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A PROPERTY OF HOMOGENEOUS ISOPARAMETRIC SUBMANIFOLDS

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Abstract. The present paper contains a new result concerning the second fundamental form of a compact, connected, homogeneous, isoparametric submanifold of codimension $h \geq 2$ in a Euclidean space.

1. Introduction

The objective of the present paper is to indicate a property of every compact, connected, homogeneous, irreducible, isoparametric submanifold M of $\mathbb{R}^{n+h}$ ($n = \dim(M)$) which, to the best of our knowledge, has not been previously noticed in the literature on the subject.

Let us consider such an isoparametric submanifold $M \subset \mathbb{R}^{n+h}$. It is a Riemannian submanifold with the induced Riemannian metric from $\mathbb{R}^{n+h}$, so we have the associated Levi-Civita connection ∇ and the corresponding second fundamental form $\alpha : T_p(M) \times T_p(M) \to T_p^{\perp}(M)$.

The definition of an *isoparametric submanifold* of a Euclidean space $M \subset \mathbb{R}^{n+h}$ has a rich and very interesting history, going back to Levi-Civita, Cartan (and beyond), and has a notable highlight in the celebrated paper by G. Thorbergsson [14][1]. A detailed account of the development of the different definitions of these submanifolds can be found in Thorbergsson's survey [15]. For a recent addition to this account, the reader may also consult the paper [16] by the same author.

The definition used in the present paper is the following. Let S be a compact (or non-compact) *irreducible* symmetric space and let $n+h$ be its dimension. Fix a point p in S and let K be the isotropy subgroup of the point p. The homogeneous isoparametric submanifolds associated to S are the principal orbits of the tangential representation of K on $T_p(S)$. They are considered submanifolds of the Euclidean space $T_p(S) \cong \mathbb{R}^{n+h}$ and are, of course, associated to the symmetric space S.[2]

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[1] It is a well-known result of G. Thorbergsson that for codimension $h \geq 3$, *all* isoparametric submanifolds are homogeneous.

[2] It is a frequent phenomenon in mathematics that an important theorem becomes a definition.

In the papers [12] and [13] it is shown that in the tangent bundle $T(M)$ of the isoparametric submanifold M there is a canonical, smooth, completely non-integrable, step 2 distribution $\mathfrak{D}(\Omega) \subset T(M)$.

Recall that a distribution $\mathfrak{D}$ of r-planes ($n > r \geq 2$) in a connected manifold M is *smooth* [17, p. 41] if, for any $p \in M$, there is an open set $A = A(p)$ containing p and r *smooth* vector fields $\{X_1, \ldots, X_r\}$ defined on A such that, for all $q \in A$ and $1 \leq j \leq r$, $X_j(q) \in \mathfrak{D}(q) \subset T_q(M)$ and $\mathfrak{D}(q) = \text{span}_{\mathbb{R}} \{X_j(q) : 1 \leq j \leq r\}$. The distribution $\mathfrak{D}$ is said to be *completely non-integrable of step 2* if, for each $p \in M$, the vector fields defined in $A(p)$ satisfy

$$\text{Span}_{\mathbb{R}} \{X_j(q),\ [X_k, X_j](q) : 1 \leq k, j \leq r\} = T_q(M) \quad \forall q \in A, \tag{1.1}$$

i.e., for $\mathfrak{D}(q) = \text{span}_{\mathbb{R}} \{X_j(q)\}$ we have $\mathfrak{D}^2(q) = (\mathfrak{D} + [\mathfrak{D}, \mathfrak{D}])(q) = T_q(M)$ $\forall q \in A \subset M$.

We consider, on each $\mathfrak{D}(p)$, the restriction of the inner product $\langle *, * \rangle_p$ on $T_p(M)$. The triple $(M, \mathfrak{D}, \langle *, * \rangle)$ is called a *sub-Riemannian manifold*. The mentioned property of the isoparametric submanifold M is the content of the following theorem.

Theorem 1.1. *Let M be a compact, connected, homogeneous, irreducible, isoparametric submanifold M of $\mathbb{R}^{n+h}$ with codimension $h \geq 2$, and let $\alpha(X, Y)$ be its second fundamental form in $\mathbb{R}^{n+h}$. Then, for any point $p \in M$ and any $X \in \mathfrak{D}(p) \subset T_p(M)$, we have*

$$\left(\overline{\nabla}_X \alpha\right)(X, X) = 0,$$

where $\left(\overline{\nabla}_X \alpha\right)(Y, Z) = \nabla_X^{\perp}(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$ is the usual covariant derivative of α. ☐

Then, since $\left(\overline{\nabla}_X \alpha\right)(Y, Z)$ *is symmetric by Codazzi's equation*, as a consequence we have

Corollary 1.2. *The restriction of the second fundamental form α from $T_p(M) \times T_p(M)$ to $\mathfrak{D}(\Omega) \times \mathfrak{D}(\Omega)$ is parallel, i.e., $\left(\overline{\nabla}_X \alpha\right)(Y, Z) = 0$ for $X, Y, Z \in \mathfrak{D}(p) \subset T_p(M)$ for all $p \in M$.* ☐

This corollary may be rephrased by saying:

The sub-Riemannian manifold $(M, \mathfrak{D}, \langle *, * \rangle)$ is parallel.

Furthermore, we also have:

Theorem 1.3. *Under the hypothesis of Theorem* 1.1*, given any pair of points p and q in M, there exists a C^∞ horizontal curve $\gamma : [0, b] \to M$ such that*

(i) *$\gamma(0) = p$, $\gamma(b) = q$, and $\gamma'(t) \neq 0$ for all $t \in [0, b]$ (i.e., γ is regular);*
(ii) *the curve γ is a* geodesic *of the sub-Riemannian metric defined on M;*
(iii) *α is parallel along the curve γ, that is, $\left(\overline{\nabla}_{\gamma'(t)} \alpha\right)(\gamma'(t), \gamma'(t)) = 0$ for all $t \in [0, b]$.* ☐

The organization of the paper is as follows. In the next section, we mention the essential facts concerning the isoparametric submanifolds under consideration, while in Section 3 we recall the definition of the distribution $\mathfrak{D}(\Omega)$. That section also contains Lemma 3.1, which is a consequence of the existence of $\mathfrak{D}(\Omega)$ and a fundamental property of the distribution. Lemma 3.1 is indicated in [12] as *well known*; however, because of its importance for Theorem 1.3, we include a proof in Section 5. The proof of Theorem 1.1 is given in Section 4. In that proof we make essential use of formula (A.8), which, for completeness, is proved in the Appendix.

2. The submanifolds under consideration

The mentioned homogeneous isoparametric submanifolds of codimension $h \geq 2$ in Euclidean spaces are the principal orbits of the tangential representation, at a basic point, of a compact (or non-compact dual) symmetric space. To obtain one of these submanifolds, we consider a *real simple non-compact* Lie algebra $\mathfrak{g}_0$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and Cartan involution θ. Then $\mathfrak{k}_0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}_0$ [6, Prop. 7.4, p. 184]. Let K be the analytic subgroup of $\operatorname{Int}(\mathfrak{g}_0)$ corresponding to the subalgebra $ad_{\mathfrak{g}_0}(\mathfrak{k}_0)$ of $ad_{\mathfrak{g}_0}(\mathfrak{g}_0)$ which is compact and let B_θ be the *positive definite*, *symmetric* bilinear form on $\mathfrak{g}_0$ defined by

$$B_\theta(x,y) := \langle x,y\rangle_\theta = -B(x,\theta y), \tag{2.1}$$

where B is the Killing form of $\mathfrak{g}_0$.

The *principal orbits* of the representation of K on $\mathfrak{p}_0$ are the isoparametric submanifolds M of $\mathbb{R}^{n+h} = (\mathfrak{p}_0, \langle *,*\rangle_\theta)$.

This is a consequence of our definition of a isoparametric submanifold and the connection between irreducible symmetric spaces and the Cartan decomposition of simple Lie algebras. This fact seems to have been indicated for the first time in [11].[3]

Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ and consider the set $\Phi(\mathfrak{g}_0,\mathfrak{a}_0)$ of roots restricted to $\mathfrak{a}_0$ (see [12] for details and notation). Let $\Delta(\mathfrak{g}_0,\mathfrak{a}_0)$ be a system of simple roots in $\Phi(\mathfrak{g}_0,\mathfrak{a}_0)$. For $\lambda \in \Phi(\mathfrak{g}_0,\mathfrak{a}_0)$, it is usual to define the subspaces associated to the Cartan decomposition

$$\begin{aligned}\mathfrak{k}_{0,\lambda} &= \left\{x \in \mathfrak{k}_0 : (ad\,(h))^2\, x = \lambda^2(h)x \ \ \forall h \in \mathfrak{a}_0\right\},\\ \mathfrak{p}_{0,\lambda} &= \left\{x \in \mathfrak{p}_0 : (ad\,(h))^2\, x = \lambda^2(h)x \ \ \forall h \in \mathfrak{a}_0\right\},\end{aligned}$$

for which obviously $\mathfrak{k}_{0,\lambda} = \mathfrak{k}_{0,(-\lambda)}$, $\mathfrak{p}_{0,\lambda} = \mathfrak{p}_{0,(-\lambda)}$. With them, with respect to B_θ (2.1), we have the orthogonal decompositions

$$\mathfrak{k}_0 = \mathfrak{m}_0 \oplus \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{k}_{0,\lambda}, \qquad \mathfrak{p}_0 = \mathfrak{a}_0 \oplus \sum_{\lambda\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{p}_{0,\lambda},$$

where $\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)$ is the set of roots written with non-negative coefficients in terms of $\Delta(\mathfrak{g}_0,\mathfrak{a}_0)$ and $\mathfrak{m}_0$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$.

Recall that for any pair $\lambda,\mu \in \Phi(\mathfrak{g}_0,\mathfrak{a}_0)$, we have the formulae

$$\begin{aligned}&[\mathfrak{k}_{0,\lambda},\mathfrak{p}_{0,\mu}] \subset \mathfrak{p}_{0,(\lambda+\mu)} + \mathfrak{p}_{0,(\lambda-\mu)},\\ &[\mathfrak{k}_{0,\lambda},\mathfrak{p}_{0,\lambda}] \subset \mathfrak{p}_{0,(2\lambda)} + \mathfrak{a}_0,\\ &[\mathfrak{k}_{0,\lambda},\mathfrak{a}_0] = \mathfrak{p}_{0,\lambda},\end{aligned} \tag{2.2}$$

[3] We thank the anonymous referee for this observation.

where $\mathfrak{p}_{0,\delta} = \{0\}$ and $\mathfrak{k}_{0,\delta} = \{0\}$ if δ is not a root.

Let us fix a *regular* element $E \in \mathfrak{a}_0 \subset \mathfrak{p}_0$, call $M = Ad(K)E \subset \mathfrak{p}_0$ its orbit and let K_E be the isotropy subgroup of K at E. The regularity of E implies that the isotropy subalgebra (corresponding to) K_E is $\mathfrak{k}_{0,E} = \mathfrak{m}_0$. Furthermore, the tangent and normal spaces of M at E are

$$T_E(M) = \sum_{\lambda \in \Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} [\mathfrak{k}_{0,\lambda}, E] = \sum_{\lambda \in \Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)} \mathfrak{p}_{0,\lambda} \quad \text{and} \quad T_E^{\perp}(M) = \mathfrak{a}_0. \tag{2.3}$$

2.1. **The manifolds of complete flags $M = K/T^n$.** An important special case of the above isoparametric submanifolds are the submanifolds of the form $M = K/T^n$, where T^n is a maximal torus of the compact, connected, simple, adjoint (i.e., centerless) Lie group K. Let $\mathfrak{u}_0$ be the compact simple Lie algebra corresponding to K and let $\mathfrak{u}_0^{\mathbb{C}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$ be the complexification $\mathfrak{u}_0^{\mathbb{C}}$ of $\mathfrak{u}_0$ *considered as a real Lie algebra*. Then $\mathfrak{g}_0 = (\mathfrak{u}_0^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0 \simeq \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$ [6, Cor. 7.5, p. 185] and we may consider, as in the previous case, the principal orbits of the adjoint action of K which are isoparametric submanifolds of $\mathfrak{p}_0 = i\mathfrak{u}_0$. They are *manifolds of complete flags* of the form $M = K/T^n$ for a maximal torus of the group K.

Let us take a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{u}_0$, so $i\mathfrak{t}_0 \subset i\mathfrak{u}_0$ is a maximal abelian subspace of $i\mathfrak{u}_0$ and $\mathfrak{h} = (\mathfrak{t}_0 \oplus i\mathfrak{t}_0) \subset \mathfrak{u}_0 \oplus i\mathfrak{u}_0 = \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. We have the roots in $\Phi(\mathfrak{g}_0, \mathfrak{h})$ and the restricted roots are those in $\Phi(\mathfrak{g}_0, i\mathfrak{t}_0)$. They are just the roots of $\mathfrak{u}_0$ with respect to $\mathfrak{t}_0$. Also in this case, we shall use the general notation indicated above.

3. Distribution

The roots of $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ are written in terms of $\Delta(\mathfrak{g}_0, \mathfrak{a}_0)$ as a $\mathbb{Z}$ linear combination with non-negative coefficients. It is usual to define the *height* of a root as the sum of these coefficients, and we may consider in $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)$ the subsets Ω and Γ of roots of odd and even height, respectively. Then $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Omega \cup \Gamma$ and we may consider, associated to the set Ω, a subspace $\mathfrak{D}_E(\Omega) \subset T_E(M)$ (2.3) defined by $\mathfrak{D}_E(\Omega) = \sum_{\lambda \in \Omega} \mathfrak{p}_{0,\lambda}$. This subspace is *invariant* by the action of the isotropy subgroup at E. Hence $\mathfrak{D}_E(\Omega)$ defines a *distribution* $\mathfrak{D}(\Omega)$ on the manifold M by translation with the action of the group K. Then, at each point $q = Ad(g)E \in M$, we have: $\mathfrak{D}_q = \mathfrak{D}_q(\Omega) = Ad(g)\mathfrak{D}_E(\Omega) \subset T_q(M)$. It is clear that the distribution $\mathfrak{D}(\Omega)$ is well defined. As we indicated above, in [12] it is shown that $\mathfrak{D}(\Omega)$ is a smooth and completely non-integrable distribution of step 2 in M.

The presence in M of the completely non-integrable step 2 distribution $\mathfrak{D}_q(\Omega)$ has the following consequence. Recall that, given the distribution $\mathfrak{D}$, a curve $\gamma : [0, b] \to M$ is said to be *horizontal for* $\mathfrak{D}$ if $\gamma'(t) \in \mathfrak{D}(\gamma(t))$ for all $t \in [0, b]$, and *regular* if $\gamma'(t) \neq 0$ for all t.

Lemma 3.1. *Let M be a compact, connected, homogeneous isoparametric submanifold of $\mathbb{R}^{n+h} = (\mathfrak{p}_0, \langle *, * \rangle_\theta)$ and consider in M the smooth distribution $\mathfrak{D}(\Omega)$, defined above, which is completely non-integrable of step 2. Then, for any two* *points p, q in M, there exists a horizontal, C^∞, regular curve $\gamma : [0, b] \to M$ such that $\gamma(0) = p$, $\gamma(b) = q$.*

A proof of Lemma 3.1 is included in Section 5.

4. Proof of Theorem 1.1

Let us consider our isoparametric submanifold $M = Ad(K)E \subset \mathfrak{p}_0$ and take two points a and c in M. Let us take the C^∞ regular curve $\gamma : [0, b] \to M$ (i.e., $\gamma'(t) \neq 0$ for all t) such that $\gamma(0) = a$, $\gamma(b) = c$ given by Lemma 3.1. As usual, to say that the curve γ is C^∞ in $[0, b]$ means that it is defined and is C^∞ in an open interval containing $[0, b]$. Let $t \in [0, b]$. We need to show that the second fundamental form α of M in $\mathfrak{p}_0$ satisfies

$$\left(\overline{\nabla}_{\gamma'(t)}\alpha\right)(\gamma'(t), \gamma'(t)) = 0. \tag{4.1}$$

Since $\gamma(t) \in M$, there exists $w \in K$ such that

$$\gamma(t) = Ad(w)E =: w(E), \qquad \gamma'(t) \in \mathfrak{D}_{\gamma(t)} = Ad(w)\mathfrak{D}_E(\Omega) \subset T_{\gamma(t)}(M).$$

Since K acts on the ambient space $(\mathfrak{p}_0, \langle x, y\rangle_\theta)$ by isometries, for each $Y \in T_E(M)$ the derivative $w_*|_E : T_E(\mathfrak{p}_0) \to T_{\gamma(t)}(\mathfrak{p}_0)$ satisfies

$$\left(\overline{\nabla}_{w_*|_E Y}\alpha\right)(w_*|_E Y, w_*|_E Y) = w_*|_E \left(\overline{\nabla}_Y\alpha\right)(Y, Y), \qquad Y \in T_E(M),$$

and since

$$w_*|_E Z = Ad(w)Z \qquad \forall Z \in T_E(\mathfrak{p}_0),$$

in order to prove (4.1) we just need to show that

$$\left(\overline{\nabla}_{\gamma'(0)}\alpha\right)(\gamma'(0), \gamma'(0)) = 0, \qquad \gamma'(0) \in \mathfrak{D}_E(\Omega) \subset T_E(M), \tag{4.2}$$

and so, in turn, to prove (4.2), it is enough to show that

$$\left(\overline{\nabla}_Y\alpha\right)(Y, Y) = 0 \qquad \forall\, Y \in \mathfrak{D}_E(\Omega) \subset T_E(M). \tag{4.3}$$

To initiate the proof of (4.3), we need to introduce some general notation. Recall that $\Phi^+(\mathfrak{g}_0, \mathfrak{a}_0) = \Omega \cup \Gamma$, and let us set

$$\begin{aligned} \mathfrak{u}_0 &= \sum_{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)} \mathfrak{k}_{0,\lambda}, \qquad \mathfrak{k}_0 = \mathfrak{m}_0 \oplus \mathfrak{u}_0, \\ \mathfrak{q}_0 &= \sum_{\lambda \in \Phi^+(\mathfrak{g}_0, \mathfrak{a}_0)} \mathfrak{p}_{0,\lambda}, \qquad \mathfrak{p}_0 = \mathfrak{a}_0 \oplus \mathfrak{q}_0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \mathfrak{u}_0(\Omega) &= \sum_{\lambda \in \Omega} \mathfrak{k}_{0,\lambda}, \qquad \mathfrak{q}_0(\Omega) = \sum_{\lambda \in \Omega} \mathfrak{p}_{0,\lambda}, \\ \mathfrak{u}_0(\Gamma) &= \sum_{\lambda \in \Gamma} \mathfrak{k}_{0,\lambda}, \qquad \mathfrak{u}_0 = \mathfrak{u}_0(\Omega) \oplus \mathfrak{u}_0(\Gamma), \\ \mathfrak{q}_0(\Gamma) &= \sum_{\lambda \in \Gamma} \mathfrak{p}_{0,\lambda}, \qquad \mathfrak{q}_0 = \mathfrak{q}_0(\Omega) \oplus \mathfrak{q}_0(\Gamma), \\ &\mathfrak{d}_0(\Gamma) = \mathfrak{a}_0 \oplus \mathfrak{q}_0(\Gamma). \end{aligned} \tag{4.5}$$

To compute the covariant derivative of the second fundamental form $\left(\overline{\nabla}_Y \alpha\right)(Y,Y)$, we use the formula (A.8), which is proved in the Appendix. That is,

$$\left(\overline{\nabla}_{[X,E]}\alpha\right)([X,E],[X,E]) = -2\left(\left[X,([X,[X,E]])_{\mathfrak{q}_0}\right]_{\mathfrak{a}_0}\right) \qquad \text{for } X \in \mathfrak{u}_0 \subset \mathfrak{k}_0.$$

Let us take any *non-zero* $Y \in \mathfrak{q}_0(\Omega) = \mathfrak{D}_E(\Omega) \subset T_E(M) \subset \mathfrak{p}_0$. Since, by (2.2) and (4.5), $\mathfrak{q}_0(\Omega) = [\mathfrak{u}_0(\Omega), \mathfrak{a}_0]$, we may take $X \in \mathfrak{u}_0(\Omega)$ such that $Y = [X,E]$. Then, *for that* X, we have to evaluate $\left(\left[X,([X,[X,E]])_{\mathfrak{q}_0}\right]\right)_{\mathfrak{a}_0}$. Let us start by noticing that

$$[X,[X,E]] \in [\mathfrak{u}_0(\Omega), \mathfrak{q}_0(\Omega)].$$

Then we have to study the product $[\mathfrak{u}_0(\Omega), \mathfrak{q}_0(\Omega)]$, and to that end compute

$$[\mathfrak{u}_0(\Omega), \mathfrak{q}_0(\Omega)] = \sum_{\lambda,\gamma\in\Omega} [\mathfrak{k}_{0,\lambda}, \mathfrak{p}_{0,\gamma}],$$

so we need to compute the brackets $[\mathfrak{k}_{0,\lambda}, \mathfrak{p}_{0,\gamma}]$ (for $\lambda,\gamma \in \Omega$). By (2.2), for any pair $\lambda,\gamma \in \Omega$ we have

$$[\mathfrak{k}_{0,\lambda}, \mathfrak{p}_{0,\gamma}] \subset \begin{cases} \mathfrak{p}_{0,(2\lambda)} + \mathfrak{a}_0 & \text{if } \gamma = \lambda, \\ \mathfrak{p}_{0,(\lambda+\mu)} + \mathfrak{p}_{0,(\lambda-\mu)} & \text{if } \gamma \neq \lambda. \end{cases}$$

Clearly, if $\lambda+\gamma$ is a root (or 2λ is a root), it is positive and belongs to Γ. On the other hand, if $\gamma-\lambda$ is a root then $|\gamma-\lambda|$ is a root and since $\mathfrak{p}_{0,(\gamma-\lambda)} = \mathfrak{p}_{0,|\gamma-\lambda|}$, it also belongs to Γ. Therefore, for any pair $\lambda,\gamma \in \Omega$, we have

$$[\mathfrak{k}_{0,\lambda}, \mathfrak{p}_{0,\gamma}] \subset \mathfrak{a}_0 \oplus \sum_{\mu\in\Gamma} \mathfrak{p}_{0,\mu} = \mathfrak{d}_0(\Gamma).$$

Then we have the inclusion

$$[\mathfrak{u}_0(\Omega), \mathfrak{q}_0(\Omega)] \subset \mathfrak{d}_0(\Gamma),$$

and in turn

$$[X,[X,E]] \in \mathfrak{d}_0(\Gamma) = \mathfrak{a}_0 \oplus \mathfrak{q}_0(\Gamma).$$

This clearly yields

$$([X,[X,E]])_{\mathfrak{q}_0} = ([X,[X,E]])_{(\mathfrak{d}_0(\Gamma)\cap\mathfrak{q}_0)} \in \mathfrak{d}_0(\Gamma).$$

Now, multiplying again by X, we have

$$\left[X,([X,[X,E]])_{\mathfrak{q}_0}\right] \in [\mathfrak{u}_0(\Omega), \mathfrak{d}_0(\Gamma)]. \tag{4.6}$$

Then we need to compute the product $[\mathfrak{u}_0(\Omega), \mathfrak{d}_0(\Gamma)]$. Recalling (4.4), we observe that, by the definitions of $\mathfrak{u}_0(\Omega)$ and $\mathfrak{d}_0(\Gamma)$, we have

$$\begin{aligned}[\mathfrak{u}_0(\Omega), \mathfrak{d}_0(\Gamma)] &= \left[\left(\sum_{\lambda\in\Omega} \mathfrak{k}_{0,\lambda}\right), \left(\mathfrak{a}_0 \oplus \sum_{\mu\in\Gamma} \mathfrak{p}_{0,\mu}\right)\right] \\ &= \sum_{\lambda\in\Omega} [\mathfrak{k}_{0,\lambda}, \mathfrak{a}_0] + \sum_{\lambda\in\Omega}\sum_{\mu\in\Gamma} [\mathfrak{k}_{0,\lambda}, \mathfrak{p}_{0,\mu}],\end{aligned}$$

so we study each of the last two terms separately. By (2.2), for the first term we have

$$\sum_{\lambda\in\Omega}[\mathfrak{k}_{0,\lambda},\mathfrak{a}_0]=\sum_{\lambda\in\Omega}\mathfrak{p}_{0,\lambda}\subset\mathfrak{q}_0,$$

while, again by (2.2), for the second one we similarly have

$$\begin{aligned}&\sum_{\lambda\in\Omega}\sum_{\mu\in\Gamma}[\mathfrak{k}_{0,\lambda},\mathfrak{p}_{0,\mu}]\\ &\quad\subset\sum_{\lambda\in\Omega}\sum_{\mu\in\Gamma}\left(\mathfrak{p}_{0,(\lambda+\mu)}+\mathfrak{p}_{0,|\lambda-\mu|}\right)=\sum_{\lambda\in\Omega}\sum_{\mu\in\Gamma}\mathfrak{p}_{0,(\lambda+\mu)}+\sum_{\lambda\in\Omega}\sum_{\mu\in\Gamma}\mathfrak{p}_{0,|\lambda-\mu|}\\ &\quad\subset\sum_{\delta\in\Phi^+(\mathfrak{g}_0,\mathfrak{a}_0)}\mathfrak{p}_{0,\delta}=\mathfrak{q}_0.\end{aligned}$$

Then we obtain

$$[\mathfrak{q}_0(\Omega),\mathfrak{d}_0(\Gamma)]\subset\mathfrak{q}_0,$$

and in turn, by (4.6), we have

$$\left[X,([X,[X,E]])_{\mathfrak{q}_0}\right]\in\mathfrak{q}_0.$$

Then we clearly get

$$\left(\left[X,([X,[X,E]])_{\mathfrak{q}_0}\right]\right)_{\mathfrak{a}_0}=0,$$

because $\mathfrak{q}_0$ and $\mathfrak{a}_0$ are orthogonal to each other. So our $Y=[X,E]\in\mathfrak{q}_0(\Omega)$ satisfies

$$\left(\overline{\nabla}_{[X,E]}\alpha\right)([X,E],[X,E])=\left(\overline{\nabla}_Y\alpha\right)(Y,Y)=0.$$

Then we have proved (4.3) and Theorem 1.1. □

5. Proof of Lemma 3.1

Let $\langle *,*\rangle_p$ ($p\in M$) be an inner product defined on $\mathfrak{D}(p)$ varying smoothly with p. The triple $(M,\mathfrak{D},\langle *,*\rangle)$ is called a *sub-Riemannian manifold*. A particular case is when (as it is the situation here) we start with a Riemannian manifold M and take the restriction of the inner product in each tangent space to $\mathfrak{D}$. A Lipschitz curve $\gamma:[0,b]\to M$ is called *horizontal* if $\gamma'(t)\in\mathfrak{D}(\gamma(t))$ for almost every $t\in[0,b]$. The length of γ is defined in the usual way as $L(\gamma)=\int_0^b\langle\gamma'(t),\gamma'(t)\rangle_{\gamma(t)}\,dt$, and for two points $x,y\in M$ it is usual to define

$$d(x,y)=\inf\{L(\gamma):\gamma\text{ is horizontal, }\gamma(0)=x\text{ and }\gamma(b)=y\}. \tag{5.1}$$

If the set of horizontal Lipschitz curves joining x with y is nonempty for any (x,y), then d is a distance on M, called the *Carnot–Carathéodory distance* [4]. Since M is connected, and the distribution satisfies (1.1) (the Hörmander condition), the Chow–Rashevski theorem [8, Thm. 2.2], [4, p. 95] implies that the set of Lipschitz curves joining x and y is nonempty for every pair (x,y). Furthermore, by the topological theorem [8, Thm. 2.3] the topology of the Carnot–Carathéodory distance coincides with the manifold topology. Since M is compact, every sequence in M has a convergent subsequence, hence every Cauchy sequence converges. Therefore M with the Carnot–Carathéodory distance d is complete. If (M,d) is complete, the closed balls are compact and the argument in [1, Cor. 3.49] shows that the infimum in (5.1) is attained. That is, there is at least one horizontal curve γ joining x with y such that $L(\gamma) = d(x, y)$ (in general, γ is not unique). This γ is called a *length minimizing curve* (or *length minimizer*). This curve is Lipschitz (differentiable almost everywhere on $[0, b]$).

In our situation, we have on the manifold M a distribution $\mathfrak{D}$ that is a *bracket generating of step 2*, which means that for every point $p \in M$ and some local set of fields $\{X_1, \ldots, X_r\}$ (defined in an open set A containing p) generating $\mathfrak{D}(p)$ for all $p \in A \subset M$ we have: $\mathfrak{D}^2(p) = (\mathfrak{D} + [\mathfrak{D}, \mathfrak{D}])(p) = T_p(M)$. That is, (1.1) is satisfied.

The point to be mentioned here is that length minimizers can be of two types, respectively called *normal* and *abnormal* (see [7, p. 329]). But it is well known that distributions of step 2 have not abnormal length minimizers (see, for instance, [9, Thm. 4], and also [5, Section 1.4] and [2, Thms. 3.7 and 3.8]), so we are left with the normal ones. The *normal length minimizers* are those which satisfy the geodesic equation [1, Thm. 4.25]. Then, given two points x, y in M, there is a normal minimizer joining them. This curve in M is C^∞ and parametrized by constant non-zero speed, hence, it is regular in M [1, Cor. 4.27]. Then, it can be parametrized by arc length. Compare also [9, Thm. 4].

This yields the proof of Lemma 3.1. □

Appendix A. Computation of $\overline{\nabla}\alpha$

In the present appendix we give a proof of formula (A.8), which is used in an essential way in the proof of Theorem 1.1 (see Section 4). It is important to make here the following remark.

Remark A.1. The isoparametric submanifolds under consideration are R-spaces (orbits of s-representations) since, in fact, they are all principal orbits of the tangential representations of symmetric spaces.

We have the Euclidean covariant derivative ∇^E in $(\mathfrak{p}, \langle *, * \rangle)$ and the Levi-Civita connection ∇ associated to the induced metric on M. Also on M we are going to consider the *canonical connection* determined by the decomposition $\mathfrak{k}_0 = \mathfrak{m}_0 \oplus \mathfrak{u}_0$, (4.4) which we shall denote by ∇^C [3, p. 200]. We have the second fundamental form of M on $\mathfrak{p}$ and Gauss formula

$$\nabla^E_U W = \nabla_U W + \alpha(U, W).$$

We need to compute $\nabla^E_U W$. Let us take, for some $X \in \mathfrak{u}_0$, the curve in M of the form

$$\gamma(t) = (Ad\,(\exp(tX))\,E)\,. \tag{A.1}$$

Its tangent vector at E is $\gamma'(0) = [X, E]$, and if we take $t_1 > 0$ then we may compute the derivative $\gamma'(t_1)$ by

$$\begin{aligned}\gamma'(t_1) &= \frac{d}{dt}\Big|_{t=0} \left(Ad\left(\exp\left((t_1+t)X\right)\right)E\right)\\ &= \frac{d}{dt}\Big|_{t=0} \left(Ad\left(\exp\left((t_1)X\right)\right)Ad\left(\exp\left((t)X\right)\right)E\right)\\ &= Ad\left(\exp\left((t_1)\,X\right)\right)\frac{d}{dt}\Big|_{t=0} Ad\left(\exp\left((t)X\right)\right)E\\ &= Ad\left(\exp\left((t_1)\,X\right)\right)[X, E],\end{aligned} \tag{A.2}$$

and so this gives the tangent field along $\gamma(t)$.

It is well known that curves like γ in (A.1) are ∇^C-geodesics, and that the ∇^C-parallel translation along these geodesics is precisely given by (A.2) [3, p. 200].

Let us take a tangent vector at E,

$$[Y, E] \in T_E(M) = [\mathfrak{k}_0, E] = [\mathfrak{u}_0, E],$$

and extend it to a field along γ by

$$[Y, E]^* = Ad\left(\exp\left(tX\right)\right)[Y, E]. \tag{A.3}$$

Let us compute now, for $X, Y \in \mathfrak{u}_0$, in (4.4)

$$\nabla^E_{[X,E]}[Y, E]^* = \frac{d}{dt}\Big|_{t=0} \left(Ad\left(\exp\left(tX\right)\right)[Y, E]\right) = [X, [Y, E]] \in \mathfrak{p}_0,$$

and writing the Gauss formula for $[X, E]$ and $[Y, E]^*$, we have

$$\nabla^E_{[X,E]}[Y, E]^* = \nabla_{[X,E]}[Y, E]^* + \alpha\left([X, E], [Y, E]\right).$$

Then, since $\mathfrak{p}_0 = \mathfrak{a}_0 \oplus \mathfrak{q}_0$, we may take the component in each one of these subspaces. That is,

$$\begin{aligned}\nabla_{[X,E]}[Y, E]^* &= \left([X, [Y, E]]\right)_{\mathfrak{q}_0},\\ \alpha\left([X, E], [Y, E]\right) &= \left([X, [Y, E]]\right)_{\mathfrak{a}_0}.\end{aligned} \tag{A.4}$$

Now, we need to compute the covariant derivative of α, which, by definition, is

$$\begin{aligned}&\left(\overline{\nabla}_{[X,E]}\alpha\right)\left([Y, E], [Z, E]\right) \\ &\quad = \nabla^{\perp}_{[X,E]}\alpha\left([Y, E], [Z, E]\right) - \alpha\left(\nabla_{[X,E]}\,[Y, E]\,, [Z, E]\right) - \alpha\left([Y, E], \nabla_{[X,E]}\,[Z, E]\right).\end{aligned} \tag{A.5}$$

In [10] (see also [3, p. 212]), the *canonical covariant derivative of the second fundamental form* α was introduced as follows:

$$\begin{aligned}&\left(\nabla^{\mathbf{C}}_{[X,E]}\alpha\right)([Y, E], [Z, E])\\ &\quad = \nabla^{\perp}_{[X,E]}\alpha([Y, E], [Z, E]) - \alpha\left(\nabla^{\mathbf{C}}_{[X,E]}[Y, E], [Z, E]\right) - \alpha\left([Y, E], \nabla^{\mathbf{C}}_{[X,E]}[Z, E]\right).\end{aligned}$$

Recall now that a central result of the paper [10] is that the condition

$$\left(\nabla^{\mathbf{C}}_{[X,E]}\alpha\right)\left([Y, E], [Z, E]\right) = 0$$

characterizes R-spaces and, since M, as indicated in Remark A.1, is an R-space, we have

$$0 = \nabla^{\perp}_{[X,E]}\alpha([Y,E],[Z,E]) - \alpha\big(\nabla^{\mathbf{C}}_{[X,E]}[Y,E],[Z,E]\big) - \alpha\big([Y,E],\nabla^{\mathbf{C}}_{[X,E]}[Z,E]\big). \tag{A.6}$$

Subtracting now (A.6) from (A.5) we get

$$\begin{aligned}(\overline{\nabla}_{[X,E]}\alpha)([Y,E],[Z,E])\\ &= -\alpha(D([X,E],[Y,E]),[Z,E]) - \alpha([Y,E],D([X,E],[Z,E])),\end{aligned}$$

where

$$D = \nabla - \nabla^{\mathbf{C}}$$

is the difference tensor of the two connections ∇ and $\nabla^{\mathbf{C}}$. Now, if we take $X = Y = Z$, we obtain, for $X \in \mathfrak{u}_0 \subset \mathfrak{k}_0$,

$$\big(\overline{\nabla}_{[X,E]}\alpha\big)([X,E],[X,E]) = -2\alpha(D([X,E],[X,E]),[X,E]). \tag{A.7}$$

Now, to complete our computation of $\overline{\nabla}\alpha$, we need to study the difference tensor $D\left([X,E],[Y,E]\right)$. To that end we use the fact that the tangent field $[Y,E]^*$ in (A.3) is parallel with respect to the canonical connection $\nabla^{\mathbf{C}}$ along γ. Then we have

$$D\left([X,E],[Y,E]\right) = \nabla_{[X,E]}[Y,E]^* - \nabla^{C}_{[X,E]}[Y,E]^* = \nabla_{[X,E]}[Y,E]^*$$

and going back to (A.4) we obtain

$$D([X,E],[Y,E]) = \nabla_{[X,E]}[Y,E]^* = Ta\left([X,[Y,E]]\right) = \left([X,[Y,E]]\right)_{\mathfrak{q}_0}.$$

Therefore the equality (A.7) becomes

$$\big(\overline{\nabla}_{[X,E]}\alpha\big)([X,E],[X,E]) = -2\big(\big[X,([X,[X,E]])_{\mathfrak{q}_0}\big]_{\mathfrak{a}_0}\big), \tag{A.8}$$

and we have the sought formula.

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