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VARIABLE CALDERÓN–HARDY SPACES ON THE HEISENBERG GROUP

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ABSTRACT. We introduce Calderón–Hardy type spaces with variable exponents on the Heisenberg group and investigate their properties. As an application, we prove that the Heisenberg sub-Laplacian is a bijective mapping from variable Calderón–Hardy spaces onto the corresponding variable Hardy spaces.

1. Introduction

The Heisenberg group $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$, whose (noncommutative) group law is given by

$$(x,t) \cdot (y,s) = (x+y, t+s+x^t J y), \tag{1.1}$$

where J is the $2n \times 2n$ skew-symmetric matrix given by

$$J = 2 \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n denoting the $n \times n$ identity matrix.

The dilation group on $\mathbb{H}^n$ is defined by

$$r \cdot (x,t) = (rx, r^2 t), \quad r > 0.$$

With this structure we have that $e = (0,0)$ is the neutral element, $(x,t)^{-1} = (-x,-t)$ is the inverse of (x,t), and $r \cdot ((x,t)\cdot(y,s)) = (r\cdot(x,y)) \cdot (r \cdot (y,s))$.

The *Koranyi norm* on $\mathbb{H}^n$ is the function $\rho : \mathbb{H}^n \to [0,\infty)$ defined by

$$\rho(x,t) = \left(|x|^4 + t^2\right)^{1/4}, \quad (x,t) \in \mathbb{H}^n, \tag{1.2}$$

where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^{2n}$. Moreover, ρ is continuous on $\mathbb{H}^n$ and is smooth on $\mathbb{H}^n \setminus \{e\}$.

The ρ-ball centered at $z_0 \in \mathbb{H}^n$ with radius $\delta > 0$ is defined by

$$B(z_0, \delta) := \{w \in \mathbb{H}^n : \rho(z_0^{-1} \cdot w) < \delta\}.$$

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The topology in $\mathbb{H}^n$ induced by the ρ-balls coincides with the Euclidean topology of $\mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ (see [3, Proposition 3.1.37]). So, the borelian sets of $\mathbb{H}^n$ are identified with those of $\mathbb{R}^{2n+1}$. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$, thus $L^p(\mathbb{H}^n) \equiv L^p(\mathbb{R}^{2n+1})$ for every $0 < p \leq \infty$. Moreover, for $f \in L^1(\mathbb{H}^n)$ and for $r > 0$ fixed, we have

$$\int_{\mathbb{H}^n} f(r \cdot z)\, dz = r^{-Q} \int_{\mathbb{H}^n} f(z)\, dz,$$

where $Q = 2n + 2$. The number $2n + 2$ is known as the *homogeneous dimension* of $\mathbb{H}^n$ (we observe that the *topological dimension* of $\mathbb{H}^n$ is $2n + 1$).

If f and g are measurable functions on $\mathbb{H}^n$, their convolution $f * g$ is defined by

$$(f * g)(z) := \int_{\mathbb{H}^n} f(w) g(w^{-1} \cdot z)\, dw$$

when the integral is finite.

A measurable function $p(\cdot) : \mathbb{H}^n \to (0, \infty)$ is called an exponent on $\mathbb{H}^n$, and we adopt the standard notation for variable exponents. Given a measurable set $E \subset \mathbb{H}^n$, let

$$p_-(E) = \operatorname*{ess\,inf}_{z \in E} p(z) \quad \text{and} \quad p_+(E) = \operatorname*{ess\,sup}_{z \in E} p(z).$$

When $E = \mathbb{H}^n$, we will simply write $p_- := p_-(\mathbb{H}^n)$ and $p_+ := p_+(\mathbb{H})$. Throughout this paper, we will assume that $0 < p_- \leq p_+ < \infty$. We also define $\underline{p} = \min\{p_-, 1\}$.

Given an exponent $p(\cdot) : \mathbb{H}^n \to (0, \infty)$, we define the modular function $\kappa_{p(\cdot)}$ on the set of the all measurable functions f by

$$\kappa_{p(\cdot)}(f) = \int_{\mathbb{H}^n} |f(z)|^{p(z)}\, dz. \tag{1.3}$$

By $L^{p(\cdot)}(\mathbb{H}^n)$ we denote the space of all measurable functions f on $\mathbb{H}^n$ such that, for some $\lambda > 0$,

$$\kappa_{p(\cdot)}(f/\lambda) < \infty.$$

We set

$$\|f\|_{p(\cdot)} = \inf\big\{\lambda > 0 : \kappa_{p(\cdot)}(f/\lambda) \leq 1\big\}.$$

We see that $\big(L^{p(\cdot)}(\mathbb{H}^n), \|.\|_{p(\cdot)}\big)$ is a quasi normed space. These spaces are referred to as the Lebesgue spaces with variable exponents or variable Lebesgue spaces (see [1]).

In [5], for $0 < p < \infty$, G. Folland and E. M. Stein defined the Hardy spaces $H^p(\mathbb{H}^n)$ on the Heisenberg group with the norm given by

$$\|f\|_{H^p(\mathbb{H}^n)} = \Big\| \sup_{t>0} \sup_{\phi \in \mathcal{F}_N} |f * \phi_t| \Big\|_p,$$

where $\phi_t(z) = t^{-Q} \phi(t^{-1} \cdot z)$ with $t > 0$, and $\mathcal{F}_N$ is a suitable family of smooth functions. In the paper [2], J. Fang and J. Zhao defined the variable Hardy spaces on the Heisenberg group $H^{p(\cdot)}(\mathbb{H}^n)$, by replacing L^p with $L^{p(\cdot)}$ in the above norm, and they investigated several of their properties.

Let $L^q_{\text{loc}}(\mathbb{H}^n)$, $1 < q < \infty$, be the space of all measurable functions g on $\mathbb{H}^n$ that belong locally to L^q for compact sets of $\mathbb{H}^n$. We endow $L^q_{\text{loc}}(\mathbb{H}^n)$ with the topology generated by the seminorms

$$|g|_{q,B} = \left(|B|^{-1} \int_B |g(w)|^q \, dw \right)^{1/q},$$

where B is a ρ-ball in $\mathbb{H}^n$ and $|B|$ denotes its Haar measure.

For $g \in L^q_{\text{loc}}(\mathbb{H}^n)$, we define a maximal function $\eta_{q,\gamma}(g; z)$ as

$$\eta_{q,\gamma}(g; z) = \sup_{r>0} r^{-\gamma} |g|_{q,B(z,r)}, \tag{1.4}$$

where γ is a positive real number and $B(z, r)$ is the ρ-ball centered at z with radius r.

Let k be a nonnegative integer and $\mathcal{P}_k$ the subspace of $L^q_{\text{loc}}(\mathbb{H}^n)$ formed by all the polynomials of homogeneous degree at most k. We denote by E^q_k the quotient space of $L^q_{\text{loc}}(\mathbb{H}^n)$ by $\mathcal{P}_k$. If $G \in E^q_k$, we define the seminorm $\|G\|_{q,B} = \inf \{|g|_{q,B} : g \in G\}$. The family of all these seminorms induces on E^q_k the quotient topology.

Given a positive real number γ, we can write $\gamma = k + t$, where k is a nonnegative integer and $0 < t \leq 1$. This decomposition is unique.

For $G \in E^q_k$, we define a maximal function $N_{q,\gamma}(G; z)$ as

$$N_{q,\gamma}(G; z) = \inf \{\eta_{q,\gamma}(g; z) : g \in G\}. \tag{1.5}$$

Such maximal function is lower semicontinuous; see [12, Lemma 4.1].

Definition 1.1. Let $p(\cdot) : \mathbb{H}^n \to (0, \infty)$ be an exponent such that $0 < p_- \leq p_+ < \infty$. We say that an element $G \in E^q_k$ belongs to the variable Calderón–Hardy space on the Heisenberg group $\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{H}^n)$ if the maximal function $N_{q,\gamma}(G; \cdot) \in L^{p(\cdot)}(\mathbb{H}^n)$. The "norm" of G in $\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{H}^n)$ is defined as $\|G\|_{\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{H}^n)} = \|N_{q,\gamma}(G; \cdot)\|_{p(\cdot)}$.

The Calderón–Hardy spaces were defined in the setting of the classical Lebesgue spaces by A. B. Gatto, J. G. Jiménez and C. Segovia [6]; they characterized the solution of the equation $\Delta^m F = f$, $m \in \mathbb{N}$, for $f \in H^p(\mathbb{R}^n)$. Moreover, they proved that the iterated Laplace operator Δ^m is a bijective mapping from the Calderón–Hardy spaces onto $H^p(\mathbb{R}^n)$.

The equation $\Delta^m F = f$, $m \in \mathbb{N}$, for $f \in H^{p(\cdot)}(\mathbb{R}^n)$ and $f \in H^p(\mathbb{R}^n, w)$, was studied by the author in [9] and [11], respectively, obtaining analogous results to those of Gatto, Jiménez and Segovia. Lately, Z. Liu, Z. He, and H. Mo [7] extended the definition of Calderón–Hardy spaces to the Orlicz setting. These new Orlicz Calderón–Hardy spaces can cover classical Calderón–Hardy spaces in [6]. As an application, they solved the equation $\Delta^m F = f$ when $f \in H^\Phi(\mathbb{R}^n)$, where $H^\Phi(\mathbb{R}^n)$ are the Orlicz–Hardy spaces defined in [8].

Recently, the author studied an analogous problem on the Heisenberg group [12]. More precisely, we proved that the equation $\mathcal{L}F = f$ for $f \in H^p(\mathbb{H}^n)$ has a unique solution F in $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$, where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{H}^n$, $1 < q < \frac{n+1}{n}$, and $Q\left(2 + \frac{Q}{q}\right)^{-1} < p \leq 1$.

The purpose of this work is to extend the results obtained in [12] to the variable setting. To do so, we must take into account certain aspects inherent to the variable spaces.

We say that an exponent function $p(\cdot) : \mathbb{H}^n \to (0,\infty)$ such that $0 < p_- \leq p_+ < \infty$ belongs to $\mathcal{P}^{\log}(\mathbb{H}^n)$, if there exist positive constants C, C_∞, and p_∞ such that $p(\cdot)$ satisfies the local log-Hölder continuity condition, i.e.,

$$|p(z) - p(w)| \leq \frac{C}{-\log(\rho(z^{-1} \cdot w))} \quad \text{for } \rho(z^{-1} \cdot w) \leq \frac{1}{2},$$

and is log-Hölder continuous at infinity, i.e.,

$$|p(z) - p_\infty| \leq \frac{C_\infty}{\log(e + \rho(z))} \quad \text{for all } z \in \mathbb{H}^n.$$

Here ρ is the *Koranyi norm* given by (1.2).

Let $G \in \mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)$; then $N_{q,2}(G; z_0) < \infty$ a.e. $z_0 \in \mathbb{H}^n$. By Lemma 2.4 (i) below, there exists $g \in G$ such that $N_{q,2}(G; z_0) = \eta_{q,2}(g; z_0)$. Now, from Proposition 2.6 below it follows that $g \in \mathcal{S}'(\mathbb{H}^n)$. So the sub-Laplacian of g, $\mathcal{L}g$, is well defined in the sense of distributions. On the other hand, since any two representatives of G differ in a polynomial of homogeneous degree at most 1, we have that $\mathcal{L}g$ is independent of the representative $g \in G$ chosen. Therefore, for $G \in E^q_1$, we shall define $\mathcal{L}G$ as the distribution $\mathcal{L}g$, where g is any representative of G.

Our main results are the following.

Theorem 3.1. *Let $p(\cdot)$ be an exponent that belongs to $\mathcal{P}^{\log}(\mathbb{H}^n)$ and $1 < q < \frac{n+1}{n}$. If $\underline{p} > Q(2 + Q/q)^{-1}$, then the sub-Laplacian $\mathcal{L}$ is a bijective mapping from $\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)$ onto $H^{p(\cdot)}(\mathbb{H}^n)$. Moreover, there exist two positive constants c_1 and c_2 such that*

$$c_1 \|G\|_{\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)} \leq \|\mathcal{L}G\|_{H^{p(\cdot)}(\mathbb{H}^n)} \leq c_2 \|G\|_{\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)}$$

hold for all $G \in \mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)$.

The case $p_+ \leq Q(2 + Q/q)^{-1}$ is trivial.

Theorem 3.2. *If $p(\cdot)$ is an exponent function on $\mathbb{H}^n$ such that $p_+ \leq Q(2+Q/q)^{-1}$, then $\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n) = \{0\}$.*

In Section 2 we state the basics of the Heisenberg group and variable Lebesgue spaces, together with some auxiliary lemmas and propositions related to variable Calderón–Hardy and variable Hardy spaces on the Heisenberg group. We also recall the definition and atomic decomposition of variable Hardy spaces on $\mathbb{H}^n$ given in [2]. Finally, in Section 3 we prove our main results.

Notation. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some constant c, and $A \sim B$ stands for $B \lesssim A \lesssim B$. We denote by $B(z_0, \delta)$ the ρ-ball centered at $z_0 \in \mathbb{H}^n$ with radius δ. Given $\beta > 0$ and a ρ-ball $B = B(z_0, \delta)$, we set $\beta B = B(z_0, \beta\delta)$. For a measurable subset $E \subseteq \mathbb{H}^n$ we denote by $|E|$ and χ_E the Haar measure of E and the characteristic function of E, respectively.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

2.1. **Basics on the Heisenberg group.** Let $\mathbb{H}^n$ be the Heisenberg group with group law given by (1.1). If $B(z_0,\delta)$ is a ρ-ball of $\mathbb{H}^n$, then its Haar measure is

$$|B(z_0,\delta)| = c\delta^Q,$$

where $c = |B(e,1)|$ and $Q = 2n+2$. Given $\lambda > 0$, we put $\lambda B = \lambda B(z_0,\delta) = B(z_0,\lambda\delta)$. So $|\lambda B| = \lambda^Q |B|$.

The Hardy–Littlewood maximal operator M on the Heisenberg group is defined by

$$Mf(z) = \sup_{B\ni z} |B|^{-1} \int_B |f(w)|\, dw,$$

where f is a locally integrable function on $\mathbb{H}^n$ and the supremum is taken over all the ρ-balls B containing z.

For every $i = 1,2,\dots,2n+1$, X_i denotes the left invariant vector field on $\mathbb{H}^n$ given by

$$X_i = \frac{\partial}{\partial x_i} + 2x_{i+n}\frac{\partial}{\partial t}, \quad i = 1,2,\dots,n;$$
$$X_{i+n} = \frac{\partial}{\partial x_{i+n}} - 2x_i\frac{\partial}{\partial t}, \quad i = 1,2,\dots,n;$$

and

$$X_{2n+1} = \frac{\partial}{\partial t}.$$

The sub-Laplacian on $\mathbb{H}^n$, denoted by $\mathcal{L}$, is the counterpart of the Laplacian Δ on $\mathbb{R}^n$. The sub-Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = -\sum_{i=1}^{2n} X_i^2,$$

where X_i, $i = 1,\dots,2n$, are the left invariant vector fields defined above.

Given a multi-index $I = (i_1, i_2, \dots, i_{2n}, i_{2n+1}) \in (\mathbb{N}\cup\{0\})^{2n+1}$, we set

$$|I| = i_1 + i_2 + \cdots + i_{2n} + i_{2n+1}, \qquad d(I) = i_1 + i_2 + \cdots + i_{2n} + 2\, i_{2n+1}.$$

The amount $|I|$ is called the length of I, and $d(I)$ the homogeneous degree of I. We adopt the following multi-index notation for higher-order derivatives and for monomials on $\mathbb{H}^n$. If $I = (i_1, i_2, \dots, i_{2n+1})$ is a multi-index, $X = \{X_i\}_{i=1}^{2n+1}$, and $z = (x,t) = (x_1,\dots,x_{2n},t) \in \mathbb{H}^n$, we put

$$X^I := X_1^{i_1} X_2^{i_2} \cdots X_{2n+1}^{i_{2n+1}} \quad \text{and} \quad z^I := x_1^{i_1} \cdots x_{2n}^{i_{2n}} \cdot t^{i_{2n+1}}.$$

Every polynomial p on $\mathbb{H}^n$ can be written as a unique finite linear combination of the monomials z^I, that is

$$p(z) = \sum_{I\in\mathbb{N}_0^n} c_I z^I, \tag{2.1}$$

where all but finitely many of the coefficients $c_I \in \mathbb{C}$ vanish. The *homogeneous degree* of a polynomial p written as (2.1) is $\max\{d(I) : I \in \mathbb{N}_0^n \text{ with } c_I \neq 0\}$. Let $k \in \mathbb{N} \cup \{0\}$. We recall that $\mathcal{P}_k$ denotes the subspace formed by all the polynomials of homogeneous degree at most k. So, every $p \in \mathcal{P}_k$ can be written as $p(z) = \sum_{d(I) \leq k} c_I \, z^I$, with $c_I \in \mathbb{C}$.

The Schwartz space $\mathcal{S}(\mathbb{H}^n)$ is defined by

$$\mathcal{S}(\mathbb{H}^n) = \left\{ \phi \in C^\infty(\mathbb{H}^n) : \sup_{z \in \mathbb{H}^n} (1+\rho(z))^N |(X^I \phi)(z)| < \infty \; \forall \, N \in \mathbb{N}_0, \, I \in (\mathbb{N}_0)^{2n+1} \right\}.$$

We topologize the space $\mathcal{S}(\mathbb{H}^n)$ with the following family of seminorms:

$$\|\phi\|_{\mathcal{S}(\mathbb{H}^n), \, N} = \sum_{d(I) \leq N} \sup_{z \in \mathbb{H}^n} (1 + \rho(z))^N |(X^I \phi)(z)| \quad (N \in \mathbb{N}_0).$$

With $\mathcal{S}'(\mathbb{H}^n)$ we denote the dual space of $\mathcal{S}(\mathbb{H}^n)$.

2.2. **Basics on variable Lebesgue spaces.** Given an exponent $p(\cdot) : \mathbb{H}^n \to (0, \infty)$, we consider the variable Lebesgue space $L^{p(\cdot)}(\mathbb{H}^n)$ defined above. It is not so hard to see the following:

(1) $\|f\|_{p(\cdot)} \geq 0$, and $\|f\|_{p(\cdot)} = 0$ if and only if $f \equiv 0$.

(2) $\|c \, f\|_{p(\cdot)} = |c| \, \|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.

(3) $\|f + g\|_{p(\cdot)}^{\underline{p}} \leq \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}}$.

(4) $\|f\|_{p(\cdot)}^s = \||f|^s\|_{p(\cdot)/s}$, for every $s > 0$.

A direct consequence of the $\underline{p}$-triangle inequality is the quasi-triangle inequality

$$\|f + g\|_{p(\cdot)} \leq 2^{1/\underline{p}-1} \left(\|f\|_{p(\cdot)} + \|g\|_{p(\cdot)} \right)$$

for all f, $g \in L^{p(\cdot)}(\mathbb{H}^n)$.

The following Fefferman–Stein vector-valued maximal inequality on $L^{p(\cdot)}(\mathbb{H}^n)$ was proved in [2].

Theorem 2.1 ([2, Theorem 4.2]). *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{H}^n)$ with $1 < p_- \leq p_+ < \infty$. Then for every $\theta \in (1, \infty)$, we have*

$$\left\| \left(\sum_{j=1}^{\infty} (M f_j)^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}(\mathbb{H}^n)} \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^\theta \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}(\mathbb{H}^n)}$$

for all sequences of bounded measurable functions with compact support $\{f_j\}_{j=1}^\infty$.

The following two results refer to the modular function $\kappa_{p(\cdot)}$ given by (1.3).

Lemma 2.2. *Let $p(\cdot)$ be an exponent on $\mathbb{H}^n$ such that $0 < p_- \leq p_+ < \infty$. Then $f \in L^{p(\cdot)}(\mathbb{H}^n)$ if and only if $\kappa_{p(\cdot)}(f) < \infty$.*

Proof. Clearly, if $\kappa_{p(\cdot)}(f) < \infty$, then $f \in L^{p(\cdot)}(\mathbb{H}^n)$. Conversely, if $f \in L^{p(\cdot)}(\mathbb{H}^n)$, then we have that $\kappa_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 1$. Then

$$\kappa_{p(\cdot)}(f) = \int_{\mathbb{H}^n} \left| \frac{\lambda f(z)}{\lambda} \right|^{p(z)} dz \leq \lambda^{p_+} \kappa_{p(\cdot)}(f/\lambda) < \infty. \qquad \square$$

Lemma 2.3. *Let $p(\cdot)$ be an exponent on $\mathbb{H}^n$ such that $0 < p_- \le p_+ < \infty$. If $\{f_j\}$ is a sequence of measurable functions on $\mathbb{H}^n$ such that $\kappa_{p(\cdot)}(f_j) \to 0$, then $\|f_j\|_{p(\cdot)} \to 0$.*

Proof. Suppose that $\kappa_{p(\cdot)}(f_j) \to 0$. Given $0 < \epsilon < 1$, for sufficiently large j we have $\kappa_{p(\cdot)}(f_j) \le \epsilon$, and so

$$\kappa_{p(\cdot)}\left(f_j \kappa_{p(\cdot)}(f_j)^{-1/p_+}\right) \le \kappa_{p(\cdot)}(f_j)^{-1}\kappa_{p(\cdot)}(f_j) = 1.$$

From this it follows that $\|f_j\|_{p(\cdot)} \le \kappa_{p(\kappa)}(f_j)^{1/p_+} \le \epsilon^{1/p_+}$. Thus, $\|f_j\|_{p(\cdot)} \to 0$. □

2.3. **Variable Calderón–Hardy spaces on $\mathbb{H}^n$.** Let $1 < q < \infty$ and $\gamma > 0$. In this section we establish some results concerning the maximal functions $\eta_{q,\gamma}(g;\cdot)$ and $N_{q,\gamma}(G;\cdot)$ defined in (1.4) and (1.5), respectively. We recall that the maximal function $N_{q,\gamma}(G;\cdot)$ is used to define the variable Calderón–Hardy spaces on $\mathbb{H}^n$ (see Definition 1.1).

Lemma 2.4. *Let $G \in E_k^q$ with $N_{q,\gamma}(G;z_0) < \infty$ for some $z_0 \in \mathbb{H}^n$. Then*

(i) *There exists a unique $g \in G$ such that $\eta_{q,\gamma}(g;z_0) < \infty$ and, therefore, $\eta_{q,\gamma}(g;z_0) = N_{q,\gamma}(G;z_0)$.*

(ii) *For any ρ-ball B, there is a constant c depending on z_0 and B such that if g is the unique representative of G given in* (i)*, then*

$$\|G\|_{q,B} \le |g|_{q,B} \le c\,\eta_{q,\gamma}(g;z_0) = c\,N_{q,\gamma}(G;z_0).$$

The constant c can be chosen independently of z_0 provided that z_0 varies in a compact set.

Proof. The proof is similar to the one given in [6, Lemma 3]. □

Lemma 2.5. *Let $\{G_j\}$ be a sequence in E_k^q such that for a given point $z_0 \in \mathbb{H}^n$, the series $\sum_j N_{q,\gamma}(G_j;z_0)$ is finite. Then*

(i) *The series $\sum_j G_j$ converges in E_k^q to an element G and*

$$N_{q,\gamma}(G;z_0) \le \sum_j N_{q,\gamma}(G_j;z_0).$$

(ii) *If g_j is the unique representative of G_j satisfying $\eta_{q,\gamma}(g_j;z_0) = N_{q,\gamma}(G_j;z_0)$, then $\sum_j g_j$ converges in $L^q_{\text{loc}}(\mathbb{H}^n)$ to a function g that is the unique representative of G satisfying $\eta_{q,\gamma}(g;z_0) = N_{q,\gamma}(G;z_0)$.*

Proof. The proof is similar to the one given in [6, Lemma 4]. □

Proposition 2.6 ([12, Proposition 4.7])**.** *If $g \in L^q_{\text{loc}}(\mathbb{H}^n)$, $1 < q < \infty$, and there is a point $z_0 \in \mathbb{H}^n$ such that $\eta_{q,\gamma}(g;z_0) < \infty$, then $g \in \mathcal{S}'(\mathbb{H}^n)$.*

Given an exponent $p(\cdot) : \mathbb{H}^n \to (0,\infty)$, $1 < q < \infty$, and $\gamma > 0$, we consider the variable Calderón–Hardy space $\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{H}^n)$. The following result states the completeness of variable Calderón–Hardy spaces.

Proposition 2.7. *The space $\mathcal{H}^{p(\cdot)}_{q,\gamma}(\mathbb{H}^n)$, $0 < p_- \le p_+ < \infty$, is complete.*

Proof. The proof is similar to the one given in [9, Proposition 9]. □

2.4. **Variable Hardy spaces on $\mathbb{H}^n$.** In the paper [2], J. Fang and J. Zhao give a variety of distinct approaches, based on differing definitions, which all lead to the same notion of the variable Hardy space $H^{p(\cdot)}(\mathbb{H}^n)$.

We recall some terminology and notation from the study of maximal functions used in [2]. Given $N \in \mathbb{N}$, define

$$\mathcal{F}_N = \left\{ \phi \in \mathcal{S}(\mathbb{H}^n) : \sum_{d(I) \leq N} \sup_{z \in \mathbb{H}^n} (1+\rho(z))^N \, |(X^I \phi)(z)| \leq 1 \right\}.$$

For any $f \in \mathcal{S}'(\mathbb{H}^n)$, the grand maximal function of f is given by

$$\mathcal{M}_{\mathcal{F}_N} f(z) = \sup_{t>0} \sup_{\phi \in \mathcal{F}_N} \left| \left(f * t^{-Q} \phi(t^{-1} \cdot) \right)(z) \right|,$$

where N is a large and fixed integer.

Definition 2.8. Given an exponent function $p(\cdot) : \mathbb{H}^n \to (0,\infty)$ with $0 < p_- \leq p_+ < \infty$, we define the integer $\mathcal{D}_{p(\cdot)}$ by

$$\mathcal{D}_{p(\cdot)} := \min\{k \in \mathbb{N} \cup \{0\} : (2n+k+3)p_- > 2n+2\}.$$

For $N \geq \mathcal{D}_{p(\cdot)} + 1$, define the variable Hardy space $H^{p(\cdot)}(\mathbb{H}^n)$ to be the collection of $f \in \mathcal{S}'(\mathbb{H}^n)$ such that $\|\mathcal{M}_{\mathcal{F}_N} f\|_{L^{p(\cdot)}(\mathbb{H}^n)} < \infty$. Then, the "norm" on the space $H^{p(\cdot)}(\mathbb{H}^n)$ is taken to be $\|f\|_{H^{p(\cdot)}} := \|\mathcal{M}_{\mathcal{F}_N} f\|_{L^{p(\cdot)}}$.

Remark 2.9. For $N \geq \mathcal{D}_{p(\cdot)} + 1$, the $H^{p(\cdot)}$-norm defined above does not depend on N (see e.g. [5]; that reference considers the case when $p(\cdot)$ is constant, and the variable case is similar).

Definition 2.10. Let $p(\cdot) : \mathbb{H}^n \to (0,\infty)$, $0 < p_- \leq p_+ < \infty$, and $p_0 > 1$. Fix an integer $D \geq \mathcal{D}_{p(\cdot)}$. A measurable function $a(\cdot)$ on $\mathbb{H}^n$ is called a $(p(\cdot), p_0, D)$-atom centered at a ρ-ball $B = B(z_0, \delta)$ if it satisfies the following conditions:

(a_1) $\text{supp}(a) \subset B$;

(a_2) $\|a\|_{L^{p_0}(\mathbb{H}^n)} \leq \dfrac{|B|^{\frac{1}{p_0}}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{H}^n)}}$;

(a_3) $\displaystyle\int_{\mathbb{H}^n} a(z)\, z^I \, dz = 0$ for all multi-index I such that $d(I) \leq D$.

Indeed, every $(p(\cdot), p_0, D)$-atom $a(\cdot)$ belongs to $H^{p(\cdot)}(\mathbb{H}^n)$. Moreover, there exists a universal constant $C > 0$ such that $\|a\|_{H^{p(\cdot)}} \leq C$ for all $(p(\cdot), p_0, D)$-atom $a(\cdot)$.

The following results deal with the size of the ρ-balls in the $L^{p(\cdot)}(\mathbb{H}^n)$-norm.

Lemma 2.11 ([2, Lemma 4.1])**.** *Suppose that $p(\cdot)$ is an exponent function such that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{H}^n)$ and $0 < p_- \leq p_+ < \infty$.*

(1) *For all ρ-balls $B = B(z, \delta)$ with $z \in \mathbb{H}^n$ and $|B| \leq 1$, we have*

$$|B|^{\frac{1}{p_-(B)}} \sim |B|^{\frac{1}{p_+(B)}} \sim |B|^{\frac{1}{p(z)}} \sim \|\chi_B\|_{L^{p(\cdot)}(\mathbb{H}^n)}.$$

(2) *For all ρ-balls $B = B(z, \delta)$ with $z \in \mathbb{H}^n$ and $|B| \geq 1$ we have*

$$|B|^{\frac{1}{p_\infty}} \sim \|\chi_B\|_{L^{p(\cdot)}(\mathbb{H}^n)} .$$

Here the implicit constants in $\sim$ do not depend on z and $r > 0$.

Definition 2.12. Let $p(\cdot) : \mathbb{H}^n \to (0, \infty)$ be an exponent such that $0 < p_- \leq p_+ < \infty$. Given a sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ and a family of ρ-balls $\{B_j\}_{j=1}^{\infty}$, we define

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\right) := \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{\lambda_j \chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}} .$$

The space $H_{\text{atom}}^{p(\cdot),p_0,D}(\mathbb{H}^n)$ is the set of all distributions $f \in S'(\mathbb{H}^n)$ that can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \tag{2.2}$$

in $S'(\mathbb{H}^n)$, where $\{\lambda_j\}_{j=1}^{\infty}$ is a sequence of nonnegative numbers, the a_j's are $(p(\cdot), p_0, D)$-atoms supported on the ρ-ball B_j, and $\mathcal{A}\big(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\big) < \infty$. One defines

$$\|f\|_{H_{\text{atom}}^{p(\cdot),p_0,D}} = \inf \mathcal{A}\big(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\big),$$

where the infimum is taken over all admissible expressions as in (2.2).

Definition 2.13. Given a collection of sets $\{E_j\}_{j\in\mathbb{N}}$, we say that the family $\{E_j\}_{j\in\mathbb{N}}$ has the bounded intersection property if there exists $L \in \mathbb{N}$ such that no point of $\bigcup_{k\in\mathbb{N}} E_k$ lies in more than L of the sets E_j.

Next, we establish the following version of [2, Theorem 4.5].

Theorem 2.14. *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{H}^n)$ and $p_0 > \max\{p_+, 1\}$ is sufficiently large, then the quantities $\|f\|_{H_{\text{atom}}^{p(\cdot),p_0,D}}$ and $\|f\|_{H^{p(\cdot)}}$ are comparable. Moreover, f admits an atomic decomposition $f = \sum_{j=1}^{\infty} \lambda_j a_j$ such that*

$$\mathcal{A}\big(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\big) \leq C \|f\|_{H^{p(\cdot)}},$$

where C does not depend on f and the family of ρ-balls $\{B_j\}_{j=1}^{\infty}$ has the bounded intersection property.

The following two results were proved in Lemma 5.7 and Proposition 3.3 of [10], respectively.

Lemma 2.15. *Let $p(\cdot) : \mathbb{H}^n \to (0, \infty)$ be an exponent function with $0 < p_- \leq p_+ < \infty$, and let $\{B_j\}$ be a family of ρ-balls which satisfies the bounded intersection property. If $0 < p_* < \underline{p}$, then*

$$\left\| \left\{ \sum_j \left(\frac{\lambda_j \chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \right)^{p_*} \right\}^{1/p_*} \right\|_{L^{p(\cdot)}} \sim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\right)$$

for any sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$.

Proposition 2.16. *Let $p(\cdot) : \mathbb{H}^n \to (0,\infty)$ such that $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{H}^n)$ and $0 < p_- \leq p_+ < \infty$. Let $s > 1$ and $0 < p_* < \underline{p}$ such that $sp_* > p_+$, and let $\{b_j\}_{j=1}^{\infty}$ be a sequence of nonnegative functions in $L^s(\mathbb{H}^n)$ such that each b_j is supported in a ρ-ball $B_j \subset \mathbb{H}^n$ and*

$$\|b_j\|_{L^s(\mathbb{H}^n)} \leq A_j |B_j|^{1/s},$$

where $A_j > 0$ for all $j \geq 1$. Then, for any sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^{\infty}$ we have

$$\left\| \sum_{j=1}^{\infty} \lambda_j b_j \right\|_{L^{p(\cdot)/p_*}(\mathbb{H}^n)} \leq C \left\| \sum_{j=1}^{\infty} A_j \lambda_j \chi_{B_j} \right\|_{L^{p(\cdot)/p_*}(\mathbb{H}^n)},$$

where C is a positive constant that does not depend on $\{b_j\}_{j=1}^{\infty}$, $\{A_j\}_{j=1}^{\infty}$, and $\{\lambda_j\}_{j=1}^{\infty}$.

For our next result, we first introduce two maximal operators on $\mathbb{H}^n$. The first is a discrete maximal operator, and the second is a non-tangential maximal one. Given $\phi \in \mathcal{S}(\mathbb{H}^n)$, we define

$$(M_\phi^{\text{dis}} f)(z) := \sup\{|(f * \phi_{2^j})(z)| : j \in \mathbb{Z}\},$$

where $\phi_{2^j}(z) = 2^{-jQ}\phi(2^{-j}z)$, and put

$$(M_\phi f)(z) := \sup\big\{|(f * \phi_t)(w)| : \rho(w^{-1} \cdot z) < t,\, 0 < t < \infty\big\}.$$

The following pointwise inequality is obvious:

$$(M_\phi^{\text{dis}} f)(z) \leq (M_\phi f)(z)$$

for all $z \in \mathbb{H}^n$. Now, this inequality and [12, Proposition 4.8] lead to the next result.

Proposition 2.17. *Let $g \in L^q_{\text{loc}} \cap \mathcal{S}'(\mathbb{H}^n)$ and $f = \mathcal{L}g$ in $\mathcal{S}'(\mathbb{H}^n)$. If $\phi \in \mathcal{S}(\mathbb{H}^n)$ and $N > Q + 2$, then*

$$(M_\phi^{\text{dis}} f)(z) \leq C \|\phi\|_{\mathcal{S}(\mathbb{H}^n), N}\; \eta_{q,2}(g; z)$$

holds for all $z \in \mathbb{H}^n$.

A fundamental solution for the sub-Laplacian on $\mathbb{H}^n$ was obtained by G. Folland in [4]. More precisely, he proved the following result.

Theorem 2.18. *$c_n\, \rho^{-2n}$ is a fundamental solution for $\mathcal{L}$ with source at 0, where*

$$\rho(x,t) = (|x|^4 + t^2)^{1/4}$$

and

$$c_n = \left[n(n+2) \int_{\mathbb{H}^n} |x|^2 (\rho(x,t)^4 + 1)^{-(n+4)/2}\, dx\, dt \right]^{-1}.$$

In other words, for any $u \in \mathcal{S}(\mathbb{H}^n)$, $\big(\mathcal{L}u, c_n \rho^{-2n}\big) = u(0)$.

If a is a bounded function with compact support on $\mathbb{H}^n$, its potential b, defined as

$$b(z) := \left(a * c_n\, \rho^{-2n}\right)(z) = c_n \int_{\mathbb{H}^n} \rho(w^{-1}\cdot z)^{-2n} a(w)\, dw,$$

is a locally bounded function and, by Theorem 2.18, $\mathcal{L}b = a$ in the sense of distributions. For these potentials, we have the following result.

In what follows, β is the constant in [5, Corollary 1.44]; we observe that $\beta \geq 1$ (see [5, p. 29]).

Lemma 2.19. *Let $a(\cdot)$ be a $(p(\cdot), p_0, D)$-atom centered at the ρ-ball $B = B(z_0, \delta)$. If $b(z) = (a * c_n\rho^{-2n})(z)$, then for $\rho(z_0^{-1}z) \geq 2\beta^2\delta$ and every multi-index I there exists a positive constant C_I such that*

$$\left|(X^I b)(z)\right| \leq C_I\, \delta^{2+Q}\, \|\chi_B\|_{p(\cdot)}^{-1}\, \rho(z_0^{-1}\cdot z)^{-Q-d(I)}$$

holds.

Proof. The proof is similar to the one given in [12, Lemma 4.11], but considering now Definition 2.10 above. □

Proposition 2.20. *Let $a(\cdot)$ be a $(p(\cdot), p_0, D)$-atom centered at the ρ-ball $B = B(z_0, \delta)$. If $b(z) = (a * c_n\rho^{-2n})(z)$, then for all $z \in \mathbb{H}^n$*

$$\begin{aligned} N_{q,2}\big(\widetilde{b}; z\big) \lesssim \|\chi_B\|_{p(\cdot)}^{-1} \left[(M\chi_B)(z)\right]^{\frac{2+Q/q}{Q}} + \chi_{4\beta^2 B}(z)(Ma)(z) \\ + \chi_{4\beta^2 B}(z) \sum_{d(I)=2} (T_I^* a)(z), \end{aligned} \tag{2.3}$$

where $\widetilde{b}$ is the class of b in E_1^q, M is the Hardy–Littlewood maximal operator, and

$$(T_I^* a)(z) = \sup_{\epsilon>0} \left| \int_{\rho(w^{-1}\cdot z)>\epsilon} (X^I \rho^{-2n})(w^{-1}\cdot z) a(w)\, dw \right|.$$

Proof. We point out that the argument used in the proof of [12, Proposition 4.12] to obtain the pointwise inequality (16) there works in this setting as well, provided that one considers conditions (a_1), (a_2), and (a_3) given in Definition 2.10 of a $(p(\cdot), p_0, D)$-atom. These conditions are similar to those of the atoms in the classical context (see [5, pp. 71–72]). Therefore, this observation and Lemma 2.19 allow us to get (2.3). □

3. Main results

In this section we will prove our main results.

Theorem 3.1. *Let $p(\cdot)$ be an exponent that belongs to $\mathcal{P}^{\log}(\mathbb{H}^n)$ and $1 < q < \frac{n+1}{n}$. If $\underline{p} > Q(2 + Q/q)^{-1}$, then the sub-Laplacian $\mathcal{L}$ is a bijective mapping from $\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$ onto $H^{p(\cdot)}(\mathbb{H}^n)$. Moreover, there exist two positive constants c_1 and c_2 such that*

$$c_1 \|G\|_{\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)} \leq \|\mathcal{L}G\|_{H^{p(\cdot)}(\mathbb{H}^n)} \leq c_2 \|G\|_{\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)}$$

hold for all $G \in \mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$.

Proof. Let $G \in \mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)$. Since $N_{q,2}(G; z)$ is finite a.e. $z \in \mathbb{H}^n$, by Lemma 2.4 (i) and Proposition 2.6, the unique representative g of G (which depends on z) satisfying $\eta_{q,2}(g; z) = N_{q,2}(G; z)$ is a function in $L^q_{\text{loc}}(\mathbb{H}^n) \cap \mathcal{S}'(\mathbb{H}^n)$. In particular, for a radial function $\phi \in \mathcal{S}(\mathbb{H}^n)$ with $\int \phi = 1$, by Proposition 2.17 we get

$$M^{\text{dis}}_\phi(\mathcal{L}G)(z) \le C_\phi \, N_{q,2}(G; z).$$

Thus, this inequality and [2, Theorem 3.2] give $\mathcal{L}G \in H^{p(\cdot)}(\mathbb{H}^n)$ and

$$\|\mathcal{L}G\|_{H^{p(\cdot)}(\mathbb{H}^n)} \le C \, \|G\|_{\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)}.$$

This proves the continuity of the sub-Laplacian $\mathcal{L}$ from $\mathcal{H}^{p(\cdot)}_{q,2}(\mathbb{H}^n)$ into $H^{p(\cdot)}(\mathbb{H}^n)$.

Now we shall see that the operator $\mathcal{L}$ is onto. By Theorem 2.14, given $f \in H^{p(\cdot)}(\mathbb{H}^n)$ there exists a sequence of nonnegative numbers $\{\lambda_j\}_{j=1}^\infty$ and a sequence of ρ-balls $\{B_j\}_{j=1}^\infty$ (with the bounded intersection property) and $(p(\cdot), p_0, D)$-atoms a_j centered at B_j, such that $f = \sum_{j=1}^\infty \lambda_j a_j$ and

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{B_j\}_{j=1}^\infty, p(\cdot)\right) \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{H}^n)}.$$

For each $j \in \mathbb{N}$, we put $b_j(z) = (a_j * c_n \rho^{-2n})(z)$, and from Proposition 2.20 we have

$$\begin{aligned} N_{q,2}\big(\widetilde{b}_j;\, z\big) \lesssim \|\chi_{B_j}\|^{-1}_{p(\cdot)} \left[(M\chi_{B_j})(z)\right]^{\frac{2+Q/q}{Q}} + \chi_{4\beta^2 B}(z)(Ma_j)(z) \\ + \chi_{4\beta^2 B_j}(z) \sum_{d(I)=2} (T^*_I a_j)(z). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^\infty k_j N_{q,2}\big(\widetilde{b}_j;\, z\big) &\lesssim \sum_{j=1}^\infty \lambda_j \|\chi_{B_j}\|^{-1}_{p(\cdot)} \left[(M\chi_{B_j})(z)\right]^{\frac{2+Q/q}{Q}} \\ &\quad + \sum_{j=1}^\infty \lambda_j \chi_{4\beta^2 B_j}(z)(Ma_j)(z) \\ &\quad + \sum_{j=1}^\infty \lambda_j \chi_{4\beta^2 B_j}(z) \sum_{d(I)=2} (T^*_I a_j)(z) \\ &= I + II + III. \end{aligned}$$

To study I, by hypothesis, we have that $(2 + Q/q)\underline{p} > Q$. Then

$$\begin{aligned} \|I\|_{p(\cdot)} &= \left\| \sum_{j=1}^\infty \lambda_j \|\chi_{B_j}\|^{-1}_{p(\cdot)} \left[(M\chi_{B_j})(\cdot)\right]^{\frac{2+Q/q}{Q}} \right\|_{p(\cdot)} \\ &= \left\| \left\{ \sum_{j=1}^\infty \lambda_j \|\chi_{B_j}\|^{-1}_{p(\cdot)} \left[(M\chi_{B_j})(\cdot)\right]^{\frac{2+Q/q}{Q}} \right\}^{\frac{Q}{2+Q/q}} \right\|^{\frac{2+Q/q}{Q}}_{\frac{2+Q/q}{Q} p(\cdot)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \lambda_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \chi_{B_j}(\cdot) \right\}^{\frac{Q}{2+Q/q}} \right\|_{\frac{2+Q/q}{Q} p(\cdot)}^{\frac{2+Q/q}{Q}} \\
&= \left\| \sum_{j=1}^{\infty} \lambda_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \chi_{B_j}(\cdot) \right\|_{p(\cdot)} \\
&\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left(\lambda_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \chi_{B_j}(\cdot) \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{p(\cdot)} \\
&= \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot) \right) \lesssim \|f\|_{H^{p(\cdot)}},
\end{aligned}
$$

where the first inequality follows from Theorem 2.1, since $(2 + Q/q)\underline{p} > Q$ and $(2+Q/q)/Q > 1$. The embedding $\ell^{\underline{p}} = \ell^{\min\{p_-,1\}} \hookrightarrow \ell^1$ gives the second inequality.

To study II, let $0 < p_* < \underline{p}$ be fixed and $p_0 > \max\{p_+, 1\}$. Since the Hardy–Littlewood maximal operator M is bounded on $L^{p_0}(\mathbb{H}^n)$ (see [13, Theorem 1, p. 13]), we have

$$
\|(Ma_j)^{p_*}\|_{L^{p_0/p_*}(4\beta B_j)} \lesssim \|a_j\|_{p_0}^{p_*} \lesssim \frac{|B_j|^{p_*/p_0}}{\|\chi_{B_j}\|_{p(\cdot)}^{p_*}} \lesssim \frac{|4\beta B_j|^{p_*/p_0}}{\|\chi_{4\beta B_j}\|_{p(\cdot)/p_*}},
$$

where the third inequality holds since the quantities $\|\chi_{4\beta B_j}\|_{p(\cdot)}$ and $\|\chi_{B_j}\|_{p(\cdot)}$ are comparable (see Lemma 2.11). Now, since $0 < p_* < 1$, we apply the p_*-inequality and Proposition 2.16 with $b_j = \left(\chi_{4\beta B_j}(Ma_j)^{p_*}\right)$, $A_j = \left\|\chi_{4\beta B_j}\right\|_{L^{p(\cdot)/p_*}}^{-1}$, and $s = p_0/p_*$, to obtain

$$
\begin{aligned}
\|II\|_{L^{p(\cdot)}} &\lesssim \left\| \sum_j \left(\lambda_j \, \chi_{4\beta B_j} \, Ma_j \right)^{p_*} \right\|_{L^{p(\cdot)/p_*}}^{1/p_*} \\
&\lesssim \left\| \sum_j \left(\frac{\lambda_j}{\left\|\chi_{4\beta B_j}\right\|_{L^{p(\cdot)}}} \right)^{p_*} \chi_{4\beta B_j} \right\|_{L^{p(\cdot)/p_*}}^{1/p_*}.
\end{aligned}
$$

It is easy to check that $\chi_{4\beta B_j} \leq (M\chi_{B_j})^2$. From this inequality, Theorem 2.1, and Lemma 2.11, we have

$$
\begin{aligned}
\|II\|_{L^{p(\cdot)}} &\lesssim \left\| \left\{ \sum_j \left(\frac{\lambda_j^{p_*/2}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}^{p_*/2}} (M\chi_{B_j}) \right)^2 \right\}^{1/2} \right\|_{L^{2p(\cdot)/p_*}}^{2/p_*} \\
&\lesssim \left\| \left\{ \sum_j \left(\frac{\lambda_j \chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \right)^{p_*} \right\}^{1/p_*} \right\|_{L^{p(\cdot)}}.
\end{aligned}
$$

Finally, Lemma 2.15 gives

$$
\|II\|_{L^{p(\cdot)}} \lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot) \right) \lesssim \|f\|_{H^{p(\cdot)}}.
$$

Now we study III. By [4, Theorem 3] and [13, Corollary 2, p. 36] (see also [13, §2.5, p. 11]), we have, for every multi-index I with $d(I) = 2$, that the operator T_I^* is bounded on $L^{p_0}(\mathbb{H}^n)$ for every $1 < p_0 < \infty$. Proceeding as in the estimate of II, we get

$$\|III\|_{L^{p(\cdot)}} \lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{B_j\}_{j=1}^{\infty}, p(\cdot)\right) \lesssim \|f\|_{H^{p(\cdot)}}.$$

Thus, we have

$$\left\|\sum_{j=1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j;\, \cdot\big)\right\|_{p(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}}.$$

By Lemma 2.2, we obtain $\kappa_{p(\cdot)}\left(\sum_{j=1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j;\cdot\big)\right) < \infty$. Hence

$$\sum_{j=1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j;\, z\big) < \infty \quad \text{a.e. } z \in \mathbb{H}^n \tag{3.1}$$

and

$$\kappa_{p(\cdot)}\left(\sum_{j=M+1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j;\, \cdot\big)\right) \to 0 \quad \text{as } M \to \infty. \tag{3.2}$$

From (3.1) and Lemma 2.5, there exists a function G such that $\sum_{j=1}^{\infty} \lambda_j \widetilde{b}_j = G$ in E_1^q and

$$N_{q,2}\left(\left(G - \sum_{j=1}^{M} \lambda_j \widetilde{b}_j\right); z\right) \leq c \sum_{j=M+1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j; z\big).$$

This estimate, together with (3.2) and Lemma 2.3, implies

$$\left\|G - \sum_{j=1}^{M} \lambda_j \widetilde{b}_j\right\|_{\mathcal{H}_{q,2}^{p(\cdot)}} \to 0 \quad \text{as } M \to \infty.$$

By Proposition 2.7, we have that $G \in \mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$ and $G = \sum_{j=1}^{\infty} \lambda_j \widetilde{b}_j$ in $\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$. Since $\mathcal{L}$ is a continuous operator from $\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$ into $H^{p(\cdot)}(\mathbb{H}^n)$, we get

$$\mathcal{L}G = \sum_j \lambda_j \mathcal{L}\widetilde{b}_j = \sum_j \lambda_j a_j = f$$

in $H^{p(\cdot)}(\mathbb{H}^n)$. This shows that $\mathcal{L}$ is onto $H^{p(\cdot)}(\mathbb{H}^n)$. Moreover,

$$\|G\|_{\mathcal{H}_{q,2}^{p(\cdot)}} = \left\|\sum_{j=1}^{\infty} \lambda_j \widetilde{b}_j\right\|_{\mathcal{H}_{q,2}^{p(\cdot)}} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j N_{q,2}\big(\widetilde{b}_j; \cdot\big)\right\|_{p(\cdot)} \lesssim \|f\|_{H^{p(\cdot)}} = \|\mathcal{L}G\|_{H^{p(\cdot)}}.$$

To conclude the proof, we will show that the operator $\mathcal{L}$ is injective on $\mathcal{H}_{q,2}^{p(\cdot)}$. Let $\mathcal{O} = \{z : N_{q,2}(G; z) > 1\}$. The set $\mathcal{O}$ is open because $N_{q,2}(G; \cdot)$ is lower semicontinuous. We take a constant $r > 0$ such that $r \geq \max\{q, p_+\}$. Since $N_{q,2}(G; \cdot) \in L^{p(\cdot)}(\mathbb{H}^n)$, it follows that $|\mathcal{O}|$ is finite and $N_{q,2}(G; \cdot) \in L^r(\mathbb{H}^n \setminus \mathcal{O})$; thus, by following the proof of [12, Theorem 4.10], we have that if $G \in \mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$ and $\mathcal{L}G = 0$, then $G \equiv 0$. This proves the injectivity of $\mathcal{L}$. □

Therefore, Theorem 3.1 allows us to conclude, for $Q(2+Q/q)^{-1} < \underline{p}$, that the equation

$$\mathcal{L}F = f, \quad f \in H^{p(\cdot)}(\mathbb{H}^n)$$

has a unique solution in $\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$, namely $F := \mathcal{L}^{-1}f$.

We shall now see that the case $0 < p_+ \leq Q\left(2 + \frac{Q}{q}\right)^{-1}$ is trivial.

Theorem 3.2. *If $p(\cdot)$ is an exponent function on $\mathbb{H}^n$ such that $p_+ \leq Q(2+Q/q)^{-1}$, then $\mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n) = \{0\}$.*

Proof. Let $G \in \mathcal{H}_{q,2}^{p(\cdot)}(\mathbb{H}^n)$ and assume $G \neq 0$. Then there exists $g \in G$ that is not a polynomial of homogeneous degree less or equal to 1. It is easy to check that there exist a positive constant c and a ρ-ball $B = B(e,r)$ with $r > 1$ such that

$$\int_B |g(w) - P(w)|^q \, dw \geq c > 0$$

for every $P \in \mathcal{P}_1$.

Let z be a point such that $\rho(z) > r$ and let $\delta = 2\rho(z)$. Then $B(e,r) \subset B(z,\delta)$. If $h \in G$, then $h = g - P$ for some $P \in \mathcal{P}_1$ and

$$\delta^{-2}|h|_{q,B(z,\delta)} \geq c\rho(z)^{-2-Q/q}.$$

So $N_{q,2}(G;\, z) \geq c\,\rho(z)^{-2-Q/q}$ for $\rho(z) > r$. Since $p_+ \leq Q(2+Q/q)^{-1}$, we have that

$$\kappa_{p(\cdot)}\left(N_{q,2}(G;\cdot)\right) \geq c \int_{\rho(z)>r} \rho(z)^{-(2+Q/q)p_+} \, dz = \infty,$$

which gives a contradiction. Thus $\mathcal{H}_{q,2}^{p}(\mathbb{H}^n) = \{0\}$ if $p_+ \leq Q(2+Q/q)^{-1}$. □

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