_1 + k_2 \leq \tfrac{n-2}{2} \text{ and } k_1 > k_2,\\
&\Lambda_{k_1,k_2}^n c_{1,1,0}^{k_1,k_2,n} + (k_1 + 1) c_{0,1,1}^{k_1+1,k_2,n} - (-1)^{|F|}(k_2 + 1) c_{1,0,1}^{k_1,k_2+1,n} = 0,\\
&\qquad\qquad\qquad\qquad k_1 + k_2 \leq \tfrac{n-4}{2} \text{ and } k_1 \geq k_2\\
&\Lambda_{k_1,k_2-2\lambda}^n c_{1,0,1}^{k_1,k_2,n} + (k_1 + 1) c_{0,0,0}^{k_1+1,k_2,n} + (-1)^{|F|}(k_2 - 2) c_{1,1,0}^{k_1,k_2,n} = 0,\\
&\qquad\qquad\qquad\qquad k_1 + k_2 \leq \tfrac{n-2}{2},
\end{aligned} \tag{3.4}$$

where $\Lambda_{k_1,k_2}^n = (-1)^{|F|+|G|}(\frac{n}{2} - k_1 - k_2)$. According to formulae (3.2) and (3.3), we deduce that $c_{0,0,0}^{k_1,k_1,n} = 0$. Now, we can see, with the help of Maple, that the system (3.4) is linearly independent. Further, by (3.2), we can see that all the coefficients $c_\varepsilon^{k_1,k_2,n}$ can be expressed in terms of

$$\begin{aligned}
&c_{1,0,1}^{k_1,k_2,n} && \text{with } k_1 \geq 2 \text{ and } k_2 \geq 3,\\
&c_{0,0,0}^{k_1,k_2,n} && \text{with } k_1 > k_2 \geq 3,\\
&c_{1,1,0}^{k_1,k_2,n} && \text{with } k_1 \geq k_2 \geq 2.
\end{aligned} \tag{3.5}$$

So, we deduce that the dimension of the space of solutions is equal to

number of coefficients $c_{\varepsilon}^{k_1,k_2,n}$ given by (3.5) − number of equations given by (3.4).

A straightforward computation shows that the space of $\mathfrak{osp}(1|2)$-invariant operators has the following structure:

(i) For $n = 4p$, it is $(\frac{n}{2} - 5)$-dimensional and can be spanned by

$$c_{1,0,1}^{2,3,n},\ c_{1,0,1}^{2,4,n},\ \dots,\ c_{1,0,1}^{2,\frac{n}{2}-3,n}.$$

(ii) For $n = 4p + 2$, it is $(\frac{n}{2} - 4)$-dimensional and can be spanned by $c_{0,0,0}^{\frac{n}{2}-3,3,n}$, $c_{1,1,0}^{2,2,n}$ and

$$c_{1,0,1}^{2,4,n},\ c_{1,0,1}^{2,5,n},\ \dots,\ c_{1,0,1}^{2,\frac{n}{2}-3,n}.$$

The case where n is odd.

In this case, the invariance property of c with respect to $X_{x\theta}$ leads to the following relations:

$$c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|,|G|) = (-1)^{\varepsilon_2+\varepsilon_3} c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|+1,|G|) = (-1)^{\varepsilon_3} c_{\varepsilon_1,\varepsilon_2,\varepsilon_3}^{k_1,k_2,n}(|F|,|G|+1). \tag{3.6}$$

and

$$\Lambda^n_{k_1,k_2+\frac{1}{2}-2\lambda} c_{0,0,1}^{k_1,k_2,n} + (k_1-2) c_{1,0,0}^{k_1,k_2,n} + (-1)^{|F|}(k_2-2) c_{0,1,0}^{k_1,k_2,n} = 0,$$
$$k_1 + k_2 \le \tfrac{n-1}{2} \text{ and } k_1 > k_2,$$
$$\Lambda^n_{k_1,k_2+\frac{3}{2}-2\lambda} c_{1,1,1}^{k_1,k_2,n} - (k_1+1) c_{0,1,0}^{k_1+1,k_2,n} + (-1)^{|F|}(k_2+1) c_{1,0,0}^{k_1,k_2+1,n} = 0,$$
$$k_1 + k_2 \le \tfrac{n-3}{2} \text{ and } k_1 \ge k_2,$$
$$\Lambda^n_{k_1,k_2+\frac{1}{2}} c_{1,0,0}^{k_1,k_2,n} - (k_1+1) c_{0,0,1}^{k_1+1,k_2,n} - (-1)^{|F|}(k_2-2) c_{1,1,1}^{k_1,k_2,n} = 0,$$
$$k_1 + k_2 \le \tfrac{n-3}{2}.$$

According to formulae (3.2) and (3.6), we deduce that $c_{0,0,1}^{k_1,k_1,n} = 0$. Now, the same arguments as in the previous case show that the space of $\mathfrak{osp}(1|2)$-invariant operators has the following structure:

(i) For $n = 4p + 1$, it is $(\frac{n-9}{2})$-dimensional and can be spanned by $c_{1,0,0}^{2,\frac{n-5}{2},n}$ and

$$c_{1,1,1}^{2,2,n},\ c_{1,1,1}^{3,2,n},\ \dots,\ c_{1,1,1}^{\frac{n-9}{2},2,n}.$$

(ii) For $n = 4p + 3$, it is $(\frac{n-9}{2})$-dimensional and can be spanned by $c_{0,0,1}^{\frac{n-7}{2},3,n}$ and

$$c_{1,1,1}^{2,2,n},\ c_{1,1,1}^{3,2,n},\ \dots,\ c_{1,1,1}^{\frac{n-9}{2},2,n}.$$

Proposition 3.2 is proved. □

3.2. **The $\mathfrak{osp}(1|2)$-relative cohomology of $\mathcal{K}(1)$.** In this subsection, we will compute the second differential $\mathfrak{osp}(1|2)$-relative cohomology spaces

$$\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}).$$

Our main result is the following:

Theorem 3.3. *The space* $\mathcal{H} = \mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ *has the following structure:*

$$\mathcal{H} \simeq \begin{cases} \mathbb{R} & \text{if} \begin{cases} (\lambda,\mu) \in \left\{ \begin{array}{l} (0,3), (-\frac{5}{2}, \frac{1}{2}), (-\frac{3}{2}, 2), (-2, \frac{5}{2}) \\ (\frac{-7+\sqrt{33}}{4}, \frac{9+\sqrt{33}}{4}), (\frac{-7-\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}), \end{array} \right\}, \\ \mu-\lambda = 5 \text{ for all } \lambda, \\ \mu-\lambda = \frac{11}{2} \text{ and } \lambda \neq -\frac{5}{2}, \\ \mu-\lambda = p-\frac{3}{2} \text{ and } \lambda = 1-\frac{p}{2},\ p\in\mathbb{N}+8, \\ \mu-\lambda = p-2 \text{ and } \lambda = -\frac{2p-5\pm\sqrt{8p-15}}{4},\ p\in\mathbb{N}+8, \end{cases} \\ \mathbb{R}^2 & \text{if } \mu-\lambda = \frac{11}{2} \text{ and } \lambda = -\frac{5}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily check that all the results of Theorem 3.3 are invariant under passage to adjoint values. We refer to [2] for adjoint differential operator modules.

4. Computing the cohomology

The proof of Theorem 3.3 will be the subject of Subsection 4.2. In fact, we need first the description of $\mathfrak{osp}(1|2)$-invariant bilinear operators, from $\mathfrak{F}_{-1}\otimes\mathfrak{F}_\lambda$ to $\mathfrak{F}_{\lambda+k-1}$, called supertransvectants.

4.1. **Supertransvectants: an explicit formula.** Gieres and Theisen [12] listed the supertransvectants and they expressed them in terms of supercovariant derivatives; then Gargoubi and Ovsienko [11] gave an interpretation of these operators, and later Ben Fraj et al. [1] gave another description of the space of $\mathfrak{osp}(1|2)$-invariant bilinear differential operators

$$\begin{aligned} \mathfrak{J}^\lambda_k : \mathfrak{F}_{-1}\otimes\mathfrak{F}_\lambda &\longrightarrow \mathfrak{F}_{\lambda+k-1} \\ (F\alpha^{-1}, G\alpha^\lambda) &\mapsto \mathfrak{J}^\lambda_k(F,G)\alpha^{\lambda+k-1} \end{aligned}$$

vanishing on $\mathfrak{osp}(1|2)$, where $k\in\frac{1}{2}\mathbb{N}$. They showed that these spaces are one-dimensional for $k>2$. The operators $\mathfrak{J}^\lambda_k$, where $k\in\mathbb{N}$, are even and they are given by

$$\mathfrak{J}^\lambda_k(F,G) = \sum_{\substack{i+j=k-1\\ i\geq 2}} \Gamma^{\frac{3}{2},\lambda}_{i-2,j,\frac{2k-5}{2}} \Big((2\lambda+k-1-i)F^{(i+1)}G^{(j)} - (-1)^{|F|}(k-j)\overline{\eta}(F^{(i)})\overline{\eta}(G^{(j)})\Big). \tag{4.1}$$

The operators $\mathfrak{J}_k^\lambda$ labeled by semi-integer k are odd; set $\mathfrak{J}_{\frac{5}{2}}^\lambda(F,G)=\overline{\eta}(F'')G$ and

$$\begin{aligned}\mathfrak{J}_k^\lambda(F,G)=&\sum_{\substack{i+j=k-\frac{1}{2}\\ i\geq 3}}(-1)^{|F|}\Gamma^{\frac{3}{2},\lambda}_{i-3,j,\frac{2k-7}{2}}F^{(i)}\overline{\eta}(G^{(j)})\\&-\sum_{\substack{i+j=k-\frac{1}{2}\\ i\geq 2}}\Gamma^{\frac{3}{2},\lambda}_{i-2,j,\frac{2k-7}{2}}\overline{\eta}(F^{(i)})G^{(j)},\end{aligned}\tag{4.2}$$

with

$$\Gamma^{\tau,\lambda}_{i,j,k}=(-1)^j\binom{2\tau+[k]}{j}\binom{2\lambda+[k]}{i},$$

where $\binom{x}{i}=\frac{x(x-1)\cdots(x-i+1)}{i!}$ and $[k]$ denotes the integer part of k, $k>0$.

Remark 4.1. The supertransvectants appear in many contexts, especially in the computation of cohomology (cf. [1, 10]). We refer to [12] for their history.

In order to prove Theorem 3.3, we will study properties of the coboundaries.

Lemma 4.2. *If $\delta(B)$ belongs to $B^2(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$, where $B:\mathcal{K}(1)\to\mathfrak{D}_{\lambda,\mu}$ is an operator vanishing on $\mathfrak{osp}(1|2)$, then B is a supertransvectant.*

Proof. For all X, $Y\in\mathcal{K}(1)$ we have

$$\delta(B)(X,Y):=(-1)^{|X||B|}\mathfrak{L}_X^{\lambda,\mu}B(Y)-(-1)^{|Y|(|X|+|B|)}\mathfrak{L}_Y^{\lambda,\mu}B(X)-B([X,Y]).$$

Since $B(X)=0$ for all $X\in\mathfrak{osp}(1|2)$, we deduce that

$$(-1)^{|X||B|}\mathfrak{L}_X^{\lambda,\mu}B(Y)-B([X,Y])=0.$$

Thus, the operator B is $\mathfrak{osp}(1|2)$-invariant; therefore it coincides with the supertransvectants.

Now, clearly the coboundary $\delta(\mathfrak{J}_k^\lambda)$ has the following form:

$$\begin{aligned}&\delta(\mathfrak{J}_k^\lambda)(X_F,X_G)\\&=\sum_{\varepsilon,k_1,k_2}\beta_\varepsilon^{k_1,k_2,2k+2}(|F|,|G|)\,\overline{\eta}^{\varepsilon_1}(F^{(k_1)})\overline{\eta}^{\varepsilon_2}(G^{(k_2)})\overline{\eta}^{\varepsilon_3}\partial_x^{k+1-k_1-k_2-\frac{1}{2}\sum_{i=1}^3\varepsilon_i},\end{aligned}\tag{4.3}$$

where the coefficients $\beta_\varepsilon^{k_1,k_2,2k+2}$ are well-determined values of the coefficients $c_\varepsilon^{k_1,k_2,2k+2}$ given by (3.1). The following lemma will be useful in the proof of Theorem 3.3.

Lemma 4.3. *The following holds:*

(i) *We have*

$$\begin{aligned}\beta_{1,1,1}^{2,2,13}&=-(-1)^{|G|}30(\lambda+2),\\ \beta_{0,0,0}^{4,3,14}&=\frac{20}{3}\lambda(2\lambda+3)(\lambda+2)(\lambda+4),\\ \beta_{1,0,1}^{2,4,14}&=(-1)^{|F|+|G|}30\left(\lambda+\frac{9+\sqrt{41}}{4}\right)\left(\lambda+\frac{9-\sqrt{41}}{4}\right),\end{aligned}$$

$$\beta^{4,3,15}_{0,0,1} = (-1)^{|F|+|G|}\frac{5}{6}(2\lambda+1)(2\lambda+5)(2\lambda+9),$$
$$\beta^{2,2,15}_{1,1,1} = (-1)^{|G|}45(2\lambda+5).$$

(ii) *For* $\lambda \in \mathbb{R}$ *and* $p \in \mathbb{N}+8$, *we have*

$$\beta^{2,2,2p+1}_{1,1,1} = -(-1)^{|G|+p}\binom{p-1}{p-5}(6\lambda+3p-6),$$
$$\beta^{2,4,2p}_{1,0,1} = -\frac{(-1)^{|F|+|G|+p}}{8}\binom{p-5}{2}\binom{p-1}{p-5}\left(4\lambda+2p-5+\sqrt{8p-15}\right) \times \left(4\lambda+2p-5-\sqrt{8p-15}\right).$$

Proof. (i) To compute the coefficient $\beta^{2,2,13}_{1,1,1}$, we must compute the coboundary $\delta(\mathfrak{J}^{\lambda}_{\frac{11}{2}})(X_F, X_G)$, where

$$\mathfrak{J}^{\lambda}_{\frac{11}{2}}(X_G) = (-1)^{|G|}\left(\Gamma^{\frac{3}{2},\lambda}_{0,2,2}G^{(3)}\overline{\eta}\partial_x^2 + \Gamma^{\frac{3}{2},\lambda}_{1,1,2}G^{(4)}\overline{\eta}\partial_x + \Gamma^{\frac{3}{2},\lambda}_{2,0,2}G^{(5)}\overline{\eta}\right) - \left(\Gamma^{\frac{3}{2},\lambda}_{0,3,2}\overline{\eta}(G'')\partial_x^3 + \Gamma^{\frac{3}{2},\lambda}_{1,2,2}\overline{\eta}(G^{(3)})\partial_x^4 + \Gamma^{\frac{3}{2},\lambda}_{2,1,2}\overline{\eta}(G^{(4)})\partial_x + \Gamma^{\frac{3}{2},\lambda}_{3,0,2}\overline{\eta}(G^{(5)})\right).$$

Collecting the terms in $\overline{\eta}(F'')\overline{\eta}(G'')\overline{\eta}\partial_x$, we get

$$\beta^{2,2,13}_{1,1,1} = 3(-1)^{|G|}\left(\Gamma^{\frac{3}{2},\lambda}_{0,3,2} + \Gamma^{\frac{3}{2},\lambda}_{1,1,2}\right) = -30(-1)^{|G|}(\lambda+2).$$

To compute the coefficient $\beta^{2,4,14}_{1,0,1}$, we must compute the coboundary $\delta(\mathfrak{J}^{\lambda}_{6})(X_F, X_G)$, where

$$\begin{aligned}\mathfrak{J}^{\lambda}_{6}(X_G) = {}&\Gamma^{\frac{3}{2},\lambda}_{0,3,3}\left((2\lambda+3)G^{(3)}\partial_x^3 - 3(-1)^{|G|}\overline{\eta}(G'')\overline{\eta}\partial_x^3\right)\\ &+ \Gamma^{\frac{3}{2},\lambda}_{1,2,3}\left((2\lambda+2)G^{(4)}\partial_x^2 - 4(-1)^{|G|}\overline{\eta}(G^{(3)})\overline{\eta}\partial_x^2\right)\\ &+ \Gamma^{\frac{3}{2},\lambda}_{2,1,3}\left((2\lambda+1)G^{(5)}\partial_x - 5(-1)^{|G|}\overline{\eta}(G^{(4)})\overline{\eta}\partial_x\right)\\ &+ \Gamma^{\frac{3}{2},\lambda}_{3,0,3}\left((2\lambda)G^{(6)} - 6(-1)^{|G|}\overline{\eta}(G^{(5)})\overline{\eta}\right).\end{aligned}$$

Collecting the terms in $\overline{\eta}(F'')G^{(4)}\overline{\eta}$, we get

$$\begin{aligned}\beta^{2,4,14}_{1,0,1} &= (-1)^{|F|+|G|}\left((\lambda+1)\Gamma^{\frac{3}{2},\lambda}_{1,2,3} - \left(3\lambda+\frac{3}{2}\right)\Gamma^{\frac{3}{2},\lambda}_{0,3,3}\right)\\ &= (-1)^{|F|+|G|}\left(15(2\lambda+3)(\lambda+1) + 20\left(3\lambda+\frac{3}{2}\right)\right)\\ &= (-1)^{|F|+|G|}15(2\lambda^2+9\lambda+5)\\ &= (-1)^{|F|+|G|}30\left(\lambda+\frac{9+\sqrt{41}}{4}\right)\left(\lambda+\frac{9-\sqrt{41}}{4}\right).\end{aligned}$$

We point out that, under the coboundary operator δ, only the terms

$$-3(-1)^{|G|}\Gamma^{\frac{3}{2},\lambda}_{0,3,3}\overline{\eta}(G'')\overline{\eta}\partial_x^3,$$

$$\Gamma^{\frac{3}{2},\lambda}_{1,2,3}(2\lambda+2)G^{(4)}\partial_x^2,$$
$$\Theta(X_G) = -6(-1)^{|G|}\Gamma^{\frac{3}{2},\lambda}_{3,0,3}\overline{\eta}(G^{(5)})\overline{\eta}$$

in the expression of $\mathfrak{J}^{\lambda}_{6}$ allow us to compute the coefficient $\beta^{2,4,14}_{1,0,1}$. Note that the coefficient of $\overline{\eta}(F'')G^{(4)}\overline{\eta}$ in $\delta(\Theta)(X_F, X_G)$ vanishes.

For the coefficients $\beta^{4,3,14}_{0,0,0}$, $\beta^{4,3,15}_{0,0,1}$ and $\beta^{2,2,15}_{1,1,1}$, the proof is the same as in the previous case.

(ii) To compute the coefficient $\beta^{2,2,2p+1}_{1,1,1}$, we must compute the coboundary $\delta\big(\mathfrak{J}^{\lambda}_{p-\frac{1}{2}}\big)(X_F, X_G)$, where $\mathfrak{J}^{\lambda}_{p-\frac{1}{2}}$ is as in equation (4.2). Collecting the terms in $\overline{\eta}(F'')\overline{\eta}(G'')\overline{\eta}\partial_x^{p-5}$, we get

$$\begin{aligned}\beta^{2,2,2p+1}_{1,1,1} &= (-1)^{|G|}\left(\binom{p-3}{2}\Gamma^{\frac{3}{2},\lambda}_{0,p-3,p-4} + \frac{1}{2}\binom{p-4}{2}\Gamma^{\frac{3}{2},\lambda}_{1,p-5,p-4}\right)\\ &= -(-1)^{|G|+p}\binom{p-1}{p-5}(6\lambda+3p-6).\end{aligned}$$

We point out that, under the coboundary operator δ, only the terms

$$-\Gamma^{\frac{3}{2},\lambda}_{0,p-3,p-4}\overline{\eta}(G'')\partial_x^{p-3},$$
$$(-1)^{|G|}\Gamma^{\frac{3}{2},\lambda}_{1,p-5,p-4}G^{(4)}\partial_x^{p-3}$$

in the expression of $\mathfrak{J}^{\lambda}_{p-\frac{1}{2}}$ allow us to compute the coefficient $\beta^{2,2,2p+1}_{1,1,1}$.

Finally, to compute the coefficient $\beta^{2,4,2p}_{1,0,1}$, we must compute the coboundary $\delta(\mathfrak{J}^{\lambda}_{p-1})(X_F, X_G)$, where $\mathfrak{J}^{\lambda}_{p-1}$ is as in (4.1). Collecting the terms in $\overline{\eta}(F'')G^{(4)}\overline{\eta}\partial_x^{p-7}$, we get

$$\begin{aligned}\beta^{2,4,2p}_{1,0,1} &= (-1)^{|F|+|G|}\bigg(\frac{2\lambda+p-5}{2}\binom{p-5}{2}\Gamma^{\frac{3}{2},\lambda}_{1,p-5,\frac{2p-7}{2}}\\ &\qquad\qquad - 3\Gamma^{\frac{3}{2},\lambda}_{0,p-4,\frac{2p-7}{2}}\left(\binom{p-4}{4} + \left(\lambda+\frac{1}{2}\right)\binom{p-4}{3}\right)\bigg)\\ &= (-1)^{|F|+|G|}\binom{p-5}{2}\bigg(\frac{2\lambda+p-5}{2}\Gamma^{\frac{3}{2},\lambda}_{1,p-5,\frac{2p-7}{2}}\\ &\qquad\qquad - (p-4)\Gamma^{\frac{3}{2},\lambda}_{0,p-4,\frac{2p-7}{2}}\left(\frac{p-7}{4} + \left(\lambda+\frac{1}{2}\right)\right)\bigg)\\ &= -(-1)^{|F|+|G|+p}\binom{p-5}{2}\binom{p-1}{p-5}\left(\frac{(2\lambda+p-5)(2\lambda+p-4)}{2} + 4\lambda+p-5\right)\\ &= \frac{-(-1)^{|F|+|G|+p}}{2}\binom{p-5}{2}\binom{p-1}{p-5}\left(4\lambda^2+2\lambda(2p-5)+(p-5)(p-2)\right)\\ &= -\frac{(-1)^{|F|+|G|+p}}{8}\binom{p-5}{2}\binom{p-1}{p-5}\left(4\lambda+2p-5+\sqrt{8p-15}\right)\\ &\quad\times\left(4\lambda+2p-5-\sqrt{8p-15}\right).\end{aligned}$$

We point out that, under the coboundary operator δ, only the terms

$$\frac{2\lambda+p-5}{2}\Gamma^{\frac{3}{2},\lambda}_{1,p-5,\frac{2p-7}{2}}G^{(4)}\partial_x^{p-5},$$
$$3(-1)^{|G|+1}\Gamma^{\frac{3}{2},\lambda}_{0,p-4,\frac{2p-7}{2}}\overline{\eta}(G'')\overline{\eta}\partial_x^{p-4},$$
$$\widetilde{\Theta}(X_G)=6(-1)^{|G|+1}\Gamma^{\frac{3}{2},\lambda}_{3,p-7,\frac{2p-7}{2}}\overline{\eta}(G^{(5)})\overline{\eta}\partial_x^{p-7}$$

in the expression of $\mathfrak{J}^{\lambda}_{p-1}$ allow us to compute the coefficient $\beta^{2,4,2p}_{1,0,1}$. Note that the coefficient of $\overline{\eta}(F'')G^{(4)}\overline{\eta}\partial_x^{p-7}$ in $\delta(\widetilde{\Theta})(X_F,X_G)$ vanishes. □

4.2. **Proof of Theorem 3.3.** According to Lemma 3.1, any 2-cocycle of $\mathcal{K}(1)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}$ vanishing on $\mathfrak{osp}(1|2)$ is $\mathfrak{osp}(1|2)$-invariant. So, by Theorem 3.2, it is identically zero if $\mu-\lambda<3$ and expressed as in (3.1) for $\mu-\lambda\in\frac{1}{2}\mathbb{N}+3$.

For $\mu-\lambda\in\frac{1}{2}\mathbb{N}+3$, the proof of Theorem 3.3 consists of two steps. First, we investigate operators that belong to $Z^2(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$. The 2-cocycle condition imposes conditions on the coefficients $c_{\varepsilon}^{k_1,k_2,n}$: we get a linear system for $c_{\varepsilon}^{k_1,k_2,n}$. Second, taking into account these conditions, we eliminate all coefficients underlying coboundaries. Gluing these bits of information together we deduce that $\dim \mathrm{H}^2$ is equal to the number of independent coefficients $c_{\varepsilon}^{k_1,k_2,n}$ remaining in the expression of the 2-cocycle (3.1).

4.2.1. *The case where* $\mu-\lambda=3$. In this case, according to Theorem 3.2, the 2-cocycle (3.1) can be expressed as follows:

$$c(X_F,X_G)=c^{2,2,10}_{1,1,0}\gamma(X_F,X_G),$$

where

$$\gamma(X_F,X_G)=\overline{\eta}(F'')\overline{\eta}(G'').$$

By (3.3), we deduce that, up to a scalar factor, $c^{2,2,10}_{1,1,0}=(-1)^{|F|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to Subsection 4.1, we can see that any coboundary $\delta(B)\in B^2(\mathcal{K}(1),\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$ can be expressed as follows:

$$\delta(B)=\tau\delta(\mathfrak{J}^{\lambda}_4),\quad \tau\in\mathbb{C}, \tag{4.4}$$

where

$$\mathfrak{J}^{\lambda}_4(X_G)=-2(2\lambda+1)\left(2(-1)^{|G|}\overline{\eta}(G^{(3)}\overline{\eta}-2G^{(3)}\overline{\eta}^2-\lambda G^{(4)}\right)-12(-1)^{|G|}\overline{\eta}(G'')\overline{\eta}^3.$$

The expression (4.4) reads

$$\delta(B)(X_F,X_G)=\tau\left(\mathfrak{L}^{\lambda,\lambda+3}_{X_F}\mathfrak{J}^{\lambda}_4(X_G)-(-1)^{|F||G|})\mathfrak{L}^{\lambda,\lambda+3}_{X_G}\mathfrak{J}^{\lambda}_4(X_F)-\mathfrak{J}^{\lambda}_4([X_F,X_G])\right).$$

Using the graded Leibniz formula:

$$\overline{\eta}^j\circ F=\sum_{i=0}^{j}\binom{j}{i}_s(-1)^{|F|(j-i)}\overline{\eta}^i(F)\overline{\eta}^{j-i},$$

where

$$\binom{j}{i}_s = \begin{cases} \binom{[\frac{j}{2}]}{[\frac{i}{2}]} & \text{if } i \text{ is even or } j \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

we can check the following equation:

$$\delta(B)(X_F, X_G) = \tau(-1)^{|F|} 6\lambda(2\lambda+5)\gamma(X_F, X_G).$$

So, for $\lambda \neq 0, -\frac{5}{2}$, we can see that the coefficient $c^{2,2,10}_{1,1,0}$ can be eliminated by adding a coboundary: indeed, the constant τ can be chosen such that $\Upsilon = \delta(B)$. Hence, the cohomology is zero-dimensional. In contrast, for $\lambda = 0$ or $-\frac{5}{2}$, the coefficient $c^{2,2,10}_{1,1,0}$ cannot be eliminated by adding a coboundary. Hence, the cohomology space is one-dimensional.

4.2.2. *The case where* $\mu - \lambda = \frac{7}{2}$. In this case, according to Theorem 3.2, the 2-cocycle (3.1) can be expressed as follows:

$$c(X_F, X_G) = c^{2,2,11}_{1,1,1}\gamma(X_F, X_G),$$

where

$$\gamma(X_F, X_G) = \overline{\eta}(F'')\overline{\eta}(G'')\overline{\eta} + (-1)^{|G|}\frac{\lambda}{3}\left((-1)^{|F|}F^{(3)}\overline{\eta}(G'') - \overline{\eta}(F'')G^{(3)}\right).$$

By (3.6), we deduce that, up to a scalar factor, $c^{2,2,11}_{1,1,1} = (-1)^{|G|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to Subsection 4.1, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ can be expressed as follows:

$$\delta(B) = \tau\delta(\mathfrak{J}^{\lambda}_{\frac{9}{2}}), \quad \tau \in \mathbb{C}, \tag{4.5}$$

where

$$\begin{aligned} \mathfrak{J}^{\lambda}_{\frac{9}{2}}(X_G) = (-1)^{|G|}\left(4G^{(3)}\overline{\eta}^3 + (2\lambda+1)G^{(4)}\overline{\eta}\right) - 6\overline{\eta}(G'')\overline{\eta}^4 \\ - (2\lambda+1)\left(4\overline{\eta}(G^{(3)})\overline{\eta}^2 + \lambda\overline{\eta}(G^{(4)})\right). \end{aligned}$$

The expression (4.5) reads

$$\begin{aligned} &\delta(B)(X_F, X_G) \\ &= \tau\left((-1)^{|F|}\mathfrak{L}^{\lambda,\lambda+\frac{7}{2}}_{X_F}\mathfrak{J}^{\lambda}_{\frac{9}{2}}(X_G) - (-1)^{(|F|+1)|G|)}\mathfrak{L}^{\lambda,\lambda+\frac{7}{2}}_{X_G}\mathfrak{J}^{\lambda}_{\frac{9}{2}}(X_F) - \mathfrak{J}^{\lambda}_{\frac{9}{2}}([X_F, X_G])\right). \end{aligned}$$

Using the graded Leibniz formula, we can check the following expressions for nonzero $\beta^{k_1,k_2,11}_{\varepsilon}$:

$$\begin{aligned} \beta^{2,2,11}_{1,1,1} &= \tau(-1)^{|G|}3(2\lambda+3), \\ \beta^{2,3,11}_{1,0,0} &= -\tau\lambda(2\lambda+3), \\ \beta^{3,2,11}_{0,1,0} &= \tau(-1)^{|F|}\lambda(2\lambda+3). \end{aligned}$$

Thus, we deduce the following equation:

$$\delta(B)(X_F, X_G) = \tau(-1)^{|G|}3(2\lambda+3)\gamma(X_F, X_G).$$

So, for $\lambda \neq -\frac{3}{2}$, we can see that the coefficient $c_{1,1,1}^{2,2,11}$ can be eliminated by adding a coboundary. Hence, the cohomology is zero-dimensional. Now, for $\lambda = -\frac{3}{2}$, clearly the coefficient $c_{1,1,1}^{2,2,11}$ cannot be eliminated by adding a coboundary. Hence the cohomology is one-dimensional.

4.2.3. *The case where $\mu - \lambda = 4$.* In this case, according to Theorem 3.2, the 2-cocycle (3.1) can be expressed as follows:

$$c(X_F, X_G) = c_{1,0,1}^{2,3,12}\gamma(X_F, X_G),$$

where

$$\begin{aligned}\gamma(X_F, X_G) &= \overline{\eta}(F'')G^{(3)}\overline{\eta} - (-1)^{|F|}F^{(3)}\overline{\eta}(G'')\overline{\eta} + 6(-1)^{|G|}\overline{\eta}(F'')\overline{\eta}(G'')\partial_x \\ &\quad - (-1)^{|G|}2\lambda\left(\overline{\eta}(F^{(3)})\overline{\eta}(G'') + \overline{\eta}(F'')\overline{\eta}(G^{(3)})\right).\end{aligned}$$

By (3.3), we deduce that, up to a scalar factor, $c_{1,0,1}^{2,3,12} = (-1)^{|F|+|G|}$. Therefore, by a direct computation, we can see that the 2-cocycle condition is always satisfied. Let us study the triviality of this 2-cocycle. According to Subsection 4.1, we can see that any coboundary $\delta(B) \in B^2(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ can be expressed as follows:

$$\delta(B) = \tau\delta(\mathfrak{J}_5^{\lambda}), \quad \tau \in \mathbb{C},$$

where

$$\begin{aligned}\mathfrak{J}_5^{\lambda}(X_G) &= 20(\lambda+1)\left(-2(-1)^{|G|}\overline{\eta}(G^{(3)}\overline{\eta}^3 + G^{(3)}\overline{\eta}^4\right) \\ &\quad - 5(\lambda+1)(2\lambda+1)\left((-1)^{|G|}\overline{\eta}(G^{(4)}\overline{\eta} - 2G^{(4)}\overline{\eta}^2\right) \\ &\quad - 30(-1)^{|G|}\overline{\eta}(G'')\overline{\eta}^5 + 2\lambda(\lambda+1)(2\lambda+1)G^{(5)}.\end{aligned}$$

Using the same arguments as before, we obtain the expressions for nonzero $\beta_{\varepsilon}^{k_1,k_2,12}$:

$$\tfrac{1}{6}(-1)^{|G|}\beta_{1,1,0}^{2,2,12} = (-1)^{|F|+1}\beta_{0,1,1}^{3,2,12} = \beta_{1,0,1}^{2,3,12} = -5\tau(-1)^{|F|+|G|}(2\lambda^2 + 7\lambda + 2),$$

$$\beta_{1,1,0}^{3,2,12} = \beta_{1,1,0}^{2,3,12} = 10\tau(-1)^{|F|}\lambda(2\lambda^2 + 7\lambda + 2).$$

Thus, we deduce the following equation:

$$\delta(B)(X_F, X_G) = -\tau(-1)^{|F|+|G|}10\left(\lambda + \frac{7+\sqrt{33}}{4}\right)\left(\lambda + \frac{7-\sqrt{33}}{4}\right)\gamma(X_F, X_G).$$

So, for $\lambda \neq \frac{-7\pm\sqrt{33}}{4}$, we can see that the coefficient $c_{1,0,1}^{2,3,12}$ can be eliminated by adding a coboundary. Hence, the cohomology is zero-dimensional. In contrast, for $\lambda = \frac{-7+\sqrt{33}}{4}$ or $\frac{-7-\sqrt{33}}{4}$, the coefficient $c_{1,0,1}^{2,3,12}$ cannot be eliminated by adding a coboundary. Hence, the cohomology space is one-dimensional.

4.2.4. *The case where* $\mu-\lambda\geq\frac{9}{2}$. Here, a straightforward computation shows that the 2-cocycle condition is equivalent to formulas (3.2)–(3.4) and (3.6) corresponding to $\mathfrak{osp}(1|2)$-invariant operators together with the following systems:

For $n=2(\mu-\lambda+2)$ **even:**

$$(-1)^{|F|+|G|}\left(\binom{\alpha+a}{\alpha}c_{1,1,0}^{\beta,\gamma,n}+\binom{\beta+a}{\beta}c_{1,1,0}^{\alpha,\gamma,n}+\binom{\gamma+a}{\gamma}c_{1,1,0}^{\alpha,\beta,n}\right)\\ +\binom{\alpha+\beta}{\alpha}c_{0,1,1}^{\alpha+\beta,\gamma,n}+\binom{\alpha+\gamma}{\alpha}c_{0,1,1}^{\alpha+\gamma,\beta,n}+\binom{\beta+\gamma}{\beta}c_{0,1,1}^{\beta+\gamma,\alpha,n}=0, \quad (4.6)$$

where $a+\alpha+\beta+\gamma=\frac{n}{2}-1$, $\alpha\geq\beta\geq\gamma\geq 2$ and $a\in\{0,1\}$, which is obtained from the coefficient of $\overline{\eta}(F^{(\alpha)})\overline{\eta}(G^{(\alpha)})\overline{\eta}(H^{(\gamma)})\overline{\eta}\partial_x^a$,

$$2\left(\tfrac{a}{\alpha}+\lambda\right)\binom{a+\alpha-1}{\alpha-1}c_{1,1,0}^{\beta,\gamma,n}\\ +(-1)^{|F|+|G|}\left(\left(\tfrac{a}{\beta}+2\lambda\right)\binom{a+\beta-1}{\beta-1}c_{0,1,1}^{\alpha,\gamma,n}+\left(\tfrac{a}{\gamma}+2\lambda\right)\binom{a+\gamma-1}{\gamma-1}c_{0,1,1}^{\alpha,\beta,n}\right)\\ +\left(2-\tfrac{\alpha}{\beta}\right)\binom{\alpha+\beta-1}{\alpha}c_{1,1,0}^{\alpha+\beta-1,\gamma,n}+\left(2-\tfrac{\alpha}{\gamma}\right)\binom{\alpha+\gamma-1}{\alpha}c_{1,1,0}^{\alpha+\gamma-1,\beta,n}\\ -(-1)^{|F|}\binom{\beta+\gamma}{\beta}c_{0,0,0}^{\beta+\gamma,\alpha,n}=0, \quad (4.7)$$

where $a+\alpha+\beta+\gamma=\frac{n}{2}$, $\alpha\geq\beta\geq\gamma\geq 2$, $\alpha\geq 3$ and $a\in\{0,1\}$, which is obtained from the coefficient of $F^{(\alpha)}\overline{\eta}(G^{(\beta)})\overline{\eta}(H^{(\gamma)})\partial_x^a$.

For $n=2(\mu-\lambda+2)$ **odd:**

$$\left(\tfrac{a}{\alpha}+2\lambda\right)\binom{a+\alpha-1}{\alpha-1}c_{1,1,1}^{\beta,\gamma,n}+\left(\tfrac{a}{\beta}+2\lambda\right)\binom{a+\beta-1}{\beta-1}c_{1,1,1}^{\alpha,\gamma,n}+\left(\tfrac{a}{\gamma}+2\lambda\right)\binom{a+\gamma-1}{\gamma-1}c_{1,1,1}^{\alpha,\beta,n}\\ -(-1)^{|F|+|G|}\left(\binom{\alpha+\beta}{\alpha}c_{0,1,0}^{\alpha+\beta,\gamma,n}+\binom{\alpha+\gamma}{\alpha}c_{0,1,0}^{\alpha+\gamma,\beta,n}+\binom{\beta+\gamma}{\beta}c_{0,1,0}^{\beta+\gamma,\alpha,n}\right)=0, \quad (4.8)$$

where $a+\alpha+\beta+\gamma=\frac{n-1}{2}$, $\alpha\geq\beta\geq\gamma\geq 2$ and $a\in\{0,1\}$, which is obtained from the coefficient of $\overline{\eta}(F^{(\alpha)})\overline{\eta}(G^{(\beta)})\overline{\eta}(H^{(\gamma)})\partial_x^a$,

$$(-1)^{|F|+|G|}\left(\tfrac{a}{2\alpha}+\lambda\right)\binom{a+\alpha-1}{\alpha-1}c_{0,0,1}^{\beta,\gamma,n}-\left(\tfrac{a}{\beta}+\lambda\right)\binom{a+\beta-1}{\beta-1}c_{1,0,0}^{\alpha,\gamma,n}\\ +\left(\tfrac{a}{\gamma}+\lambda\right)\binom{a+\gamma-1}{\gamma-1}c_{1,0,0}^{\alpha,\beta,n}+\left(\tfrac{a}{2\alpha}-1\right)\binom{\alpha+\beta-1}{\alpha-1}c_{1,0,0}^{\alpha+\beta-1,\gamma,n}\\ -\left(\tfrac{a}{2\alpha}-1\right)\binom{\alpha+\gamma-1}{\alpha-1}c_{1,0,0}^{\alpha+\gamma-1,\beta,n}+(-1)^{|F|}\left(\tfrac{\gamma}{\beta}-1\right)\binom{\beta+\gamma-1}{\beta-1}c_{0,1,0}^{\beta+\gamma-1,\alpha,n}=0, \quad (4.9)$$

where $a+\alpha+\beta+\gamma=\frac{n+1}{2}$, $\alpha\geq 2$, $\beta\geq\gamma\geq 3$ and $a\in\{0,1\}$, which is obtained from the coefficient of $\overline{\eta}(F^{(\alpha)})G^{(\beta)}H^{(\gamma)}\partial_x^a$.

Of course, these systems have at least one solution in which the solutions $c_\varepsilon^{k_1,k_2,n}$ are just the coefficients $\beta_\varepsilon^{k_1,k_2,n}$ of the coboundary (4.3).

4.2.5. *The case where* $\mu-\lambda=\frac{9}{2}$. In this case, according to Theorem 3.2, the space of solutions is spanned by $c_{1,1,1}^{2,2,13}$ and $c_{1,0,0}^{2,4,13}$. Moreover, by (4.8), we readily obtain

$$\lambda c_{1,1,1}^{2,2,13}-3(-1)^{|F|+|G|}c_{0,1,0}^{4,2,13}=0.$$

According to formula (3.2), we deduce that the coefficients of every 2-cocycle are expressed in terms of $c_{1,1,1}^{2,2,13}$. But this general formula may contain coboundaries.

We explain how the coboundaries can be removed. Consider any coboundary given as in (4.3). We discuss the following cases:

(1) $\lambda \neq -2$. Then, by Lemma 4.3 part (i), the coefficient $c^{2,2,13}_{1,1,1}$ can be eliminated by adding the coboundary (4.3) because $\beta^{2,2,13}_{1,1,1}$ is nonzero. Hence, the cohomology is zero-dimensional.

(2) $\lambda = -2$. Then the constant $\beta^{2,2,13}_{1,1,1}$ vanishes. Hence the coefficient $c^{2,2,13}_{1,1,1}$ cannot be eliminated by adding the coboundary (4.3). Therefore the cohomology is one-dimensional.

4.2.6. *The case where* $\mu - \lambda = 5$. In this case, according to Theorem 3.2, together with formula (4.6), we check that the coefficients of every 2-cocycle are expressed in terms of $c^{4,3,14}_{0,0,0}$ and $c^{2,4,14}_{1,0,1}$. On the other hand, by Lemma 4.3 part (i), one of the coefficients $c^{4,3,14}_{0,0,0}$, $c^{2,4,14}_{1,0,1}$ can be eliminated by adding the coboundary (4.3) because $\beta^{4,3,14}_{0,0,0}$ or $\beta^{2,4,14}_{1,0,1}$ is nonzero. Hence, the cohomology is one-dimensional.

4.2.7. *The case where* $\mu - \lambda = \frac{11}{2}$. In this case, by Theorem 3.2 together with formula (4.8), we check that the coefficients of every 2-cocycle are expressed in terms of $c^{2,2,15}_{1,1,1}$ and $c^{4,3,15}_{0,0,1}$. We discuss the following cases:

1) $\lambda \neq -\frac{5}{2}$. Then, by Lemma 4.3 part (i), one of the coefficients $c^{2,2,15}_{1,1,1}$, $c^{4,3,15}_{0,0,1}$ can be eliminated by adding the coboundary (4.3) because $\beta^{2,2,15}_{1,1,1}$ or $\beta^{4,3,15}_{0,0,1}$ is nonzero. Hence, the cohomology is one-dimensional.

2) $\lambda = -\frac{5}{2}$. Then, by Lemma 4.3 part (i), the coefficients $\beta^{2,2,15}_{1,1,1}$ and $\beta^{4,3,15}_{0,0,1}$ vanish simultaneously. Hence the coefficients $c^{2,2,15}_{1,1,1}$ and $c^{4,3,15}_{0,0,1}$ cannot be eliminated by adding the coboundary (4.3). Therefore the cohomology is two-dimensional.

4.2.8. *The case where* $\mu - \lambda = p - 2$ *with* $p \in \mathbb{N} + 8$. According to Theorem 3.2 together with formulas (4.6)–(4.7), we check that the coefficients of every 2-cocycle are expressed in terms of $c^{2,4,2p}_{1,0,1}$. On the other hand, by Lemma 4.3 part (ii), we can see that

$$\beta^{2,4,2p}_{1,0,1} = 0 \quad \text{if } \lambda = -\frac{2p - 5 \pm \sqrt{8p - 15}}{4}.$$

So, in the same way as before, we deduce that the cohomology space is one-dimensional for $\lambda = -\frac{2p-5\pm\sqrt{8p-15}}{4}$ and zero-dimensional otherwise.

4.2.9. *The case where* $\mu - \lambda = p - \frac{3}{2}$ *with* $p \in \mathbb{N} + 8$. According to Theorem 3.2 together with formulas (4.8)–(4.9), we check that the coefficients of every 2-cocycle are expressed in terms of $c^{2,2,2p+1}_{1,1,1}$. On the other hand, by Lemma 4.3 part (ii), we can see that $\beta^{2,2,2p+1}_{1,1,1} = 0$ if $\lambda = 1 - \frac{p}{2}$. So, in the same way as before, we deduce that the cohomology space is one-dimensional for $\lambda = 1 - \frac{p}{2}$ and zero-dimensional otherwise. This completes the proof of Theorem 3.3.

5. Extensions of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda,\mu}$

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. The second space $\mathrm{H}^2(\mathfrak{g}, V)$ classifies the nontrivial extensions of the Lie algebra $\mathfrak{g}$ by the module V:

$$0 \longrightarrow V \longrightarrow \mathfrak{g}_V \longrightarrow \mathfrak{g} \longrightarrow 0,$$

the Lie structure on $\mathfrak{g}_V = \mathfrak{g} \oplus V$ being given by

$$[(g_1, a), (g_2, b)] = ([g_1, g_2], g_1.b - g_2.a + c(g_1, g_2)),$$

where c is a 2-cocycle with values in V.

Here we consider a natural class of "non-central" extensions of $\mathcal{K}(1)$, namely extensions by the module $\mathfrak{D}_{\lambda,\mu}$ of linear differential operators acting on weighted densities. We will be interested in the projectively invariant extensions which are given by projectively invariant 2-cocycles c. The cocycle c in this case represents a nontrivial cohomology class of the second cohomology space $\mathrm{H}^2_{\mathrm{diff}}(\mathcal{K}(1), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$. We mention that the same problem was considered in [16, 19] in the classical setting.

Corollary 5.1. *The following statements hold.*

(1) For $(\lambda, \mu) \in J$ or $\mu - \lambda = 5$ or $\mu - \lambda = \frac{11}{2}$ and $\lambda \neq -\frac{5}{2}$ or $\mu - \lambda = p - \frac{3}{2}$ and $\lambda = 1 - \frac{p}{2}$ with $p \in \mathbb{N} + 8$ or $\mu - \lambda = p - 2$ and $\lambda = -\frac{2p-5\pm\sqrt{8p-15}}{4}$ with $p \in \mathbb{N}+8$, there exists a unique non-trivial extension of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda,\mu}$, where

$$J = \Big\{(0,3), \Big(-\frac{5}{2}, \frac{1}{2}\Big), \Big(-\frac{3}{2}, 2\Big), \Big(-2, \frac{5}{2}\Big), \Big(\frac{-7+\sqrt{33}}{4}, \frac{9+\sqrt{33}}{4}\Big), \Big(\frac{-7-\sqrt{33}}{4}, \frac{9-\sqrt{33}}{4}\Big)\Big\}.$$

(2) For $\mu - \lambda = \frac{11}{2}$ and $\lambda = -\frac{5}{2}$, there exist two non-isomorphic non-trivial extensions of $\mathcal{K}(1)$ by $\mathfrak{D}_{\lambda,\mu}$.

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